Problem set 10 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

Exercise 1. Let G be a finite group. Let (V, π) and (W, ρ) be representations of G. Prove that if $Ker(\rho) \not\supset Ker(\pi)$ then (W, ρ) is not isomorphic to a direct summand of $(V^{\otimes n}, \pi^{\otimes n})$ for any $n \geq 0$. Here $Ker(\pi)$ denotes the kernel of the group homomorphism $\pi : G \to \mathrm{GL}(V)$.

Exercise 2. Let G be a finite group. Let (V, π) and (W, ρ) be representations of G. Denote χ_π and χ_ρ their characters and denote $\chi_{\pi^{\otimes n}}$ the character of the nth tensor power of (V, π); for $n = 0$ you get the character of the 1-dimensional trivial representation by convention. Recall that for class functions f_1, f_2 on G we set

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum\nolimits_{g \in G} f_1(g) \overline{f_2(g)}$$

(1) Set

$$H(t) = \sum\nolimits_{n \geq 0} \langle \chi_{\pi^{\otimes n}}, \chi_\rho \rangle t^n \quad \text{and} \quad H'(t) = \frac{1}{|G|} \sum_{g \in G} \frac{\overline{\chi_\rho(g)}}{1 - \chi_\pi(g) t}$$

Prove that $H(t) = H'(t)$.

(2) From $H = H'$ conclude that $H(t)$ is nonzero if $Ker(\rho) \supset Ker(\pi)$. Hint: show that $H = H'$ has a pole.

(3) Conclude that if $Ker(\rho) \supset Ker(\pi)$ and (W, ρ) is irreducible, then (W, ρ) is isomorphic to a direct summand of $(V^{\otimes n}, \pi^{\otimes n})$ for some $n \geq 0$.

Exercise 3. Let G be a finite group and let (V, π) be a faithful representation, i.e., the map $\pi : G \to GL(V)$ is injective. Prove that every irreducible representation (W, ρ) is isomorphic to a direct summand of $(V^{\otimes n}, \pi^{\otimes n})$ for some $n \geq 0$. Hint: Use the result of Exercise 2.