Problem set 2 for Representations of Finite Groups

If you find errors in this text, please email me, thanks! More generally, please keep sending me feedback on the problem sets!

Restricting representations to subgroups. Let (V, π) be a representation of a group G, so $\pi : G \to GL(V)$ is a homomorphism of groups. Let $H \subset G$ be a subgroup. Then the restriction $\pi|_H : H \to GL(V)$ is also a homomorphism. Thus the pair $(V, \pi|_H)$ is a representation of H[1].

Exercise 1. Let (V, π) be a finite dimensional representation of a finite group G over the complex numbers. Let $H \subset G$ be a subgroup. Recall from the lectures that both (V, π) and $(V, \pi|_H)$ are completely reducible. So we can write

$$(V, \pi) = \bigoplus\nolimits_{i=1,\ldots,n} (V_i, \pi_i)$$

with (V_i, π_i) an irreducible representation of G and and

$$(V, \pi|_H) = \bigoplus\nolimits_{j=1,\ldots,m} (W_j, \rho_j)$$

with (W_j, ρ_j) an irreducible representation of H. Carefully prove that $n \leq m$.

Exercise 2. Let G be a finite group. In each of the following cases, explain briefly why there does not exist a finite dimensional $\mathbf{C}$-representation π of G with character χ_π having the stated properties:

(1) $\chi_\pi(1) = -1$ where $1 \in G$ is the identity element.
(2) $\chi_\pi(1) = 1/2$,
(3) $\chi_\pi(1) = 5$ and $\chi_\pi(g) = 6$ for some $g \in G$,
(4) $\chi_\pi(1) = 2$ and $\chi_\pi(g) = 1/11$ for some $g \in G$,
(5) $\chi_\pi(g) = 4$ and $\chi_\pi(g^{-1}) = -4$ for some $g \in G$.

Exercise 3. Let

$$G = \langle a, b \text{ with relations } b^{11} = 1,\ aba^{-1} = b^2,\ a^{10} = 1\rangle$$

This group has 110 elements, which can all be written in the form $a^i b^j$ with $0 \leq i \leq 9$ and $0 \leq j \leq 10$. In fact, G is the semi-direct product $G = C_{11} \rtimes C_{10}$ where C_{11} is generated by b and the generator a of C_{10} acts on C_{11} by multiplication by 2. Prove that if (V, π) is a finite dimensional complex representation such that $\pi(b)$ is not the identity then $\dim(V) \geq 10$. Hint: look at the eigenvalues of b.

Exercise 4. Let $n \geq 1$. Let $\zeta = \exp(2i\pi/n) \in \mathbf{C}$ be the usual primitive nth root of 1. Consider the $n \times n$ complex matrix

$$A = \operatorname{diag}(1, \zeta, \zeta^2, \ldots) = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 0 & \zeta & 0 & \ldots \\ 0 & 0 & \zeta^2 & \ldots \\ \ldots & & & \end{pmatrix}$$

[1]There is a generalization of this where $\tau : H \to G$ is a homorphism of groups and one considers $(V, \pi \circ \tau)$ which is also a representation of H.

and the permutation matrix corresponding to the n-cycle $(12\dots n)$, namely

$$B = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & & \\ 0 & 1 & 0 & \dots & & \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

Prove that

(1) A and B generate a finite subgroup G of $GL_n(\mathbf{C})$
(2) compute the number of elements of G
(3) the representation of G on $\mathbf{C}^n$ is irreducible.