Exercise: finiteness class group in function field case

If you find errors in this text, please email me, thanks!

List of notations and things you may use

(1) p is an odd prime
(2) q is a positive integer power of p
(3) k is a finite field with q elements
(4) $A = k[x]$ is the polynomial ring in 1 variable over k
(5) A is a principal ideal domain
(6) $F = k(x)$ is the fraction field of A
(7) $f \in A$ is a squarefree monic polynomial of odd degree
(8) F'/F is the degree two extension you get by adjoining a square root y of f
(9) $A' = A[y]/(y^2 - f)$ and is the integral closure of A in F',
(10) A' is a Dedekind domain
(11) for a nonzero ideal $I \subset A'$ the quotient A'/I is a finite group
(12) we set $N(I) = \#(A'/I)$ for I as above
(13) then $N(IJ) = N(I)N(J)$ for nonzero ideals I, J of A'
(14) for a nonzero element $\alpha \in A'$ we set $N(\alpha) = N((\alpha)) = N(\alpha A')$

Please just use this notation and use these facts.

Exercise. Prove that the class group $Cl(A')$ is finite using the following steps:

a Show that it suffices to show the following: There exists a constant $c > 0$ such that given a nonzero ideal $I \subset A'$ there is a nonzero element $\alpha \in I$ such that $N(\alpha) \leq cN(I)$.

b Show that if $M \subset N$ is an inclusion of free A-modules of rank 2 then the number of elements of N/M is finite and can be computed as follows: choose a basis e_1, e_2 of M, choose a basis f_1, f_2 of N, write $e_i = a_{i1}f_1 + a_{i2}f_2$, let $a = \det(a_{ij}) = a_{11}a_{22} - a_{12}a_{21}$. Then $\#(N/M) = q^{\deg(a)}$. Hints: use that the degree of the determinant is independent of the choice of bases, and then use Smith normal form, see Proposition 2 of `https://mattbaker.blog/2022/11/09/finitely-generated-modules-over-a-p-i-d/`. Of course this argument works for any rank, not just for rank 2. You might be able to do an easier proof for rank 2, but I am not sure.

c Show that there exists a constant c' such that if $\alpha = a + by$ with $a, b \in A$ not both zero, then $N(\alpha) \leq c' q^{2\max(\deg(a), \deg(b))}$.

d Write $N(I) = q^m$. Show that there exists a nonzero $\alpha = a + by$ in I with $\max(\deg(a), \deg(b)) \leq 1 + m/2$. Hint: count!

e Finish the proof.

Extra credit. Compute the class group of A' when $q = p = 3$ and $f = x(x-1)(x-2)$. (I didn't try this myself so it could be wild! In order to do this, it might help if you found relatively "good" constants in your proofs above.)