Midterm Problem Set Algebraic Number Theory, Spring 2023

Try as much as possible to work on this by yourself and try to not use the text. The reason being that the final exam will be an in class exam similar to this.

Exercise 0.1 (Definitions)**.** Provide brief definitions of the italicized concepts.

(1) a *number field* K,
(2) the *ring of integers* $\mathcal{O}_K$ of a number field K,
(3) a *Noetherian ring* R,
(4) a *Dedekind domain* A,
(5) a *lattice* Λ in an n-dimensional $\mathbf{Q}$-vector space V.

Exercise 0.2 (Theorems)**.** Precisely and succintly state a nontrivial fact discussed in the lectures related to each item (if there is more than one then just pick one of them).

(1) the ring of integers $\mathcal{O}_K$ of a number field K,
(2) factorization in a Dedekind domain A,
(3) the ring of Gaussian integers $\mathbf{Z}[i]$,
(4) the ring of integers in the cyclotomic number field $\mathbf{Q}(\zeta_n)$,
(5) norms of ideals in rings of integers of number fields.

Exercise 0.3. Let $d_1, d_2 \in \mathbf{Z}\backslash\{0,1\}$, $d_1 \neq d_2$ be squarefree. Let $K = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2})$. You may use that $[K : \mathbf{Q}] = 4$. Compute the discriminant of the lattice $\mathbf{Z}[\sqrt{d_1}, \sqrt{d_2}]$.

Exercise 0.4. Let $K = \mathbf{Q}(\sqrt{-13})$. The ring of integers of K equal to $\mathbf{Z}[\sqrt{-13}]$.

(1) Factor the ideal $I = (1 + \sqrt{-13})$ in $\mathcal{O}_K$ as a product of prime ideals.
(2) Are any of the primes you found principal ideals?

Exercise 0.5. Let $K = \mathbf{Q}(\alpha)$ where α is a complex root of $x^3 + x + 1/7 = 0$.

(1) Show that the number field K has degree 3 over $\mathbf{Q}$.
(2) Find the smallest $d \in \{1, 2, 3, \ldots\}$ such that $d\alpha$ is an algebraic integer.

Exercise 0.6. Let K be a number field of degree n. Suppose that $\mathcal{O}_K = \mathbf{Z}[\alpha]$ for some $\alpha \in \mathcal{O}_K$. Let p be a prime which is completely split in K. Show that $n \leq p$. (Hint: Dedekind's factorization criterion.)

Exercise 0.7. Let K be a number field. Let $\alpha \in K$ be an algebraic integer. Let $x^d + a_1x^{d-1} + \ldots + a_d$ be the minimal polynomial of α over $\mathbf{Q}$. Suppose $a_d = 1$ or $= -1$. Show that α is a unit of $\mathcal{O}_K$.

Exercise 0.8. Let $K = \mathbf{Q}(\alpha)$ where α is a complex root of $x^4 + x^2 + 3 = 0$. You may use that $x^4 + x^2 + 3$ is irreducible in $\mathbf{Q}[x]$.

(1) Why does the number field K have degree 4?
(2) Compute the discriminant of $\mathbf{Z}[\alpha]$ up to sign[1].
(3) Conclude that 3 does not divide $\#(\mathcal{O}_K/\mathbf{Z}[\alpha])$.
(4) Compute the prime factorization of $3\mathcal{O}_K$ in $\mathcal{O}_K$.

[1]This is a bit of work.