Problem set 3 for Representations of Finite Groups

Representations of product groups. Let G and H be finite groups. Let $G \times H$ be the product group. Let (V, π) be a representation of G and let (W, θ) be a representation of H. Then $V \otimes W$ is a representation of $G \times H$ by letting (g, h) act by $\pi(g) \otimes \theta(h)$; this representation of $G \times H$ is sometimes denoted $(V \otimes W, \pi \otimes \theta)$, sometimes $V \otimes W$, and sometimes $\pi \otimes \theta$.

Exercise 1. Let G and H be finite groups. Let $G \times H$ be the product group. In this exercise all representations are finite dimensional complex representations.

(1) Express the number of conjugacy classes of $G \times H$ in terms of the number of conjugacy classes of G and H.
(2) With (V, π) and (W, θ) representations of G and H show that the character $\chi_{\pi\otimes\theta}$ is given by the rule
$$\chi_{\pi\otimes\theta}((g, h)) = \chi_\pi(g)\chi_\theta(h)$$
(3) Show that if V and W are irreducible, then $V \otimes W$ is irreducible. Hint: compute $\langle \chi_{V\otimes W}, \chi_{V\otimes W}\rangle$.
(4) Using that the number of irreducible complex representations of a finite group is equal to the number of conjugacy classes in the group, prove every irreducible representation of $G \times H$ is of this form. Hint: count.

Exercise 2. Let $\tau : G \to H$ be a homomorphism of finite groups. Let (V, ρ) be a finite dimensional complex representation of H. Let $(V, \rho \circ \tau)$ be the corresponding representation of H.

(1) Prove that if τ is surjective and (V, ρ) is irreducible, then $(V, \rho \circ \tau)$ is irreducible.
(2) Give an example to show that this doesn't hold if τ is not surjective.

Exercise 3. Let G be the group of order 12 generated by elements a and b subject to the relations
$$a^6 = 1,\ a^3 = b^2,\ b^{-1}ab = a^{-1}$$
It follows from these relations that every element of G can be uniquely written as $a^r b^s$ with $0 \le r \le 5$ and $0 \le s \le 1$. Compute the character table of G using the following steps:

(1) List the conjugacy classes of G.
(2) Show that the quotient G^{ab} of G by the normal subgroup generated by a^2 is isomorphic to the cyclic group of order 4 (generated by the image of b).
(3) Using Exercise 2, explain how this gives 4 pairwise nonisomorphic 1-dimensional representations of G.
(4) Find two more 2-dimensional representations of G, compute their characters, and show they give the remaining two irreducibles. Hint: try a diagonal matrix for a and a matrix with zeros on the diagonal for b.
(5) Write the character table.

Exercise 4. A representation (V, π) of a group G is said to be *faithful*, if the homomorphism $\pi : G \to GL(V)$ is injective. Give an example of a finite group G which does not have a faithful representation of dimension ≤ 2025. Hints: try (1) or try (2) below. (1) you can look at Exercise 3 of problem set 2 and generalize it, or (2) you can take G to be the abelian group $G = (\mathbf{Z}/2\mathbf{Z})^N \cong C_2^N$ for a sufficiently large N and show a faithful representation has to have large dimension.