Problem set 11 for Representations of Finite Groups

Representations of Symmetric Groups

The 6 problems in this set discuss some of the material from Burrow's article "A generalization of the Young diagram" and then go on to talk about the representation of a symmetric group associated to a Young tableau.

Situation. Let G be a finite group, let $R, C \subset G$ be subgroups, and let $\theta : R \to \mathbf{C}^*$ and $\varphi : C \to \mathbf{C}^*$ be group homomorphisms (sometimes called a character, or a linear character, or a degree 1 character).

Exercise 1. Show that the inner product of the characters of the induced representations $\mathrm{Ind}_R^G(\theta)$ and $\mathrm{Ind}_C^G(\varphi)$, namely

$$\langle \chi_{\mathrm{Ind}_R^G(\theta)}, \chi_{\mathrm{Ind}_C^G(\varphi)} \rangle$$

is equal to the following sum

$$\frac{1}{|R| \cdot |C|} \sum_{r \in R,\ c \in C,\ t \in G \text{ such that } c = t^{-1}rt} \theta(r)\overline{\varphi(c)}$$

Hint: use the formula for the induced representation and use reciprocity formula.

Exercise 2. For a fixed $t \in G$ consider the subgroup $R_t = R \cap tCt^{-1}$ of R. Show that the map

$$\psi_t : R_t \to \mathbf{C}^*, \quad r \mapsto \psi_t(r) := \theta(r)\overline{\varphi(t^{-1}rt)}$$

is a linear character (i.e. a group homomorphism) and show that

$$\langle \chi_{\mathrm{Ind}_R^G(\theta)}, \chi_{\mathrm{Ind}_C^G(\varphi)} \rangle = \frac{1}{|R| \cdot |C|} \sum_{t \in G} \sum_{r \in R_t} \psi_t(r)$$

Assumptions:

(A1) $\theta|_{R \cap C} = \varphi|_{R \cap C}$,

(A2) for $t \in G$, $t \notin RC$ there exist a $r \in R_t$ such that $\psi_t(r) \neq 1$.

Note that condition (A2) means that for $t \in G$, $t \notin RC$ there exists an $r \in R$, $r \in tCt^{-1}$ such that $\theta(r) \neq \varphi(t^{-1}rt)$.

Exercise 3. Under assumption (A1) prove that for $t \in RC$ the character ψ_t is trivial.

Exercise 4. Under assumptions (A1) and (A2) prove there exists a unique (up to isomorphism) irreducible representation (V, π) of G such that (V, π) occurs in both $\mathrm{Ind}_R^G(\theta)$ and $\mathrm{Ind}_C^G(\varphi)$. In fact, show that (V, π) occurs with multiplicity 1 in both $\mathrm{Ind}_R^G(\theta)$ and $\mathrm{Ind}_C^G(\varphi)$. Hint: compute the inner product from Exercise 1 in this case and use everything you know about this inner product.

Exercise 5. Let $G = S_3$ and $R = \{1, (12)\}$ and $C = \{1, (13)\}$. Let θ be the trivial character and φ the sign character.

(1) Prove (A1) and (A2).

(2) Identify the representation (V, π) of exercise 4.

This exercise is associated with the Young tableau

1	2
3	

where R is the row stabilizer and C is the colum stabilizer.

Remark. For $G = S_4$ consider the following Young tableau

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

Then let R be the row stabilizer, i.e., $R = \{1, (12), (34), (12)(34)\}$ and let C be the column stabilizer, i.e., $C = \{1, (13), (24), (13)(24)\}$. Let θ be the trivial character and φ the sign character. Then you can again prove that (A1) and (A2) hold. Thus we get a well defined irreducible representation (V, π) of S_4 associated to this Young tableau.

Exercise 6. (Open ended and extra credit.) Can you figure out which representation of S_4 is the one we found in the remark above? (You can look at your notes from a lecture or the book of Serre for a list of the irreducible representations of S_4.) What about the other tablau:

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}$$

You may assume (A1) and (A2) always hold to think about this.