Problem set 1 for Representations of Finite Groups

NOTE: If a lot of people have trouble with this set, then we'll adjust the level of abstraction and/or difficulty in future sets. Please let me and Ryan know if you are having trouble and which parts you have trouble with!

Tensor products. Let V_1, V_2 be finite dimensional complex vector spaces. Let $V = V_1 \otimes V_2$ be the tensor product. If $e_1, \ldots, e_n$ is a basis of V_1 and $f_1, \ldots, f_m$ is a basis of V_2, then $e_i \otimes f_j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ is a basis for V.

Exercise 1. Let V, W be two 2-dimensional complex vector spaces. Let $A : V \to V$ and $B : W \to W$ be linear maps both of which have exactly 1 Jordan block in their Jordan normal form. What are the sizes of the Jordan blocks for the map $A \otimes B : V \otimes W \to V \otimes W$? Please justify your answer (as usual). Hint: choose bases for V and W such that A and B are in Jordan normal form and then compute the matrix of $A \otimes B$ wrt the corresponding basis for $V \otimes W$. Then do usual linear algebra on this matrix to conclude.

Diagonalizable maps. A linear map $T : V \to V$ is called *diagonalizable* if there exists a basis of V consisting of eigenvectors of T. An $n \times n$ square matrix A is called diagonalizable if there exists an invertible matrix P such that PAP^{-1} is diagonal; this is equivalent to asking the linear map defined by A to be diagonalizable.

Exercise 2. Let V be a finite dimensional vector space over a field F (you may assume $F = \mathbf{R}$ or $F = \mathbf{C}$ if you like). Let $W \subset V$ be a subspace and denote V/W the quotient vector space. Let $T : V \to V$ be a linear map such that $T(W) \subset W$. This will induce a commutative diagram of vector spaces

$$\begin{tikzcd} W \arrow[r] \arrow[d, "T|_W"] & V \arrow[r] \arrow[d, "T"] & V/W \arrow[d, "T|_{V/W}"] \\ W \arrow[r] & V \arrow[r] & V/W \end{tikzcd}$$

Assume the following

(1) the induced linear map $T|_W : W \to W$ is diagonalizable,
(2) the induced linear map $T|_{V/W} : V/W \to V/W$ is diagonalizable,
(3) there is no overlap between the set of eigenvalues of $T|_W$ and the set of eigenvalues of $T|_{V/W}$.

Show that T is diagonizable using the following steps:

(1) Let $u \in V/W$ be an eigenvector for $T|_{V/W}$ with eigenvalue λ. Pick $v \in V$ which maps to u. Show that $w = (T - \lambda)v$ is an element of W. Then show that $w = (T - \lambda)(w')$ for some $w' \in W$. Then show that $v' = v - w'$ is an eigenvector for T with eigenvalue λ mapping to u in V/W.
(2) choose a basis of W consisting of eigenvectors $w_1, \ldots, w_r$ for $T|_W$ and choose a basis of V/W consisting of eigenvectors $u_1, \ldots, u_s$ for $T|_{V/W}$. Using (1) pick eigenvectors $v_1, \ldots, v_s \in V$ for T mapping to $u_1, \ldots, u_s$. Explain why $w_1, \ldots, w_r, v_1, \ldots, v_s$ is a basis of V consisting of eigenvectors for T.

Exercise 3. Let F be a finite field with $q = p^f$ elements (so the characteristic of F is p). Feel free to assume $F = \mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ and $q = p$ is a prime number.

(1) How many elements does $G = GL_2(F)$ have?
(2) Describe a p-Sylow subgroup of G.

Permutation representations. Let G be a finite group. Let X be a finite set. Let $G \times X \to X$, $(g, x) \mapsto g \cdot x$ be an action of G on X. Let $\mathbf{C}[X]$ be the corresponding *permutation representation* of G. What this means is this:

(a) as a vector space $\mathbf{C}[X] = \{\xi = \sum_{x \in X} a_x x \mid a_x \in \mathbf{C}\}$
(b) for $\xi = \sum a_x x \in \mathbf{C}[X]$ and $g \in G$ we define $g \cdot \xi$ by the rule

$$g \cdot \xi = \sum a_x (g \cdot x)$$

for all $x \in X$.

Please make sure you understand why this defines a representation of G, in order words, check that $g' \cdot (g \cdot \xi) = (g'g) \cdot \xi$ on a napkin.

Exercise 4. We will say a representation V of a finite group G is a *permutation representation* if it is isomorphic[1] to a representation on $\mathbf{C}[X]$ constructed from an action of G on the finite set X.

(1) Give an example of a finite group G and a (finite dimensional as always) representation V which is not a permutation representation.
(2) Show that if V_1 and V_2 are permutation representations of the same finite group G, then so is $V_1 \oplus V_2$.
(3) Show that if V_1 and V_2 are permutation representations of the same finite group G, then so is $V_1 \otimes V_2$.
(4) (Difficult; please skip if out of time.) Give an example of a group G and a permutation representation V such that $\wedge^2(V)$ is not a permutation representation.
(5) Show that a permutation representation is isomorphic to its dual. Hint: you could use that a representation V is isomorphic to its dual if and only if there exists a G-invariant nondegenerate bilinear pairing $\langle , \rangle : V \times V \to \mathbf{C}$.

[1] Not equal, just isomorphic.