Problem set 12 for Representations of Finite Groups

Exercise 1. This is exercise 13.9 from Serre. Let G be a finite group. Given a conjugacy class $C \subset G$ in G we define $C^{-1} = \{g^{-1} \mid g \in C\}$. Then C^{-1} is also a conjugacy class in G. We say C is *even* if $C = C^{-1}$ and *odd* if not.

(1) Show that the number of real-valued irreducible complex characters[1] of G is equal to the number of even conjugacy classes of G. Hint: Think about a basis for the space of class functions given by the indicator function 1_C for C even and the two functions $1_C + 1_{C^{-1}}$ and $1_C - 1_{C^{-1}}$ for C odd. Similarly, think about a basis of the space of class functions given by χ for χ a real valued irreducible complex character and $\mathrm{Re}(\chi)$ and $\mathrm{Im}(\chi)$ for when χ is not real-valued. Then think about the inner products of these.

(2) Show that, if G has odd order, the only even conjugacy class is the conjugacy class of the identity. Deduce that the only real-valued complex irreducible character is the unit character (Burnside).

Exercise 2. Using (1) of Exericise 1 show that every irreducible complex character of the symmetric group $G = S_n$ is real-valued. (We will show in the lectures that all irreducible representations of S_n are realizable over the rational numbers $\mathbf{Q}$ and a fortiori over $\mathbf{R}$.)

Exercise 3. Let G be a group as studied in Exercise 4 of Problem set 8 and Exercise 2 of Problem set 9. Let (V, π) be the unique irreducible complex representation of G of dimension > 1.

(1) Show that χ_π is real-valued.

(2) An example of such a G is the group

$$G = \langle a, b \text{ with relations } b^{11} = 1,\ aba^{-1} = b^2,\ a^{10} = 1\rangle \cong C_{11} \rtimes C_{10}$$

of Exercise 3 of Problem set 2 (you don't have to show this).

(a) Compute the character χ of (V, π) in this case.

(b) Show (V, π) is realizable over $\mathbf{R}$. Hint: use the formula in Proposition 39 of Serre's book.

(3) Extra credit / difficult: Show that (V, π) is realizable over $\mathbf{R}$ for any G.

Discussion of (3): I have a proof using the formula in Proposition 39 of Serre's book + some extra work on the structure of G to compute the terms of the formula. (The cases where p is odd/even are a bit different.) If you have a "direct" proof, please let me know.

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[1] We say $\chi : G \to \mathbf{C}$ is an irreducible complex character if $\chi = \chi_\pi$ where π is an irreducible complex representation of G.