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ALGEBRAIC STACKS DESIRABLES

Abstract. OK, here are some ideas about how to write a text about stacks, what should go into it, and fixing ideas for some of the definitions. Please email comments to the mailing list.

1. Foundational and prerequisites

The next section is more interesting. Everything that comes from outside of the project is mentioned in here (?). Note that subsections may actually be documents all by themselves in the project.

1.1. **Set Theory.** We use Zermelo-Fraenkel set theory with the axiom of choice. We do not use classes or large sets such a universes (different from SGA4). We do not stress set-theoretic issues but we make sure everything is correct (of course) and so we do not ignore them either.

1.2. **Categories.** Categories are sets. This is different from the "proof of concept" stacks-0.1 project. This means the category of rings (or of affine schemes) will mean all rings belonging to some (largish) set. We will perhaps have to introduce some notation for this, such as Rings_α, where α is some ordinal. This will obviously create some problems and difficulties later. Perhaps we will have contributors who will now and then go through the project and fix them up.

We define "catergories fibred in groupoids" here. Mention that they are always equivalent to others that are "split" (where you have pullback functors that compose on the nose). So in other words functors from the base category to the category of groupoids.

1.3. **Sites and Topoi.** Do a little bit of theory here. Talk about sheaves, morphisms of sites. The category of sheaves on a site now means all sheaves with values in Sets_α where α is suitably large (relative to the site).

Introduce the notion of topos and morphism of topoi. The notion of simplicial and strictly simplicial topos.

Ringed sites, quasi-coherent sheaves of modules. Ringed topos and morphism of ringed topoi.

Some generalities about cohomology goes in here as well. (Just with injective resolutions, nothing fancy. Allthough it might be nice to have hypercoverings here for later use.) Injective resolutions of a sheaf in a (strictly) simplicial topos. The fundamental spectral sequence relating cohomology of the individual pieces to global cohomology. The simplicial topos arising from a covering of the final object in a topos and comparison of cohomologies.

Some basic facts about cohomological descent (at least enough to deal with a flat hypercover for quasi–coherent sheaves and a proper hypercover for étale sheaves).

1.4. **Stacks.** Stacks are stacks in groupoids $p : \mathcal{S} \to \mathcal{C}$, where $\mathcal{C}$ is a site. We talk about 1-morphisms between stacks and 2-morphisms between 1-morphisms. If we want to talk about a 2-category of stacks then we choose a (large) ordinal α as before.

Introduce representable stacks. Introduce 2-fibre products. Define representable 1-morphisms of stacks. (Let's not identify an object with the functor it represents, or its associated stack.) Define the inertia stack of a stack. Define gerbes.

Talk about properties of morphisms in $\mathcal{C}$ and properties of 1-morphisms of stacks: Suppose that P is a property of morphisms in $\mathcal{C}$ that is "local on the target". Then there is a corresponding property of representable 1-morphisms. Etc.

Example result: Given a sheaf of abelian groups $\mathcal{F}$ over $\mathcal{C}$ the set of equivalence classes of gerbes with "group" $\mathcal{F}$ is bijective to $H^2(\mathcal{C}, \mathcal{F})$. In particular enlarging α above will not matter.

1.5. **Algebra.** Mainly commutative algebra here. In particular we can define what it means for a morphism of rings to be flat, étale, smooth, finite type, etc.

We should have the nice short argument (Grothendieck's I think) proving descent of modules through flaithfully flat ring maps here so that we can have a complete proof of descent for polarized projective schemes later on.

1.6. **Schemes.** Some results and definitions about schemes go here. This is a little backward because schemes will be defined later. Make sure we have a list of properties of morphisms of schemes here mirroring the list of properites of ring maps in Subsection 1.5. Also, projective and proper morphisms.

Define simplicial schemes.

1.7. **Cohomology of schemes.** Talk a little about specifically étale cohomology and flat cohomology of schemes, Galois cohomology etc.

1.8. **Deformation theory a la Schlessinger.** Maybe this should come later but we could have a short discussion of Schlessinger's paper and also discussing a tiny bit what happens if you have automorphisms (functor in groupoids case – this is somewhere in the literature SGA??).

2. Algebraic Stacks, algebraic spaces and schemes

Here we introduce algebraic stacks and say what they are. The basic setup here is that schemes, algebraic spaces and algebraic stacks are all stacks over the category of affine schemes $\mathbf{Aff}_\alpha$ with the fppf topology. This means we do not assume that we know what a scheme is. Also, as suggested by Martin Olsson, it makes sense not to impose stronger separation conditions then strictly necessary.

A technicality will be that a 1-morphism between stacks over the category of affines being representable means that it is "in reality" representable by affine schemes. In particular, if you think of a scheme as a stack (as we will) then the diagonal $X \to X \times X$ is in general not representable as a 1-morphism of stacks on the category of affine schemes. Of course it is representable if X is separated, so perhaps if makes sense to define separated schemes first? Yes, this is ugly! (See 2.1.1 below for an alternative.)

2.1. **Definition of schemes.** A separated scheme is a stack $\mathcal{S}$ over the category of affine schemes $\mathbf{Aff}_\alpha$ such that: (a) all fibre categories are equivalent to sets, (b) there is a set I and 1-morphisms $\pi_i : X_i \to \mathcal{S}$, $i \in I$ such that (c) each X_i is representable, (d) each π_i is representable by open immersions, (e) $\coprod X_i \to \mathcal{S}$ is a covering for the Zariski topology, and (f) the 1-morphism $\mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is representable by closed immersions.

A scheme is a stack with properties (a), (b), (c), (d)', (e), where (d)' = each π_i representable by separated schemes and is an open immersion.

In this subsection we explain how this is exactly the same as the notion of schemes in EGA or Hartshorne.

2.1.1. *Alternative.* Define what it means for a 1-morphism of stacks over $\mathbf{Aff}_\alpha$ to be representable by quasi-affine schemes and use that in (d) to get the definition of a scheme in 1 step.

2.2. **Definition of algebraic spaces.** Here we use (a), (b), (c), (d)" and (e)', where (d)" = each π_i is representable by schemes and étale, and in (e)' we say it is an étale covering. There is also some weak separation property (FIXME).

2.3. **Definition of algebraic stacks.** An algebraic stack is a stack that has a diagonal representable by algebraic spaces, that is the target of a surjective smooth morphism from a scheme, and that has some weak separation condition (FIXME).

The notion "Deligne-Mumford stack" will be reserved for a stack as in [DM69]. Perhaps it makes sense to reserve the term "Artin stack" for a stack such as in the papers by Artin [Art69], and [Art74]. (See also [CdJ02].) In other words, and Artin stack will be an algebraic stack with some reasonable finiteness and separatedness conditions.

2.4. **Examples of schemes, algebraic spaces, algebraic stacks.** It really is not that hard to show that $\mathcal{M}_g$ is an algebraic stack for $g \geq 2$. We should have $[X/G]$ here. We should really have a long list of moduli problems here and prove they are all algebraic stacks. (Some of them we can postpone the proof until after Artin approximation.) For example the Kontsevich moduli space in characteristic $p > 0$.

How about the algebraic space you get from the deformation theory of a general surface in $\mathbf{P}^3$ with a node? (I mean where you deform it to a general smooth surface in $\mathbf{P}^3$.)

Perhaps we can talk about some small dimensional examples here too. For example the stack where you have $\mathbf{A}^1$ with a $B(\mathbf{Z}/2)$ sitting at 0. Bugeyed covers. You name it.

2.5. **Properties of algebraic stacks.** Such as the various ways of defining what a proper algebraic stack is. Of course these things are really properties of morphisms of stacks.

We can define singularities (up to smooth factors) etc. Prove that a connected normal stack is irreducible, etc.

2.6. **Lisse etale site of an algebraic stack.** This has to be explained. Explain it is not functorial with respect to 1-morphisms of algebraic stacks. Define etale cohomology of an algebraic stack.

2.7. **Things you always wanted to know but were afraid to ask.** There are going to be lots of lemmas that you use over and over again that are usefull but aren't really mentioned specifically in the literature, or it isn't easy to find references for. Bag of tricks.

Example: Given two groupoids in schemes $R \Rightarrow U$ and $R' \Rightarrow U'$ whatdoes it mean to have a 1-morphism $[U/R] \to [U'/R']$ purely in terms of groupoids in schemes. (This is bad because surely this is in the lit somewhere.) More anybody?

2.8. **Quasi-coherent sheaves on stacks.** Define them and explain how you get them. You can define them as living on the lisse-etale site or on all of the stack and show the two notions are equivalent. Cohomology of quasi-coherent sheaves.

3. Fundamental results

Here is a list of results that could be explained. (From memory and in random order.) Actually, perhaps in stead of having the goal to fully explain these results we should look in the relevant papers for the basic tricks used in them and explain those and put them in Section 2.

3.1. **Flat and smooth.** Artin's theorem that having a flat surjection from a scheme is a replacement for the smooth surjective condition.

3.2. **Artin's representability theorem.** Title is clear enough. Perhaps we can reformulate the condition of having a deformation theory a little to adapt it more to the examples we know about, especially those where there is a perfect obstruction theory (discussions with Jason)?

3.3. **DM stacks are finitely covered by schemes.** This all begins with Gabber's lemma I think. Somewhere in Asterisque about Faltings proof of Mordell?

3.4. **Martin Olson's paper on properness.** This proves two notions of proper are the same. We can also discuss Faltings result that it suffices to use DVR's in certain cases.

3.5. **Proper pushforward of coherent sheaves.** No comments yet.

3.6. **Keel and Mori.** See [KM97]. The steps in this article also give a good way of looking at what an algebraic stack locally looks like.

3.7. **Add more here.** Please.

References

[Art69] Michael Artin. Algebraization of formal moduli. I. In *Global Analysis (Papers in Honor of K. Kodaira)*, pages 21–71. Univ. Tokyo Press, Tokyo, 1969.

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