ON THE STRUCTURE OF INSTABILITY IN MODULI THEORY

DANIEL HALPERN-LEISTNER

Abstract. We formulate a theory of instability and Harder-Narasimhan filtrations for an arbitrary moduli problem in algebraic geometry. We introduce the notion of a Θ-stratification of a moduli problem, which generalizes the Kempf-Ness stratification in GIT as well as the Harder-Narasimhan stratification for coherent sheaves on a projective scheme, and establish necessary and sufficient conditions for the existence of these stratifications. For a certain class of algebraic stacks, which we call Θ-reductive, the existence of Θ-stratifications can be reduced to solving a relatively straightforward "boundedness" problem. We apply our methods to an example that lies beyond the reach of classical methods: the stratification of the stack of objects in the heart of a Bridgeland stability condition.

One of the first fundamental observations in algebraic geometry is that small deformations of an algebro-geometric object are often controlled by finite dimensional parameter spaces which are algebraic varieties. Globalizing this observation, i.e. finding a moduli space which parameterizes all isomorphism classes of a particular type of algebro-geometric object, introduces quite a bit of machinery and ultimately leads to the theory of algebraic stacks.

The general and tautological answer, that the solution to a moduli problem is an algebraic stack, is often insufficient in moduli problems of interest. Stacks which arise in nature are rarely representable by algebraic varieties, and are often non-quasi-compact and non-separated. In this paper we propose a new standard for what it means to "solve" a moduli problem in algebraic geometry: in addition to constructing an algebraic stack $\mathfrak{X}$, one should construct a certain type of stratification of $\mathfrak{X}$, which we call a Θ-stratification.

The prototypical example is the moduli stack $\mathfrak{X}$ of vector bundles of rank r and degree d on a smooth Riemann surface. $\mathfrak{X}$ is not quasi-compact or separated. There is an open substack $\mathfrak{X}^{\mathrm{ss}} \subset \mathfrak{X}$ parameterizing *semistable* vector bundles which admits a projective *good moduli space* $\mathfrak{X}^{\mathrm{ss}} \to M$, which is a coarse moduli space if r and d are coprime. On the other hand, every unstable vector bundle (meaning one which is not semistable) has a canonical Harder-Narasimhan filtration, and the unstable locus in the moduli stack admits a stratification $\mathfrak{X}^{us} = \bigcup \mathfrak{S}_\alpha$, where $\mathfrak{S}_\alpha$ is the locally closed substack of bundles for which the associated graded pieces of the Harder-Narasimhan filtration have particular rank and degree recorded by the index α [S1]. The $\mathfrak{S}_\alpha$ fiber over moduli stacks $\mathfrak{Z}_\alpha^{\mathrm{ss}}$ of semistable graded vector bundles, and the $\mathfrak{Z}_\alpha^{\mathrm{ss}}$ again have projective good moduli spaces.

In pursuit of a general theory of instability in algebraic geometry, we will give an intrinsic description of this structure on $\mathfrak{X}$ and establish necessary and sufficient conditions for the existence of such a stratification. We have three primary goals:

(1) We develop the notion of a filtration for a point in an arbitrary moduli problem $\mathfrak{X}$;
(2) We associate a numerical invariant μ, a real valued function on the set of filtrations, to certain rational cohomology classes on $\mathfrak{X}$. The Harder-Narasimhan filtration of a point $p \in \mathfrak{X}$ is defined to be a filtration of p which maximizes the numerical invariant. We give necessary and sufficient conditions for a numerical invariant to define a Θ-stratification;
(3) We introduce the notion of a Θ-reductive stack and show that Θ-stratifications are particularly easy to construct on Θ-reductive stacks.

The notion of a numerical invariant generalizes the Hilbert-Mumford numerical invariant in geometric invariant theory [MFK], and for quotient stacks of the form X/G, where X is a scheme which is project over its affinization and G is a reductive group, the Θ-stratifications which we construct agree with the Hesselink-Kempf-Kirwan-Ness stratification in geometric invariant theory. In its simplest form, our main result is the following:

Theorem. *Any numerical invariant associated to rational cohomology classes*[1] *on a quasi-compact Θ-reductive stack $\mathfrak{X}$ defines a weak Θ-stratification of $\mathfrak{X}$.*

Let us give a more precise statement which applies to $\mathfrak{X}$ which are non-quasi-compact (as is the case for many stacks arising in nature):

Theorem (See Theorem 4.39)**.** *For a Θ-reductive stack $\mathfrak{X}$ locally of finite type with quasi-affine diagonal over a base stack B satisfying mild hypotheses (See (†)), a numerical invariant μ associated to rational cohomology classes defines a weak Θ-stratification if and only if the following boundedness condition is satisfied:*

(B2) *For any map from a finite type affine scheme $\xi : T \to \mathfrak{X}$, $\exists$ a quasi-compact substack $\mathfrak{X}' \subset \mathfrak{X}$ such that $\forall$ finite type points $p \in T(k)$ and a filtration f of $\xi(p)$ for which $\mu(f) > 0$, there is another filtration f' of $\xi(p)$ with $\mu(f') \geq \mu(f)$ and whose associated graded point $\mathrm{gr}(f')$ lies in $\mathfrak{X}'$.*

After introducing some additional notation in this introduction, we will provide a more precise main statement in Theorem 0.2 which gives necessary and sufficient conditions for the existence of Θ-stratifications even when

[1] By this we mean a cohomology class $\ell \in H^2(\mathfrak{X};\mathbb{Q})$ and a positive definite $b \in H^4(\mathfrak{X};\mathbb{Q})$, as we shall discuss below.

$\mathfrak{X}$ is not Θ-reductive. Beyond the aesthetic appeal of extending Harder-Narasimhan theory to other moduli problems, there are two main motivations for studying Θ-stratifications:

Comparing the geometry of $\mathfrak{X}$ with that of $\mathfrak{X}^{\mathrm{ss}}$:

Atiyah & Bott [AB] used the Harder-Narasimhan stratification to derive explicit formulas for the Betti numbers of the moduli of semistable G-bundles on a Riemann surface. Likewise for a smooth projective variety over $\mathbb{C}$ with an action of a reductive group G, Kirwan [K2] used the stratification of the unstable locus to derived explicit formulas for the Betti numbers of X^{ss}/G. The theme in both of these examples is that sometimes the geometry of the stack $\mathfrak{X}$ is simpler than the geometry of the semistable locus $\mathfrak{X}^{\mathrm{ss}} \subset \mathfrak{X}$, and the presence of a Θ-stratification allows one to make precise comparisons between the two.

This phenomenon is even more pronounced in Teleman and Woodward's proof of the Verlinde formula for the dimension of the cohomology of positive line bundles on the stack of semistable G-bundles on a Riemann surface [TW]. In that case the holomorphic index of certain vector bundles on the stack of all G-bundles can be computed explicitly via an abelianization formula, and the index of these vector bundles on the stack can be identified with the index of their restrictions to the semistable locus using a non-abelian localization formula, which uses the Harder-Narasimhan stratification.

Birational modifications of moduli problems:

Our numerical invariants are typically associated to rational cohomology classes on $\mathfrak{X}$. As these cohomology classes vary, points in an unstable stratum become semistable precisely when the value of the numerical invariant on that stratum tends to 0, so the stratification of the unstable locus is essential to studying how the semistable locus $\mathfrak{X}^{\mathrm{ss}}$ depends on the choice of cohomology class on $\mathfrak{X}$. Classically, this is the theory of variation of GIT quotient [DH, T2], and we formulate and prove an intrinsic version in Theorem 4.54. When $\mathfrak{X}$ is smooth, the stratification can be used to formulate precise statements as to how the topology and geometry of $\mathfrak{X}^{\mathrm{ss}}$ varies as one varies the cohomology class using the methods of [K2].

We expect many interesting examples of variation of Θ-stratification to arise via the following meta-principle: Whenever one has a birational isomorphism of algebraic spaces $M_1 \dashrightarrow M_2$, if M_1 and M_2 are good moduli spaces for algebraic stacks $\mathfrak{X}_1$ and $\mathfrak{X}_2$ parameterizing objects of geometric interest, then there should be a third stack $\mathfrak{X}$ which is Θ-reductive and also parameterizes objects of geometric interest, in which $\mathfrak{X}_i \subset \mathfrak{X}, i = 1, 2$ is the open substack of semistable points for two different numerical invariants $\mu_i, i = 1, 2$. In this case the Θ-stratifications induced by μ_i would allow one to compare the geometry of $\mathfrak{X}_1$ and $\mathfrak{X}_2$. Ideally $\mathfrak{X}$ also has a good moduli space $\mathfrak{X} \to M$, and the resulting maps $M_1 \to M \leftarrow M_2$ are projective and birational.

This perspective is implicit, for example, in the log minimal model program for the moduli of curves [FS], where one constructs different birational models of $\overline{M}_{g,n}$ as good moduli spaces for open substacks of the moduli stack of all curves (possibly subject to a mild restriction on singularity type). What is missing in our understanding of this question is whether there is a variation of Θ-stratification on the stack of all curves controlling these birational modifications.

The other papers in this series:

This is the first in a series of three papers on the structure of moduli problems in algebraic geometry. To put the methods of this paper in context, let us briefly remark on what will appear in the sequel. The second paper [AHLH] is joint work with Jarod Alper and Jochen Heinloth. Whereas here we focus on the structure of the unstable locus, in the sequel we focus on the structure of the semistable locus. Among other things, we establish necessary and sufficient conditions for a stack $\mathfrak{X}$ with affine diagonal over a field of characteristic 0 to admit a good moduli space. One key condition is that $\mathfrak{X}$ must be Θ-reductive, so this condition plays an important role both in the construction of Θ-stratifications and in the construction of good moduli spaces for the semistable locus.

The third paper [HL3] will focus on applications of Θ-stratifications, and will prove the results announced and explained in some detail in [HL2].[2]. We will prove structure theorems for the derived category of coherent sheaves $\mathrm{D}^- \operatorname{Coh}(\mathfrak{X})$ generalizing the main theorem of [HL1] to arbitrary derived stacks with a Θ-stratification. For stacks which are not smooth, the proof of this structure theorem makes use of a non-classical derived structure on the strata coming from the modular interpretation for the stratification which we develop in this paper, and in fact the need for this derived structure originally motivated the development of the theory of Θ-stratifications. In [HL3] we will use the structure theorem for $\mathrm{D}^- \operatorname{Coh}(\mathfrak{X})$ to establish a general non-abelian localization theorem for derived local-complete-intersection (i.e. quasi-smooth) stacks, give a direct proof of Mochizuki's wall crossing formulas for algebraic Donaldson invariants of surfaces [M], and prove that any two Calabi-Yau manifolds in the birational equivalence class of a moduli space of sheaves on a $K3$ surface are derived equivalent. See [HL2] for further discussion.

Contents

[2]Some sections of this paper are currently available at `http://math.cornell.edu/~danielhl`

Summary of contents. In Thomason's terminology (see [T3, Introduction]) the "thrill seaker" might want to look at the list of key terms below and then skip immediately to the statements of the main theorems: Theorem 2.7, Theorem 4.38, Theorem 4.39, then Proposition 4.21, Theorem 4.24, and Theorem 3.60. Sections 1 through 3 are mostly foundational, section 4 is the conceptual core of the paper, and section 5 contains a completely worked example, that of Bridgeland stability on a projective scheme over an arbitrary field (See Proposition 5.40). We refer the reader also to [HL2] for an expository account of the results of this paper.

List of key terms

stack of graded objects $(\mathrm{Grad}(\mathfrak{X}))$
: The stack parameterizing algebraic families of maps $B\mathbb{G}_m \to \mathfrak{X}$. 15

stack of filtered objects $(\mathrm{Filt}(\mathfrak{X}))$
: The stack parameterizing algebraic families of maps $\Theta \to \mathfrak{X}$. 17

flag space $(\mathrm{Flag}(\xi))$
: An algebraic space parameterizing algebraic families of filtrations of a given point $\xi : \mathrm{Spec}(R) \to \mathfrak{X}$. It typically has many connected components. 18

formal fan $(F_\bullet, G_\bullet, \ldots)$
: A structure analogous to the fans studied in toric geometry, except cones are not embedded in $\mathbb{R}^n$, and they can meet along arbitrary rational polyhedral subcones rather than just faces. The definition is analogous to that of a semisimplicial set. 50

degeneration fan $(\mathbf{DF}(\mathfrak{X}, p)$ or $\mathbf{DF}(p)$ for $p \in \mathfrak{X}(k))$
: Formal fan which encodes all possible filtrations of an object in a moduli problem. 61

degeneration space $(\mathscr{D}eg(\mathfrak{X}, p)$ or $\mathscr{D}eg(p)$ for $p \in \mathfrak{X}(k))$
: A topological space parameterizing filtrations of a given point in a moduli problem. It is the "projective geometric realization" of the degeneration fan. 61

component fan $(\mathbf{CF}(\mathfrak{X}))$
: Formal fan which encodes filtered objects in a stack $\mathfrak{X}$ up to algebraic equivalence, meaning two filtrations are equivalent if they occur as points in an algebraic family of filtrations over a connected base scheme. 85, 110

component space $(\mathscr{C}omp(\mathfrak{X}))$
: A topological space parameterizing filtered objects in a moduli problem $\mathfrak{X}$ up to algebraic equivalence. It is the projective geometric realization of the component fan. 85

numerical invariant $(\mu : \mathfrak{U} \to \mathbb{R}$ with $\mathfrak{U} \subset \mathscr{C}omp(\mathfrak{X}))$
: The data provided for a stack $\mathfrak{X}$ which allows one to formulate the question of existence and uniqueness of Harder-Narasimhan filtrations. 93, 95, 112

Section 1: Filtrations. The Reese construction assigns to any $\mathbb{Z}$-weighted filtered vector space $\cdots \subset V_{w+1} \subset V_w \subset \cdots \subset V$ over a field k to the torsion-free quasi-coherent sheaf $\sum_w t^{-w} V_w \subset k[t^{\pm}] \otimes V$ on the stack

$$\Theta_k := \mathbb{A}^1_k/(\mathbb{G}_m)_k = \mathrm{Spec}(k[t])/\mathbb{G}_m,$$

where t has weight -1 with respect to $\mathbb{G}_m$. This construction gives an equivalence of categories between the category of filtered finite dimensional vector spaces where $E_w = 0$ for $w \gg 0$ and the category of vector bundles on Θ_k. The latter we interpret as the category of maps $\Theta_k \to \mathrm{pt}/\mathrm{GL}_n$, where n is the dimension of V. This motivates our definition of a filtration of a point $p \in \mathfrak{X}(k)$ for any stack $\mathfrak{X}$ and any field k as a map $f : \Theta_k \to \mathfrak{X}$ along with an isomorphism $f(1) \simeq p$. Similarly, a graded point in $\mathfrak{X}$ is a map $g : \mathrm{pt}/(\mathbb{G}_m)_k \to \mathfrak{X}$.

When $\mathfrak{X}$ satisfies suitable hypotheses (†), which we fix for most of this paper, the stack $\mathrm{Filt}(\mathfrak{X}) := \underline{\mathrm{Map}}(\Theta, \mathfrak{X})$ parameterizing flat families of filtrations is algebraic and locally finitely presented, as is the stack $\mathrm{Grad}(\mathfrak{X}) := \underline{\mathrm{Map}}(\mathrm{pt}/\mathbb{G}_m, \mathfrak{X})$ parameterizing flat families of graded objects in $\mathfrak{X}$ (Proposition 1.2). We shall use several universal maps:

$$\begin{tikzcd} \mathrm{Grad}(\mathfrak{X}) \arrow[rrr, bend left, "u"] \arrow[r, shift right, "\sigma"'] & \mathrm{Filt}(\mathfrak{X}) \arrow[l, shift right, "\mathrm{ev}_0"'] \arrow[r, shift left, "\mathrm{ev}_1"] \arrow[r, shift right, "\mathrm{gr}"'] & \mathfrak{X} \end{tikzcd} \tag{1}$$

ev$_1$) "forgetting the filtration" map restricts a filtration $\Theta_k \to \mathfrak{X}$ to the point $\{1\} \in \Theta_k$.

σ) "split filtration" map pulls back a graded point $\mathrm{pt}/(\mathbb{G}_m)_k \to \mathfrak{X}$ along the projection $\Theta_k \to \mathrm{pt}/(\mathbb{G}_m)_k$.

ev$_0$) "associated graded" map restricts a filtration $f : \Theta_k \to \mathfrak{X}$ to the closed substack $\{0\}/(\mathbb{G}_m)_k \hookrightarrow \Theta_k$. $\mathrm{ev}_0 \circ \sigma \simeq \mathrm{id}_{\mathrm{Grad}(\mathfrak{X})}$, and ev_0 is an algebraic deformation retract (Lemma 1.24).

gr) "associated graded" map assigns a filtration $f : \Theta_k \to \mathfrak{X}$ to $f(0)$ without its $\mathbb{G}_m$-action, i.e. $\mathrm{gr}(f) = f(0)$; $u \circ \mathrm{ev}_0 \simeq \mathrm{gr}$ canonically.

u) "forgetful" map restricts a map $\mathrm{pt}/(\mathbb{G}_m)_k \to \mathfrak{X}$ along the map $\mathrm{pt} \to \mathrm{pt}/(\mathbb{G}_m)_k$; $u \simeq \mathrm{ev}_1 \circ \sigma \simeq \mathrm{gr} \circ \sigma$ canonically.

The fiber of the map ev_1 over an R-point $\xi \in \mathfrak{X}(R)$ is a locally finitely presented algebraic space $\mathrm{Flag}(\xi)$ over $\mathrm{Spec}(R)$ paramerizing flat families of filtrations of ξ (Definition 1.12). $\mathrm{Flag}(\xi)$ is separated if $\mathfrak{X}$ has affine diagonal (Proposition 1.41).

The monoid $\mathbb{N}^\times$ acts on the stack $\mathrm{Filt}(\mathfrak{X})$ by composing a map $\Theta_k \to \mathfrak{X}$ with the degree n ramified cover $\Theta_k \to \Theta_k$ described in coordinates by $t \mapsto t^n$. We also give an explicit description of $\mathrm{Grad}(\mathfrak{X})$ and $\mathrm{Filt}(\mathfrak{X})$ when $\mathfrak{X} = X/G$ is a quotient stack (Theorem 1.36). Each connected component of $\mathrm{Filt}(X/G)$ is the quotient of a Białynicki-Birula stratum of X associated to the action of $\mathbb{G}_m$ on X via a one parameter subgroup $\lambda : \mathbb{G}_m \to G$, the existence of which we establish in a high level of generality (Proposition 1.31), by the parabolic subgroup of G associated to λ.

Section 2: Θ-stratifications. In the example where $\mathfrak{X}$ is the stack of vector bundles on a Riemann surface, the connected components of $\mathrm{Filt}(\mathfrak{X})$ correspond to filtrations whose associated graded pieces have fixed ranks and degrees. The existence and uniqueness of the Harder-Narasimhan filtration of an unstable bundle says that for each unstable point $p \in \mathfrak{X}(k)$, there is a certain connected component of $\mathrm{Filt}(\mathfrak{X})$ such that $(\mathrm{ev}_1)^{-1}(p)$ consists of a single point. Generalizing this observation, a Θ-stratification (Definition 2.2) indexed by a well-ordered set Γ is given by a collection of open substacks $\mathfrak{S}_\alpha \subset \mathrm{Filt}(\mathfrak{X})$ for $\alpha \in \Gamma$ such that $\mathrm{ev}_1 : \mathfrak{S}_\alpha \to \mathfrak{X}$ is a locally closed immersion, the locally closed subsets $\mathrm{ev}_1(\mathfrak{S}_\alpha)$ are disjoint for different α, and $\bigcup_{\beta \geq \alpha} \mathrm{ev}_1(\mathfrak{S}_\beta)$ is closed. Our methods typically only produce a "weak Θ-stratification," which is the same data except that $\mathrm{ev}_1 : \mathfrak{S}_\alpha \to \mathfrak{X} \setminus \bigcup_{\beta > \alpha} \mathfrak{S}_\beta$ is finite and radicial rather than a closed immersion. If $\mathfrak{X}$ is defined over a field of characteristic 0, then any weak Θ-stratification is a Θ-stratification (Corollary 2.6.1).

A Θ-stratification is determined uniquely by the subset of irreducible components $S \subset \mathrm{Irred}(\mathrm{Filt}(\mathfrak{X}))$ which containing some stratum $\mathfrak{S}_\alpha$ and the indexing map $\mu : S \to \Gamma$. The main result of this section, Theorem 2.7, gives necessary and sufficient conditions for such data to define a Θ-stratification. We discuss conditions under which a Θ-stratification on $\mathfrak{X}$ induces a Θ-stratification on a stack $\mathfrak{Y}$ via a map $\mathfrak{Y} \to \mathfrak{X}$, and show that a Θ-stratification of $\mathfrak{X}$ always induces a Θ-stratification of $\mathrm{Grad}(\mathfrak{X})$ (Proposition 2.15).

Section 3: Combinatorial structures. We introduce a category of formal fans (Definition 3.1), which behave somewhat like semisimplicial sets. We define the geometric realization $|F_\bullet|$ (respectively the projective geometric realizations $\mathbb{P}(F_\bullet)$) of a formal fan $F_\bullet$, which is a topological space constructed as a union of copes of $\mathbb{R}^n_{\geq 0}$ (respectively the standard n-simplex $(\mathbb{R}^{n+1}_{\geq 0} \setminus \{0\})/\mathbb{R}^\times_{>0}$). The projective realization $\mathbb{P}(F_\bullet)$ of a quasi-separated (Definition 3.14) fan $F_\bullet$ is a compactly generated Hausdorff space (Proposition 3.20).

Given a stack $\mathfrak{X}$ and a k-point $p \in \mathfrak{X}(k)$, the *degeneration fan* $\mathbf{DF}(\mathfrak{X}, p)_\bullet$ (Definition 3.33) parameterizes filtrations of p. This object is of independent interest. When X is a toric variety and $\mathfrak{X} = X/T$, $\mathbf{DF}(\mathfrak{X}, p)_\bullet$ encodes the classical fan in the cocharacter space of T associated to X (Lemma 3.37, Lemma 3.42), and when $\mathfrak{X} = \mathrm{pt}/G$ for a split semisimple algebraic group G, $\mathbf{DF}(\mathfrak{X}, p)_\bullet$ encodes the structure of the spherical building of G (Proposition 3.46). This suggests many questions for further inquiry as to what geometric properties of a normal variety X with a single dense open orbit under the action of a reductive group G are encoded in the degeneration fan $\mathbf{DF}(\mathfrak{X}, p)_\bullet$. The degeneration space $\mathscr{D}eg(\mathfrak{X}, p) := \mathbb{P}(\mathbf{DF}(\mathfrak{X}, p)_\bullet)$ parameterizes filtrations up to the action of $\mathbb{N}^\times$, and under fairly general hypotheses on $\mathfrak{X}$ it is a compactly generated Hausdorff space (Lemma 3.40). If $\pi : \mathfrak{X} \to \mathfrak{Y}$ is a proper representable map, then $\pi_* : \mathscr{D}eg(\mathfrak{X}, p) \to \mathscr{D}eg(\mathfrak{Y}, p)$ is a homeomorphism, and if π is affine then $\pi_* : \mathscr{D}eg(\mathfrak{X}, p) \to \mathscr{D}eg(\mathfrak{Y}, p)$ is a locally convex closed embedding (Proposition 3.41).

Example 0.1. For the stack $\mathfrak{X} = \mathbb{A}^2/\mathbb{G}_m^2$ and the point $p = (1, 1)$, any pair of nonnegative integers, (a, b), defines a group homomorphism $\mathbb{G}_m \to \mathbb{G}_m^2$ which extends to a map of quotient stacks $f : \Theta \to \mathbb{A}^2/\mathbb{G}_m^2$ along with an isomorphism $f(1) \simeq (1, 1)$, and the morphism f corresponding to the pair (na, nb) is the pre-composition with the n-fold ramified cover $\Theta \to \Theta$.

Thus non-degenerate filtrations of $(1,1) \in \mathbb{A}^2/\mathbb{G}_m^2$ (up to ramified coverings) correspond to rational rays in the cone $(\mathbb{R}_{\geq 0})^2$. We identify this, in turn, with the set of rational points in the unit interval, and $\mathscr{D}eg(\mathfrak{X}, p)$ is this interval. Note that $\mathscr{D}eg(\mathfrak{X}, p)$ parameterizes filtrations of p, but a path in $\mathscr{D}eg(\mathfrak{X}, p)$ does not correspond to an algebraic family of filtrations. In this case $\operatorname{Flag}(p)$ for $p = (1,1) \in \mathbb{A}^2/\mathbb{G}_m^2$ is an infinite disjoint union of points, one for every pair of nonnegative integers (a, b).

Given a filtration $f : \Theta_k \to \mathfrak{X}$, the associated graded point $\operatorname{ev}_0(f) : \{0\}/(\mathbb{G}_m)_k \to \mathfrak{X}$ receives a canonical filtration as a point in $\operatorname{Grad}(\mathfrak{X})$. Our main structural result, Theorem 3.60, identifies a neighborhood of the canonical filtration in $\mathscr{D}eg(\operatorname{Grad}(\mathfrak{X}), \operatorname{ev}_0(f))$ with a neighborhood of the given filtration f in $\mathscr{D}eg(\mathfrak{X}, \operatorname{ev}_1(f))$. Thus small perturbations of a given filtration are identified with filtrations of $\operatorname{ev}_0(f)$ which are close to the canonical filtration.

The component fan $\mathbf{CF}(\mathfrak{X})_\bullet$ and component space $\mathscr{C}omp(\mathfrak{X}) := \mathbb{P}(\mathbf{CF}(\mathfrak{X})_\bullet)$ parameterize filtrations in $\mathfrak{X}$ up to algebraic equivalence, i.e. these objects parameterize connected components of $\operatorname{Filt}(\mathfrak{X})$ (Definition 3.72). When $\mathfrak{X}$ is quasi-compact, $\mathbf{CF}(\mathfrak{X})_\bullet$ is "bounded," which implies $\mathscr{C}omp(\mathfrak{X})$ is compact (Corollary 3.78.1). Any cohomology class in $H^{2n}(\mathfrak{X}; \mathbb{R})$ defines a continuous function on $|\mathbf{CF}(\mathfrak{X})_\bullet|$ whose restriction to each cone $\mathbb{R}^n_{\geq 0} \to |\mathbf{CF}(\mathfrak{X})_\bullet|$ is a homogeneous polynomial of degree n, and this defines a continuous function on $|\mathbf{DF}(\mathfrak{X}, p)_\bullet|$ for any $p \in \mathfrak{X}(k)$ via the canonical map of fans $\mathbf{DF}(\mathfrak{X}, p)_\bullet \to \mathbf{CF}(\mathfrak{X})_\bullet$. We call a cohomology class $b \in H^4(\mathfrak{X}; \mathbb{R})$ positive definite if the corresponding function $\hat{b} : |\mathbf{CF}(\mathfrak{X})_\bullet| \to \mathbb{R}$ is positive away from the cone point, and if $\mathfrak{X}$ is quasi-compact and admits a positive definite class, the connected components of the flag space $\operatorname{Flag}(\xi)$ are quasi-compact for any $\xi \in \mathfrak{X}(R)$ (Proposition 3.89).

Section 4: Construction of Θ-stratifications. We associate (Definition 4.9) a numerical invariant, which consists (Definition 4.1) of a realizable subset $\mathfrak{U} \subset \mathscr{C}omp(\mathfrak{X})$ and a continuous function $\mu : \mathfrak{U} \to \mathbb{R}$, to any cohomology class $\ell \in H^2(\mathfrak{X}; \mathbb{R})$ and positive semidefinite $b \in H^4(\mathfrak{X}; \mathbb{R})$ which is positive on $\mathfrak{U}$ via the formula $\mu(f) = f^*(\ell)/\sqrt{f^*(b)}$, where we are using the isomorphisms $H^2(\Theta_k; \mathbb{R}) \simeq H^4(\Theta_k; \mathbb{R}) \simeq \mathbb{R}$. The class in ℓ plays the role of the first Chern class of the G-linearized ample bundle in geometric invariant theory, and the class b plays the role of a Weyl group invariant inner product on the coweight lattice of G. For $x \in |\mathfrak{X}|$, we define $M^\mu(x) = \sup\{\mu(f) | f \in \operatorname{Filt}(\mathfrak{X}) \text{ s.t. } f(1) = x \in |\mathfrak{X}|\}$. A point $p \in |\mathfrak{X}|$ is unstable if $M^\mu(p) > 0$, and an HN filtration of an unstable point $p \in \mathfrak{X}(k)$ is a point in $|\operatorname{Flag}(p)|$ which maximizes $\mu(f)$.

A Θ-reductive stack is one for which $\operatorname{ev}_1 : \operatorname{Filt}(\mathfrak{X}) \to \mathfrak{X}$ satisfies the valuative criterion for properness with respect to discrete valuation rings (Definition 4.16). Examples include pt/G for a reductive group G, and any $\mathfrak{X}$ which admits an affine map $\mathfrak{X} \to \mathfrak{Y}$ to a Θ-reductive stack $\mathfrak{Y}$ (Corollary 4.19.1). If $\mathfrak{X}$ is Θ-reductive, then for any $p \in \mathfrak{X}(k)$ and any $\ell \in H^2(\mathfrak{X}; \mathbb{R})$, the subset of $\mathscr{D}eg(\mathfrak{X},p)$ of filtrations for which $f^*(\ell) > 0$ is convex (Proposition 4.21). It follows that the HN filtration of an unstable point in a Θ-reductive stack is unique if it exists (Corollary 4.21.1). One can restrict a numerical invariant on $\mathfrak{X}$ to one on $\mathrm{Grad}(\mathfrak{X})$ along the map $u : \mathrm{Grad}(\mathfrak{X}) \to \mathfrak{X}$ which forgets the grading, and if $\mathfrak{X}$ is Θ-reductive then a filtration $f : \Theta_k \to \mathfrak{X}$ is a HN filtration for $f(1)$ if and only if $\mathrm{ev}_0(f) \in \mathrm{Grad}(\mathfrak{X})$ is graded-semistable (Theorem 4.24). We also show that the semistable locus in a Θ-reductive stack is again Θ-reductive (Proposition 4.26).

Our main theorem on the existence of weak Θ-stratifications is the following:

Theorem 0.2 (Theorem 4.38)**.** *Let $\mathfrak{X}$ be a stack satisfying (†), and let $\mu : \mathfrak{U} \subset \mathscr{C}omp(\mathfrak{X}) \to \mathbb{R}$ be a standard numerical invariant on $\mathfrak{X}$ (some mild hypotheses omitted, see statement in main text). Then μ defines a weak Θ-stratification if and only if it satisfies the following:*

1. ***Uniqueness of HN-filtrations:*** *For all* finite type *unstable points $p \in \mathfrak{X}(k)$, if $|\mathrm{Flag}(p)|$ contains a point f which maximizes μ, then it is unique up to the action of $\mathbb{N}^\times$ on $\mathrm{Flag}(p)$. This is the* Harder-Narasimhan (HN) filtration *of p.*
2. ***HN-specialization:*** *For any discrete valuation ring R with fraction field K and residue field k, and any map $\xi : \mathrm{Spec}(R) \to \mathfrak{X}$ whose generic point is unstable and a HN filtration $f_K \in \mathrm{Flag}(\xi)(K)$ of ξ_K, one has*
$$\mu(f_K) \leq M^\mu(\xi|_{\mathrm{Spec}(k)}),$$
and when equality holds there is a unique extension of f_K to a filtration $f_R \in \mathrm{Flag}(\xi)(R)$,[3]
3. *The boundedness condition (B2) holds.*

Our most general theorem on the existence of weak Θ-stratifications (Theorem 4.44) is a relative version pertaining to polynomial valued classes $\ell \in H^2(\mathfrak{X};\mathbb{Q}[n])$ which are positive relative to a proper map $\mathfrak{X} \to \mathfrak{Y}$, where $\mathfrak{Y}$ has a Θ-stratification for which $\mathfrak{Y}^{\mathrm{ss}}$ is Θ-reductive. Another interesting class of stacks are those which admit a good moduli space $\mathfrak{X} \to X$ in the sense of [A]. An algebraic stack of finite type with affine diagonal over a field which admits a good moduli space is Θ-reductive (Lemma 4.53), and we reformulate the main theorem of geometric invariant theory (Theorem 4.54) as the fact that for such an $\mathfrak{X}$, any $\ell \in H^2(\mathfrak{X};\mathbb{Q})$ and positive definite $b \in H^4(\mathfrak{X};\mathbb{Q})$ defines a Θ-stratification of $\mathfrak{X}$ for which $\mathfrak{X}^{\mathrm{ss}}$ admits a good moduli space, which is projective over X if $\ell \propto c_1(L)$ for some $L \in \mathrm{Pic}(\mathfrak{X})$.

[3]Here it suffices to consider only discrete valuation rings R which are essentially finite type over the base B, and when equality holds it suffices show the existence and uniqueness of an extension $f_{R'}$ after passing to an arbitrary extension of discrete valuation rings $R' \supset R$.

Section 5: Bridgeland stability on a projective scheme. Our goal is to illustrate a complete example, which lies beyond the scope of geometric invariant theory, of a numerical invariant which defines a Θ-stratification on a non-quasi-compact algebraic stack: the stack of torsion-free objects in the heart of a noetherian t-structure on the bounded derived category of coherent sheaves $\mathrm{D}^b(X)$ on a projective k-scheme X. The story here is familiar to experts, but we hope it is instructive to connect our theory to this well-studied example. We review and slightly elaborate on the theory of flat families of objects in the heart $\mathrm{D}^b(X)^\heartsuit$ (Definition 5.2) following [AP, P3]. Under a "generic flatness" hypothesis on the t-structure, the moduli functor $\mathcal{M}$ of flat families in $\mathrm{D}^b(X)^\heartsuit$ is an algebraic stack (Proposition 5.10). Filtrations in $\mathcal{M}$ correspond to $\mathbb{Z}$-weighted filtrations in the abelian category $\mathrm{D}^b(X)^\heartsuit$ (Lemma 5.12), and the degeneration space $\mathscr{D}eg(\mathcal{M}, E)$ parameterizes finite descending filtrations in $\mathrm{D}^b(\mathfrak{X})^\heartsuit$ along with a choice of ascending weights in $\mathbb{R}$ (Proposition 5.15). $\mathcal{M}$ is Θ-reductive with respect to essentially finite type discrete valuation rings, and is Θ-reductive with respect to all valuation rings if $\mathcal{M}$ is algebraic and has quasi-compact flag spaces (Proposition 5.17).

We review the notion of slope stability on $\mathrm{D}^b(X)^\heartsuit$ (Definition 5.24). Both Bridgeland stability conditions and the usual notion of slope stability for coherent sheaves [HL5] fit into this framework. We identify cohomology classes (36) on $\mathcal{M}$ for which Θ-stability corresponds to slope semistability in $\mathrm{D}^b(X)^\heartsuit$ (Theorem 5.29). We recall a "boundedness of quotients" condition (Definition 5.36) which is known to hold for algebraic Bridgeland stability conditions for which the moduli of semistable objects in each numerical K-theory class is bounded (Lemma 5.38) and in particular holds for a dense subset of the stability conditions on $K3$-surfaces constructed in [B^+2]. For a noetherian algebraic stability condition which satisfies the generic flatness and boundedness of quotients hypotheses, the stack $\mathcal{M}^{\mathcal{F}}$ of torsion-free objects in $\mathrm{D}^b(X)^\heartsuit$ is algebraic, and the cohomology classes we have identified define a Θ-stratification of the stack $\mathcal{M}^{\mathcal{F}}$ which on underlying k-points agrees with the decomposition of $\mathcal{M}^{\mathcal{F}}(k)$ by the type of the Harder-Narasimhan filtration. Our treatment of Θ-stratifications for slope semistability is by no means complete, and we outline some further directions for study in Section 5.4.

Context. We will fix a base algebraic stack B which we assume is locally Noetherian and *covered by G-rings* in the sense that it admits an atlas of the form $\bigsqcup \mathrm{Spec}(R_i)$ where R_i is a G-ring. We will typically denote stacks with fraktur font ($\mathfrak{X}$,$\mathfrak{Y}$, etc.), and we will denote quotient stacks as X/G. All of our stacks $\mathfrak{X}$ will be stacks on the big étale site of affine schemes over B, or equivalently stacks $\mathfrak{X}$ on the big étale site of affine schemes with a fixed map $\mathfrak{X} \to B$. Although we will often not mention this explicitly, all conditions on $\mathfrak{X}$ will be relative to B. For instance, $\mathfrak{X}$ has quasi-affine diagonal if the map $\mathfrak{X} \to \mathfrak{X} \times_B \mathfrak{X}$ is quasi-affine. If S is a B-scheme, we will use the notation $\mathfrak{X}_S$ to denote the S-stack $S \times_B \mathfrak{X}$.

We will often make the following technical hypotheses on $\mathfrak{X}$, which for brevity we collect here:

(†) $\mathfrak{X}$ is algebraic and locally finite type over B and has quasi-affine diagonal

For instance $\mathfrak{X}$ could admit a Zariski open cover by stacks of the form X/G where X is a quasi-separated, quasi-compact algebraic space and G is a smooth affine group scheme.

Author's note. This project began as the third chapter of my PhD thesis, which contained much weaker versions of many of the results here. As I came to understand the relative ubiquity of examples of Θ-stratifications which are already studied in algebraic geometry, I began to appreciate the value of a text which would introduce a general notion and attempt to present an authoritative treatment of these structures in moduli problems. This has led me to expand, strengthen, and rewrite this paper several times over the intervening years.

As
I have had over this time: Constantin Teleman, Andrei Okounkov, Davesh Maulik, Johan de Jong, and Michael Thaddeus. This work has benefitted from useful conversations with many other researchers: Jarod Alper, Arend Bayer, Gergely Bérczi, Tristan Collins, Brian Conrad, Anand Deopurkar, Galyna Dobrovolska, Maksym Fedorchuk, Simion Filip, Joshua Jackson, Frances Kirwan, Sándor Kovács, Jochen Heinloth, Victoria Hoskins, Jacob Lurie, Johan Martens , Ian Morrison, Yuji Odaka, Martin Olsson, Alexander Polishchuk, Simon Schieder, Carlos Simpson, Pablo Solis, David Swinarski, Richard Thomas, Yukinobu Toda, Roman Travkin, Xiaowei Wang, Chris Woodward, and Chenyang Xu.

This work was partially supported by NSF grants DMS-1601967 and DMS-1303960 (MSPRF). I would like to thank Columbia University, the Institute for Advanced Study, Cornell University, and the Mathematical Sciences Research Institute for providing the stimulating environments in which I completed this work.

1. Filtrations

In this section we lay the foundational framework for the theory of stability in an algebraic stack $\mathfrak{X}$. As mentioned in the introduction, the quotient stack $\Theta := \mathbb{A}^1/\mathbb{G}_m$ plays a key role. Motivated by concept of the Rees module associated to a filtered vector space, we interpret a map $f : \Theta \to \mathfrak{X}$ as a "filtration" of the point $f(1) \in \mathfrak{X}$. This leads us to define algebraic stacks parameterizing filtered points of $\mathfrak{X}$, graded points of $\mathfrak{X}$, and flag spaces associated to a point of $\mathfrak{X}$, and to establish the basic properties of these objects.

1.0.1. *Motivating example.* Maps $S \times \Theta \to B\mathrm{GL}_N$ classify locally free sheaves on $S \times \Theta$. Recall the Rees construction:

Proposition 1.1. *Let S be a k-scheme. The category of quasicoherent sheaves on $S \times \Theta$ is equivalent to the category of diagrams of quasicoherent sheaves on S of the form $\cdots \to F_i \to F_{i-1} \to \cdots$.*

Proof. Using descent one sees that quasicoherent sheaves on $S \times \Theta$ are the same as graded $\mathcal{O}_S[t]$ modules, where t has degree -1. The equivalence assigns a diagram $\cdots \to F_i \to F_{i-1} \to \cdots$ to the module $\bigoplus F_i$ with F_i in degree i with the maps $F_i \to F_{i-1}$ corresponding to multiplication by t. $\square$

Under this equivalence, locally free sheaves on $S \times \Theta$ correspond to diagrams such that each F_i is locally free on S, $F_i \to F_{i-1}$ is injective and $\mathrm{gr}_i(F_\bullet) = F_i/F_{i+1}$ is locally free for each i, F_i stabilizes for $i \ll 0$, and $F_i = 0$ for $i \gg 0$. In other words GL_N bundles on $S \times \Theta$ correspond to locally free sheaves with decreasing, weighted filtrations, and the shape of this filtration must be constant along connected schemes.

1.1. **Graded objects, filtered objects, and flag spaces.** Given two stacks $\mathfrak{X}$ and $\mathfrak{Y}$ over a site, one can form the mapping stack $\underline{\mathrm{Map}}(\mathfrak{Y}, \mathfrak{X})$ as the presheaf of groupoids

$$\underline{\mathrm{Map}}(\mathfrak{Y}, \mathfrak{X}) : T \mapsto \mathrm{Map}(\mathfrak{Y} \times T, \mathfrak{X}), \tag{2}$$

where Map denotes groupoid of 1-morphisms between stacks. If $\mathfrak{Y} = \Theta := \mathbb{A}^1/\mathbb{G}_m$ and $\mathfrak{X}$ is an algebraic stack, then one can apply smooth descent [V] to describe $\underline{\mathrm{Map}}(\Theta, \mathfrak{X})$ more explicitly. We consider the first three levels of the simplicial scheme determined by the action of $\mathbb{G}_m$ on $\mathbb{A}^1 \times T$

$$\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1 \times T \overset{\mu}{\underset{a}{\overset{\sigma}{\rightrightarrows}}} \mathbb{G}_m \times \mathbb{A}^1 \times T \overset{\sigma}{\underset{a}{\rightrightarrows}} \mathbb{A}^1 \times T \tag{3}$$

Where μ denotes group multiplication, σ denotes the action of $\mathbb{G}_m$ on $\mathbb{A}^1$, and a forgets the leftmost group element. Then the category $\underline{\mathrm{Map}}(\Theta, \mathfrak{X})(T)$ has

- *objects:* $\eta \in \mathfrak{X}(\mathbb{A}^1 \times T)$ along with a morphism $\phi : a^*\eta \to \sigma^*\eta$ satisfying the cocycle condition $\sigma^*\phi \circ a^*\phi = \mu^*\phi$
- *morphisms:* $f : \eta_1 \to \eta_2$ such that $\phi_2 \circ a^*(f) = \sigma^*(f) \circ \phi_1 : a^*\eta_1 \to \sigma^*\eta_2$

Informally, if $\mathfrak{X}(T)$ is the groupoid of T-families of geometric objects of a certain kind, then $\underline{\mathrm{Map}}(\Theta, \mathfrak{X})(T)$ is the groupoid of $\mathbb{G}_m$-equivariant families over $\mathbb{A}^1 \times T$.

For general stacks $\mathfrak{Y}$ and $\mathfrak{X}$, the mapping stack $\underline{\mathrm{Map}}(\mathfrak{Y}, \mathfrak{X})$ is certainly not algebraic. In particular, $\underline{\mathrm{Map}}(\mathbb{A}^1, \mathfrak{X})$ is rarely algebraic, and hence it is not obvious from the description of $\underline{\mathrm{Map}}(\Theta, \mathfrak{X})$ in terms of descent that the latter is algebraic. Nevertheless [HP] establishes a theory of cohomologically proper stacks which leads to the following:

Proposition 1.2. *If $\mathfrak{X}$ satisfies (†), then for any $n \geq 1$, $\underline{\mathrm{Map}}(\Theta^n, \mathfrak{X})$ and $\underline{\mathrm{Map}}(\mathrm{pt}/\mathbb{G}_m^n, \mathfrak{X})$ are algebraic stacks, locally of finite presentation over B. If $\mathfrak{X}$ has affine diagonal then so do $\underline{\mathrm{Map}}(\Theta^n, \mathfrak{X})$ and $\underline{\mathrm{Map}}(\mathrm{pt}/\mathbb{G}_m^n, \mathfrak{X})$.*

Proof. The stacks $\mathrm{pt}/\mathbb{G}_m^n$ and Θ^n are cohomologically projective and flat over B, so this is an immediate application of the main theorem of [HP]. □

As with any mapping stack, one has a universal evaluation 1-morphism

$$\mathrm{ev} : \Theta \times \underline{\mathrm{Map}}(\Theta, \mathfrak{X}) \to \mathfrak{X}. \tag{4}$$

Θ has two canonical B-points, the generic point corresponding to $1 \in \mathbb{A}^1$ and the special point $0 \in \mathbb{A}^1$, and the restriction to the open (respectively closed) substack $\{1\} \subset \Theta$ (respectively $\{0\}/\mathbb{G}_m \hookrightarrow \Theta$) define the maps ev_1 (respectively ev_0) of Equation (1). More generally, we consider the B-point of Θ^n determined by $(1, \ldots, 1) \in \mathbb{A}^n$, and the map $\{0\}/\mathbb{G}_m^n \to \Theta^n$ determined by the point $(0, \ldots, 0) \in \mathbb{A}^n$. We denote the resulting restriction maps $\mathrm{ev}_1 : \underline{\mathrm{Map}}(\Theta^n, \mathfrak{X}) \to \mathfrak{X}$ and $\mathrm{ev}_0 : \underline{\mathrm{Map}}(\Theta^n, \mathfrak{X}) \to \underline{\mathrm{Map}}(\mathrm{pt}/\mathbb{G}_m^n, \mathfrak{X})$ as well.

Note that in our context we implicitly work relative to B, so all test schemes S are B-schemes, $\mathbb{A}^1$ and $\mathbb{G}_m$ refer to $\mathbb{A}^1_B$ and $(\mathbb{G}_m)_B$ unless otherwise specified, and the mapping stack is formed relative to B. Concretely, a T-point of $\underline{\mathrm{Map}}(\Theta^n, \mathcal{X})$ is a map of stacks $\Theta^n_T \to \mathcal{X}$ along with a factorization of the composition $\Theta^n_T \to B$ through the projection $\Theta^n_T \to T$. If B is algebraic and locally noetherian with quasi-affine diagonal, then the tannakian formalism (see Lemma 1.28 below) implies that any factorization of the map $\Theta_T \to B$ through T is unique up to unique isomorphism, so the existence of this factorization is a condition rather than additional data.

The formation of mapping stacks behaves well with respect to base change, so we note the following purely formal observation:

Lemma 1.3. *Let T be an algebraic stack over B, and let S be a T-scheme. The groupoid of S-points of $\underline{\mathrm{Map}}_T(\Theta^n_T, \mathfrak{X}_T)$ is canonically equivalent to the groupoid of maps $\Theta^n_S \to \mathfrak{X}$ relative to B, and likewise for $\underline{\mathrm{Map}}_T((\mathrm{pt}/\mathbb{G}_m^n)_T, \mathfrak{X}_T)$. This gives canonical equivalences of T-stacks*

$$\begin{aligned} \underline{\mathrm{Map}}_T(\Theta^n_T, \mathfrak{X}_T) &\simeq \underline{\mathrm{Map}}(\Theta^n, \mathfrak{X}) \times_B T \\ \underline{\mathrm{Map}}_T((\mathrm{pt}/\mathbb{G}_m^n)_T, \mathfrak{X}_T) &\simeq \underline{\mathrm{Map}}(\mathrm{pt}/\mathbb{G}_m^n, \mathfrak{X}) \times_B T \end{aligned}. \tag{5}$$

Furthermore this equivalence identifies the canonical evaluation map $\Theta^n_T \times_T \underline{\mathrm{Map}}_T(\Theta^n_T, \mathfrak{X}_T) \to \mathfrak{X}_T$ with the base change of (4), and likewise for the maps ev_0, ev_1, and σ appearing in (1).

This implies that the stacks $\underline{\mathrm{Map}}_T((\mathrm{pt}/\mathbb{G}_m^n)_T, \mathfrak{X}_T)$ and $\underline{\mathrm{Map}}_T(\Theta^n_T, \mathfrak{X}_T)$ are algebraic and locally of finite presentation over T, even if T is not locally Noetherian.

We will be using these mapping stacks so often that we introduce more concise and intuitive notation.

1.1.1. *The stack of graded objects.*

Example 1.4. Assume that B is a scheme, and let X be a projective scheme over B. Let $\mathfrak{X}$ denote the stack of flat families of coherent sheaves on X. Then $\underline{\mathrm{Map}}(\mathrm{pt}/\mathbb{G}_m^n, \mathfrak{X})$ parameterizes flat families of graded coherent sheaves, graded by the abelian group $\mathbb{Z}^n$.

Motivated by this example, we make the following

Definition 1.5. Given a B-stack T, we define the stack of $\mathbb{Z}^n$*-graded objects* of $\mathfrak{X}$ over T to be

$$\mathrm{Grad}_T^n(\mathfrak{X}) := \underline{\mathrm{Map}}(\mathrm{pt}/\mathbb{G}_m^n, \mathfrak{X}) \times_B T \simeq \underline{\mathrm{Map}}_T((\mathrm{pt}/\mathbb{G}_m^n)_T, \mathfrak{X}_T).$$

We simplify notation by writing Grad^n for Grad^n_B.

There is a canonical "forgetful" map $u : \mathrm{Grad}^n(\mathfrak{X}) = \underline{\mathrm{Map}}(\mathrm{pt}/\mathbb{G}_m^n, \mathfrak{X}) \to \mathfrak{X}$ which restricts along the map $B \to (\mathrm{pt}/\mathbb{G}_m)_B$. Concretely for any B-scheme S this assigns a map $S \times \mathrm{pt}/\mathbb{G}_m \to \mathfrak{X}$ to its restriction $\xi : S \to \mathfrak{X}$. In fact, lifting an S point $\xi \in \mathfrak{X}(S)$ along the map u is equivalent to giving a homomorphism from $\mathbb{G}_m^n$ to the automorphism group

$$\mathrm{Aut}_{\mathfrak{X}/B}(\xi) := \ker(\mathrm{Aut}_{\mathfrak{X}}(\xi) \to \mathrm{Aut}_B(\pi(\xi))),$$

where $\pi : \mathfrak{X} \to B$ is the structure map. More precisely we have

Lemma 1.6. *For any B-scheme S and any S-point $\xi : S \to \mathfrak{X}$ the following diagram is cartesian*

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{S-\mathrm{gp}}((\mathbb{G}_m^n)_S, \underline{\mathrm{Aut}}_{\mathfrak{X}/B}(\xi)) \arrow[r] \arrow[d] & \underline{\mathrm{Map}}(\mathrm{pt}/\mathbb{G}_m^n, \mathfrak{X}) \arrow[d, "\textit{restriction along } B\to(\mathrm{pt}/\mathbb{G}_m^n)_B"] \\
S \arrow[r, "\xi"] & \mathfrak{X}
\end{tikzcd}.$$

Where $\underline{\mathrm{Aut}}_{\mathfrak{X}/B}(\xi)$ denotes the sheaf of groups on S which assigns to any map $U \to S$ the group $\mathrm{Aut}_{\mathfrak{X}/B}(\xi|_U)$. In particular if $\mathfrak{X}$ satisfies (†), then the vertical maps are representable by algebraic spaces.

Proof. It suffices to show the non-sheafified version of the claim for any S-scheme T. This is basically a formal consequence of the description of the groupoid $\mathrm{Map}_B((\mathrm{pt}/\mathbb{G}_m^n)_T, \mathfrak{X})$ via descent. This groupoid is canonically equivalent to the groupoid of maps $\xi : T \to \mathfrak{X}$ relative to B along with an automorphism of the composition $(\mathbb{G}_m^n)_T \to T \to \mathfrak{X}$ relative to B satsifying a cocycle condition. The data of an automorphism of the map $(\mathbb{G}_m^n)_T \to T \to \mathfrak{X}$ relative to B is equivalent to specifying a section of the pullback of the T-sheaf $\underline{\mathrm{Aut}}_{\mathfrak{X}/B}(\xi)$ along $(\mathbb{G}_m^n)_T \to T$, which in turn is equivalent to a map of sheaves of sets $(\mathbb{G}_m^n)_T \to \underline{\mathrm{Aut}}_{\mathfrak{X}/B}(\xi)$ over T. The cocycle condition translates to the condition that this map of sheaves respects the group structure. Thus we see that $\mathrm{Map}_B((\mathrm{pt}/\mathbb{G}_m^n)_T, \mathfrak{X})$ is equivalent to the groupoid consisting of a pairs $(\xi : T \to \mathfrak{X}/B, \phi \in \mathrm{Hom}_{T-\mathrm{gp}}((\mathbb{G}_m^n)_T, \underline{\mathrm{Aut}}_{\mathfrak{X}/B}(\xi)))$

and isomrophisms $(\xi_1, \phi_1) \simeq (\xi_2, \phi_2)$ are isomorphisms $\eta : \xi_1 \simeq \xi_2$ relative to B for which the induced bijection $\mathrm{Hom}_{T-\mathrm{gp}}((\mathbb{G}_m^n)_T, \underline{\mathrm{Aut}}_{\mathfrak{X}/B}(\xi_1)) \simeq \mathrm{Hom}_{T-\mathrm{gp}}((\mathbb{G}_m^n)_T, \underline{\mathrm{Aut}}_{\mathfrak{X}/B}(\xi_2))$ maps $\phi_1 \mapsto \phi_2$. □

Remark 1.7. There are general existence theorems for schemes representing the functor $\underline{\mathrm{Hom}}_{S-\mathrm{gp}}(\mathbb{G}_m^n, G)$ for an S group scheme G, but they typically require G to be flat. Lemma 1.6 extends these results to any G which is étale locally modeled on the inertia $I_{\mathfrak{X}/B} \to \mathfrak{X}$ for some stack satisfying (†).

Let $I_{\mathfrak{X}/B} := \mathfrak{X} \times_{\mathfrak{X} \times_B \mathfrak{X}} \mathfrak{X}$ denote the inertia stack of $\mathfrak{X}$ relative to B. It is a group scheme over $\mathfrak{X}$ whose R points are $\xi \in \mathfrak{X}(R)$ along with an automorphism in $\mathrm{Aut}_{\mathfrak{X}/B}(\xi)$. The previous lemma can be interpreted as providing a canonical equivalence of stacks over $\mathfrak{X}$

$$\mathrm{Grad}^n(\mathfrak{X}) \simeq \underline{\mathrm{Hom}}_{gp}((\mathbb{G}_m^n)_{\mathfrak{X}}, I_{\mathfrak{X}/B}),$$

where the latter denotes the Hom sheaf between group sheaves over $\mathfrak{X}$.

Corollary 1.7.1. *Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a map of stacks such that the canonical map on inertia groups fits into a short exact sequence of group sheaves over $\mathfrak{X}$*

$$\{1\} \to I_{\mathfrak{X}/B} \to \mathfrak{X} \times_{\mathfrak{Y}} I_{\mathfrak{Y}/B} \to \mathcal{Q} \to \{1\},$$

where the fibers of $\mathcal{Q}$ over any field-valued point of $\mathcal{X}$ are representable and quasi-finite. Then the canonical map

$$\mathrm{Grad}^n(\mathfrak{X}) \to \mathfrak{X} \times_{\mathfrak{Y}} \mathrm{Grad}^n(\mathfrak{Y})$$

induces an equivalence of groupoids of k-points for any field k over B. Furthermore, if either $\mathcal{Q} = \{1\}$ (i.e. ϕ is inertia preserving), or $\mathfrak{X}$ and $\mathfrak{Y}$ satisfy (†) and ϕ is representable and étale, then $\mathrm{Grad}^n(\mathfrak{X}) \to \mathfrak{X} \times_{\mathfrak{Y}} \mathrm{Grad}^n(\mathfrak{Y})$ *is an isomorphism of stacks.*

Proof. This follows from Lemma 1.6 and the fact that the Hom sheaf between group schemes commutes with pullback, so $\phi^{-1}(\underline{\mathrm{Hom}}((\mathbb{G}_m^n)_{\mathfrak{Y}}, I_{\mathfrak{Y}/B})) \simeq \underline{\mathrm{Hom}}((\mathbb{G}_m^n)_{\mathfrak{X}}, I_{\mathfrak{X}/B})$. The claim when $\mathfrak{X}$ and $\mathfrak{Y}$ satisfy (†) and ϕ is representable and étale follows from the observation that $\mathrm{Grad}^n(\mathfrak{X}) \to \mathfrak{X} \times_{\mathfrak{Y}} \mathrm{Grad}^n(\mathfrak{Y})$ is a universally bijective map of algebraic stacks which induces an isomorphism on tangent complexes by Proposition 1.18. □

Applying the previous corollary gives

Corollary 1.7.2. *If $\mathfrak{X} \to \mathfrak{Y}$ is a closed immersion (resp. surjective closed immersion, resp. open immersion) relative to B, then so is the map* $\mathrm{Grad}^n(\mathfrak{X}) \to \mathrm{Grad}^n(\mathfrak{Y})$.

We can apply this as well to establish a base change result.

Corollary 1.7.3. *Let $\mathfrak{X}$ be a stack over an algebraic base stack B and consider a map* $\mathrm{Spec}(R) \to B$. *Then the canonical maps define equivalences*

$$\mathrm{Grad}^n(\mathfrak{X}_R) \simeq \mathrm{Spec}(R) \times_B \mathrm{Grad}^n(\mathfrak{X}) \simeq \underline{\mathrm{Map}}_{\mathrm{Spec}(R)}((\mathrm{pt}/\mathbb{G}_m)^n, \mathfrak{X}_R),$$

where the stack of graded objects is formed relative to B.

Proof. The second equivalence is Lemma 1.3. For the first equivalence, consider the composition $\mathfrak{X}_R \to \operatorname{Spec}(R) \to B$. This leads to a left-exact sequence of relative inertia group sheaves over $\mathfrak{X}$

$$\{1\} \to I_{\mathfrak{X}_R/\operatorname{Spec}(R)} \to I_{\mathfrak{X}_R/B} \to \pi^{-1}(I_{\operatorname{Spec}(R)/B}),$$

where $\pi : \mathfrak{X}_R \to \operatorname{Spec}(R)$ is the projection. $I_{\operatorname{Spec}(R)/B} = \{1\}$ because $\operatorname{Spec}(R) \to B$ is representable, and it follows that $I_{\mathfrak{X}_R/R} \to I_{\mathfrak{X}_R/B}$ is an equivalence. On the other hand if $p : \mathfrak{X}_R \to \mathfrak{X}$ is the projection, then the composition $I_{\mathfrak{X}_R/R} \to I_{\mathfrak{X}_R/B} \to p^{-1}(I_{\mathfrak{X}/B})$ is an equivalence, and it follows that $\mathfrak{X}_R \to \mathfrak{X}$ is inertia preserving relative to B. We can therefore apply Corollary 1.7.1. ☐

A closely related observation is the following:

Corollary 1.7.4. *Let $\mathfrak{X}$ be a stack over a base stack B. Let $B' \to B$ be a map of stacks with trivial relative inertia $I_{B'/B} = \{1\}$, and let $\mathfrak{X}' = \mathfrak{X} \times_B B'$. Then the natural map $\underline{\operatorname{Map}}_{B'}((\mathrm{pt}/\mathbb{G}_m)^n, \mathfrak{X}') \to \underline{\operatorname{Map}}_B((\mathrm{pt}/\mathbb{G}_m)^n, \mathfrak{X}')$ as stacks over $\mathfrak{X}'$ is an equivalence.*

Proof. Under Lemma 1.6, we can identify this map of stacks with the map

$$\underline{\operatorname{Hom}}_{gp}((\mathbb{G}_m^n)_{\mathfrak{X}'}, I_{\mathfrak{X}'/B'}) \to \underline{\operatorname{Hom}}_{gp}((\mathbb{G}_m^n)_{\mathfrak{X}'}, I_{\mathfrak{X}'/B})$$

of group sheaves over $\mathfrak{X}'$. The argument of Corollary 1.7.3 shows that $I_{\mathfrak{X}'/B'} \to I_{\mathfrak{X}'/B}$ is an isomorphism, so the map above is an isomorphism. ☐

1.1.2. *The stack of filtered objects.*

Definition 1.8. Given a B-stack T, we define the stack of $\mathbb{Z}^n$*-filtered objects* of $\mathfrak{X}$ over T to be

$$\operatorname{Filt}^n_T(\mathfrak{X}) := \underline{\operatorname{Map}}(\Theta^n, \mathfrak{X}) \times_B T \simeq \underline{\operatorname{Map}}_T(\Theta^n_T, \mathfrak{X}_T),$$

We simplify notation by writing Filt^n for Filt^n_B, and we say that a point of $\operatorname{Filt}^n(\mathfrak{X})$, a filtered point of $\mathfrak{X}$, is *split* if it lies in the image of $\sigma : \operatorname{Grad}^n(\mathfrak{X}) \to \operatorname{Filt}^n(\mathfrak{X})$.

Example 1.9. Continuing Example 1.4, we will see in Lemma 5.12 that $\underline{\operatorname{Map}}(\Theta^n, \mathfrak{X})$ parameterizes flat families of coherent sheaves along with a flat family of filtrations indexed by the partially ordered abelian group $\mathbb{Z}^n$. The map $\operatorname{ev}_1 : \underline{\operatorname{Map}}(\Theta^n, \mathfrak{X}) \to \mathfrak{X}$ is the map which forgets the filtration, and the map $\operatorname{ev}_0 : \underline{\operatorname{Map}}(\Theta^n, \mathfrak{X}) \to \underline{\operatorname{Map}}(\mathrm{pt}/\mathbb{G}_m^n, \mathfrak{X})$ maps a flat family of filtered coherent sheaves to its associated graded family of coherent sheaves. The map σ is the canonical map regarding a graded coherent sheaf as a filtered coherent sheaf, where the filtered pieces are $\mathcal{E}_{\geq w} = \bigoplus_{i \geq w} \mathcal{E}_i$.

1.1.3. *Flag spaces.* Finally, we can define an analog of the classical flag scheme for general moduli problems, using the following:

Lemma 1.10. *If $\mathfrak{X}$ satisfies (†), then the morphism $\operatorname{ev}_1 : \underline{\operatorname{Map}}(\Theta^n, \mathfrak{X}) \to \mathfrak{X}$ is representable by algebraic spaces, locally of finite presentation, and quasi-separated.*

Proof. Proposition 1.2 implies that ev_1 is relatively representable by algebraic stacks, locally of finite presentation, and quasi-separated. It thus suffices to show that $\underline{\mathrm{Map}}(\Theta^n, \mathfrak{X})$ is a category fibered in sets over $\mathfrak{X}$, with respect to the morphism ev_1.

Let S be a B-scheme and let $f : \Theta^n_S \to \mathfrak{X}$ be a morphism, an element of $\underline{\mathrm{Map}}(\Theta^n, \mathfrak{X})(S)$. Then $\mathrm{Aut}(f)$ is equivalent to the group of sections of $\mathfrak{Y} := \Theta^n_S \times_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X} \to \Theta^n_S$, where $\Theta^n_S \to \mathfrak{X}\times\mathfrak{X}$ in this fiber product is classified by (f, f) and $\mathfrak{X} \to \mathfrak{X}\times\mathfrak{X}$ is the diagonal. $\mathrm{ev}_1(f) \in \mathfrak{X}(S)$ is the restriction of f to $\{1\} \times S$, and automorphisms of f which induce the identity on $\mathrm{ev}_1(f)$ correspond to those sections of $\mathfrak{Y} \to \Theta^n_S$ which agree with identity on the open substack $(\mathbb{A}^1 - \{0\})^n \times S/\mathbb{G}^n_m \subset \Theta^n_S$. By hypothesis $\mathfrak{Y} \to \Theta^n$ is representable by separated algebraic spaces, so a section is uniquely determined by its restriction to $(\mathbb{A}^1 - \{0\})^n \times S$. Hence $\mathrm{Aut}(f) \to \mathrm{Aut}_{\mathfrak{X}}(\mathrm{ev}_1(f))$ has trivial kernel. $\square$

Remark 1.11. The fact that for any B-scheme S the fiber of ev_1 over an S point of $\mathfrak{X}$ is equivalent to a set does not depend on the representability of $\underline{\mathrm{Map}}(\Theta^n, \mathfrak{X})$ or even the representability of $\mathfrak{X}$. It only relies on the fact that the inertia stack $I_{\mathfrak{X}} \to \mathfrak{X}$ is representable by separated algebraic spaces.

Definition 1.12. Given a map $\xi : T \to \mathfrak{X}$ over B, we define the $\mathbb{Z}^n$-*flag space* of ξ to be

$$\mathrm{Flag}^n(\xi) := \underline{\mathrm{Map}}(\Theta^n, \mathfrak{X}) \times_{\mathrm{ev}_1, \mathfrak{X}, \xi} T \simeq \mathrm{Filt}^n_T(\mathfrak{X}) \times_{\mathrm{ev}_1, \mathfrak{X}_T, \xi_T} T,$$

which is a locally finitely presented algebraic space relative to T. When $n = 1$ we drop "n" from the notation and we drop "$\mathbb{Z}^n$-" from the terminology.

Example 1.13. Continuing Example 1.9, if $\xi : T \to \mathfrak{X}$ parameterizes a T-flat coherent sheaf $\mathcal{E}$ on $T \times X$, then $\mathrm{Flag}(\xi)$ consists of countably many connected components. Each component is isomorphic to a classical flag scheme parameterizing T-flat filtrations of $\mathcal{E}$ whose associated graded coherent sheaves have specified Hilbert polynomials. Each flag scheme of this kind, however, appears infinitely many times in $\mathrm{Flag}(\xi)$, corresponding to all of the ways to assign integer weights to the associated graded pieces.

1.2. **Deformation theory and the spectral mapping stack.** We will often consider the deformation theory of the stacks $\underline{\mathrm{Map}}(\mathrm{pt}/\mathbb{G}^n_m, \mathfrak{X})$ and $\underline{\mathrm{Map}}(\Theta^n, \mathfrak{X})$, but the easiest way to access that will be to use a bit of spectral algebraic geometry. Spectral algebraic geometry will not play a very central role, so in this section we will simply summarize the short list of results we will need and provide references to thorough treatments of the subject. We suggest the reader skip this section on a first reading of this paper.

In the context of spectral algebraic geometry, we will consider stacks as sheaves of spaces over the ∞-category CAlg^{cn} of connective E_∞-algebras with its étale topology [L4, Section 7.5]. We say that $\mathfrak{X} : \mathrm{CAlg}^{cn} \to \mathcal{S}$ is a 1-stack if for any $A \in \mathrm{CAlg}^{cn}$ with $\pi_0(A) \simeq A$, the space $\mathfrak{X}(A)$ is 1-truncated, i.e. weakly equivalent to the classifying space of a groupoid. For any 1-stack

$\mathfrak{X}$, we define the underlying classical stack $\mathfrak{X}^{\mathrm{cl}}$ as the restriction of $\mathfrak{X}$ to the category of classical rings $\mathrm{Ring} \subset \mathrm{CAlg}^{cn}$. On the other hand given a classical stack $\mathfrak{X}$, we can define $\mathfrak{X}^{\mathrm{sp}}$ to be the left Kan extension of $\mathfrak{X}$ along this same inclusion, and refer to this as the stack $\mathfrak{X}$ regarded as a spectral stack.

More concretely if $\mathfrak{X}$ is an algebraic classical stack, then one can choose a simplicial scheme $U_\bullet$ presenting $\mathfrak{X}$ such that $U_n = \mathrm{Spec}(R_n)$ is affine for every n, and $\mathfrak{X}^{\mathrm{sp}}$ is the spectral stack which is the colimit of the simplicial spectral scheme $U_\bullet^{\mathrm{sp}}$ obtained by regarding each $R_n \in \mathrm{Ring}$ as an E_∞-algebra. The functor $(-)^{\mathrm{sp}}$ is fully faithful. Furthermore these operations are adjoint to one another, in the sense that if $\mathfrak{X}$ is classical and $\mathfrak{Y}$ spectral, then

$$\mathrm{Map}(\mathfrak{X}^{\mathrm{sp}}, \mathfrak{Y}) \simeq \mathrm{Map}(\mathfrak{X}, \mathfrak{Y}^{\mathrm{cl}})$$

where the left hand side is the space of maps of spectral stacks, and the right hand side is the classifying space of the groupoid of maps of classical stacks. In particular the unit of adjunction $\mathfrak{X} \to (\mathfrak{X}^{\mathrm{sp}})^{\mathrm{cl}}$ is an equivalence of classical stacks.

In the context of spectral algebraic geometry, one defines the mapping stack via the same functor of points (2), but where T is an arbitrary affine spectral scheme, i.e. connective E_∞-algebra.

Lemma 1.14. *If $\mathfrak{X}$ is a spectral 1-stack and $\mathfrak{Y}$ is a classical stack, then the adjunction betwen $(-)^{\mathrm{sp}}$ and $(-)^{\mathrm{cl}}$ provides a canonical equivalence of classical stacks*

$$\underline{\mathrm{Map}}(\mathfrak{Y}^{\mathrm{sp}}, \mathfrak{X})^{\mathrm{cl}} \simeq \underline{\mathrm{Map}}(\mathfrak{Y}, \mathfrak{X}^{\mathrm{cl}}),$$

where the right hand side denotes the classical mapping stack.

The main theorem for mapping stacks in [HP] actually applies to the context of spectral algebraic stacks, implying that the spectral mapping stack $\underline{\mathrm{Map}}((\Theta^n)^{\mathrm{sp}}, \mathfrak{X})$ is algebraic when $\mathfrak{X}$ is a spectral algebraic 1-stack locally almost of finite presentation over B and with quasi-affine diagonal. We will be concerned mostly with the situation where we start with a classical algebraic stack $\mathfrak{X}$ satisfying (†), and we will consider the *spectral stack of $\mathbb{Z}^n$-filtered objects*

$$\mathrm{Filt}^n(\mathfrak{X}^{\mathrm{sp}}) := \underline{\mathrm{Map}}((\Theta^n)^{\mathrm{sp}}, \mathfrak{X}^{\mathrm{sp}}).$$

Note that Lemma 1.14 provides a canonical equivalence $\mathrm{Filt}^n(\mathfrak{X}^{\mathrm{sp}})^{\mathrm{cl}} \simeq \mathrm{Filt}^n(\mathfrak{X})$.

Crucially, however, the spectral stack of $\mathbb{Z}^n$-filtered objects $\mathrm{Filt}^n(\mathfrak{X}^{\mathrm{sp}})$ is *not* equivalent to $\mathrm{Filt}^n(\mathfrak{X})^{\mathrm{sp}}$ as a spectral stack. The difference lies in their deformation theory. Any spectral algebraic 1-stack $\mathfrak{X}$ over B has a canonical cotangent complex $\mathbb{L}_\mathfrak{X} \in \mathrm{D}_{qc}(\mathfrak{X})$, which is almost perfect if $\mathfrak{X}$ is locally almost finitely presented over B.[4] The cotangent complex of $\mathrm{Filt}^n(\mathfrak{X})^{\mathrm{sp}}$ in the context of spectral aglebraic geometry, as well as the cotangent complex

[4]Technically the absolute cotangent complex $\mathbb{L}_\mathfrak{X}$ in this context refers to the relative cotangent complex $\mathbb{L}_{\mathfrak{X}/B}$, but as is customary we will suppress B from the notation.

of the classical stack $\mathrm{Filt}^n(\mathfrak{X})$ defined using simplicial commutative rings as in [LMB], are hard to compute in practice. In contrast, we have

Lemma 1.15 ([HP, Proposition 4.13])**.** *At a k-point of $\mathrm{Filt}^n(\mathfrak{X}^{\mathrm{sp}})$, corresponding to a map $f : \Theta_k^n \to \mathfrak{X}$, we have a canonical quasi-isomorphism*

$$(\mathbb{L}_{\mathrm{Filt}^n(\mathfrak{X}^{\mathrm{sp}})})_f \simeq R\Gamma(\Theta_k^n, (f^*\mathbb{L}_{\mathfrak{X}})^\vee)^\vee.$$

Likewise for a graded point $g : \mathrm{Spec}(k)/(\mathbb{G}_m)^n \to \mathfrak{X}$, we have

$$(\mathbb{L}_{\mathrm{Grad}^n(\mathfrak{X}^{\mathrm{sp}})})_g \simeq (g^*(\mathbb{L}_{\mathfrak{X}}))^0,$$

where $(-)^0$ denotes the invariant piece of an object of $\mathrm{D}_{qc}(\mathrm{Spec}(k)/(\mathbb{G}_m)^n)$ regarded as a $\mathbb{Z}^n$-graded complex of vector spaces.

From this formula we can explicitly compute the cotangent complex at a point $f \in \mathrm{Filt}(\mathfrak{X}^{\mathrm{sp}})(k)$ classifying a split filtration. For a complex of graded vector spaces E, we let $E^{\geq 0}$ and $E^{<0}$ denote the summand with nonnegative (resp. negative) weights.

Lemma 1.16. *Let $g : (\mathrm{pt}/\mathbb{G}_m)_k \to \mathfrak{X}$ be a map and let $f = \sigma(g) : \Theta_k \to \mathfrak{X}$ be the corresponding split filtration. Then we have a canonical quasi-isomorphism of exact triangles*

$$\begin{tikzcd}
(\mathbb{L}_{\mathrm{Filt}(\mathfrak{X}^{\mathrm{sp}})/\mathfrak{X}^{\mathrm{sp}}}[-1])_f \arrow[r] \arrow[d, "\simeq"] & (\mathbb{L}_{\mathfrak{X}^{\mathrm{sp}}})_{f(1)} \arrow[r, "\mathrm{ev}_1^*"] \arrow[d, "\simeq"] & (\mathbb{L}_{\mathrm{Filt}(\mathfrak{X}^{\mathrm{sp}})})_f \arrow[r] \arrow[d, "\simeq"] & {} \\
(g^*\mathbb{L}_{\mathfrak{X}})^{>0} \arrow[r] & g^*\mathbb{L}_{\mathfrak{X}} \arrow[r] & (g^*\mathbb{L}_{\mathfrak{X}})^{\leq 0} \arrow[r] & {}
\end{tikzcd}$$

Proof. The top row is the exact triangle of cotangent complexes induced by the map $\mathrm{ev}_1 : \mathrm{Filt}(\mathfrak{X}^{\mathrm{sp}}) \to \mathfrak{X}^{\mathrm{sp}}$. Using Lemma 1.15 we can identify the canonical map $(\mathbb{L}_{\mathfrak{X}^{\mathrm{sp}}})_{f(1)} \to (\mathbb{L}_{\mathrm{Filt}(\mathfrak{X}^{\mathrm{sp}})})_f$ with the linear dual of the map of restriction to the point $\mathbf{1} \in \Theta_k$:

$$((\mathbb{L}_{\mathfrak{X}^{\mathrm{sp}}}|_{f(1)})^\vee)^\vee \simeq \mathbb{L}_{\mathfrak{X}^{\mathrm{sp}}}|_{f(1)} \to R\Gamma(\Theta_k, (f^*\mathbb{L}_{\mathfrak{X}^{\mathrm{sp}}})^\vee)^\vee.$$

Let $p : \Theta_k \to \mathrm{Spec}(k)/\mathbb{G}_m$ denote the projection. The fact that $f = \sigma(g)$ means, by definition, that $f = g \circ p$, which gives a canonical isomorphism $f^*(\mathbb{L}_{\mathfrak{X}^{\mathrm{sp}}}) \simeq p^*(g^*\mathbb{L}_{\mathfrak{X}^{\mathrm{sp}}})$. In particular $\mathbb{L}_{\mathfrak{X}^{\mathrm{sp}}}|_{f(1)} \simeq g^*(\mathbb{L}_{\mathfrak{X}^{\mathrm{sp}}})$ after forgetting the $\mathbb{G}_m$-action on the latter.

After chasing through a few dualizations, the claim follows from the following more concrete claim: for any graded complex of vector spaces, regarded as an object $E \in \mathrm{D}_{qc}((\mathrm{pt}/\mathbb{G}_m)_k)$, the map $R\Gamma(\Theta_k, p^*(E)) \to p^*(E)|_1 \simeq E$ is isomorphic to the inclusion of the direct summand $E^{\geq 0} \to E$. One can verify this by identifying $R\Gamma(\Theta_k, p^*(E))$ with $(E \otimes_k k[t])^{\mathbb{G}_m}$, where t has weight -1. □

For most of this paper we will be concerned instead with classical algebraic stacks. As mentioned above, the deformation theory of the spectral stack of filtrations differs from the classical stack of filtrations, so it will be useful to have a comparison theorem between the two. We introduce some temporary notation. For a spectral algebraic 1-stack $\mathfrak{X}$, we let $\mathbb{L}_{\mathfrak{X}}^{E_\infty} \in \mathrm{D}_{qc}(\mathfrak{X})$ denote its cotangent complex in the context of spectral algebraic geometry, and by a slight abuse of notation let $\mathbb{L}_{\mathfrak{X}^{\mathrm{cl}}}^{E_\infty} \in \mathrm{D}_{qc}(\mathfrak{X}^{\mathrm{cl}})$ denote the cotangent complex of the spectral algebraic stack $(\mathfrak{X}^{\mathrm{cl}})^{\mathrm{sp}}$. Here we are making use of the canonical equivalence of stable ∞-categories $\mathrm{D}_{qc}(\mathfrak{X}^{\mathrm{cl}}) \to \mathrm{D}_{qc}((\mathfrak{X}^{\mathrm{cl}})^{\mathrm{sp}})$, where the former denotes the derived category of complexes with quasi-coherent homology on the classical stack. Finally, we let $\mathbb{L}_{\mathfrak{X}^{\mathrm{cl}}}^{\mathrm{cl}} \in \mathrm{D}_{qc}(\mathfrak{X}^{\mathrm{cl}})$ denote the cotangent complex computed in the context of classical algebraic geometry.

There are two canonical maps in $\mathrm{D}_{qc}(\mathfrak{X}^{\mathrm{cl}})$

$$\mathbb{L}_{\mathfrak{X}}^{E_\infty}|_{\mathfrak{X}^{\mathrm{cl}}} \xrightarrow{\phi^1_{\mathfrak{X}}} \mathbb{L}_{\mathfrak{X}^{\mathrm{cl}}}^{E_\infty} \xrightarrow{\phi^2_{\mathfrak{X}}} \mathbb{L}_{\mathfrak{X}^{\mathrm{cl}}}^{\mathrm{cl}} ,$$

where the first is induced by the inclusion of spectral stacks $(\mathfrak{X}^{\mathrm{cl}})^{\mathrm{sp}} \hookrightarrow \mathfrak{X}$, and the second is induced by a direct comparison between the functors on $\mathrm{D}_{qc}(\mathfrak{X}^{\mathrm{cl}})$ which are corepresented by $\mathbb{L}^{\mathrm{cl}}$ and $\mathbb{L}^{E_\infty}$ respectively. The canonical cofiber sequence of cotangent complexes associated to a map of spectral stacks $\mathfrak{X} \to \mathfrak{Y}$ are comparible with these comparison maps, which induces comparison maps

$$\mathbb{L}_{\mathfrak{X}/\mathfrak{Y}}^{E_\infty}|_{\mathfrak{X}^{\mathrm{cl}}} \xrightarrow{\phi^1_{\mathfrak{X}/\mathfrak{Y}}} \mathbb{L}_{\mathfrak{X}^{\mathrm{cl}}/\mathfrak{Y}^{\mathrm{cl}}}^{E_\infty} \xrightarrow{\phi^2_{\mathfrak{X}}} \mathbb{L}_{\mathfrak{X}^{\mathrm{cl}}/\mathfrak{Y}^{\mathrm{cl}}}^{\mathrm{cl}} .$$

These maps are not equivalences, but they induce isomorphisms in high cohomological degree.

Lemma 1.17. *Given a map of quasi-separated spectral algebraic* 1*-stacks* $\mathfrak{X} \to \mathfrak{Y}$*, the canonical map on homology sheaves*

(1) $H^n(\phi^1_{\mathfrak{X}/\mathfrak{Y}}) : H^n(\mathbb{L}_{\mathfrak{X}/\mathfrak{Y}}^{E_\infty}|_{\mathfrak{X}^{\mathrm{cl}}}) \to H^n(\mathbb{L}_{\mathfrak{X}^{\mathrm{cl}}/\mathfrak{Y}^{\mathrm{cl}}}^{E_\infty})$ *is surjective for* $n \geq -1$ *and an isomorphism for* $n \geq 0$*, and*
(2) $H^n(\phi^2_{\mathfrak{X}/\mathfrak{Y}}) : H^n(\mathbb{L}_{\mathfrak{X}^{\mathrm{cl}}/\mathfrak{Y}^{\mathrm{cl}}}^{E_\infty}) \to H^n(\mathbb{L}_{\mathfrak{X}^{\mathrm{cl}}/\mathfrak{Y}^{\mathrm{cl}}}^{\mathrm{cl}})$ *is surjective for* $n \geq -2$ *and an isomorphism for* $n \geq -1$*.*

Proof. The claims are equivalent to the claim that $H^n(\mathrm{cofib}(\phi^i_{\mathfrak{X}/\mathfrak{Y}})) = 0$ for $n \geq -1$ in the case $i = 1$ and for $n \geq -2$ in the case $i = 2$. The canonical exact triangle of cotangent complexes associated to the map $\pi : \mathfrak{X} \to \mathfrak{Y}$ induces an exact triangle

$$\pi^* \mathrm{cofib}(\phi^i_{\mathfrak{Y}}) \to \mathrm{cofib}(\phi^i_{\mathfrak{X}}) \to \mathrm{cofib}(\phi^i_{\mathfrak{X}/\mathfrak{Y}}) \to,$$

which shows that the claim of the lemma can be deduced from the absolute case, i.e. the claim that $H^n(\mathrm{cofib}(\phi^i_{\mathfrak{X}})) = 0$ for n in the appropriate range and for any spectral algebraic 1-stack $\mathfrak{X}$.

Let $\mathfrak{X}$ be a spectral stack and conisder a smooth surjective map $X \to \mathfrak{X}$ from a spectral scheme X. The exact triangle of cotangent complexes associated to the map $X \to \mathfrak{X}$ allows one to reduce both claims to the corresponding claim for $\mathbb{L}_X$ and $\mathbb{L}_{X/\mathfrak{X}}$. All three versions of the cotangent complex $\mathbb{L}_{X^{\mathrm{cl}}/\mathfrak{X}^{\mathrm{cl}}}^{\mathrm{cl}}$, $\mathbb{L}_{X^{\mathrm{cl}}/\mathfrak{X}^{\mathrm{cl}}}^{E_\infty}$, and $\mathbb{L}_{X/\mathfrak{X}}^{E_\infty}|_{\mathfrak{X}^{\mathrm{cl}}}$ are compatible with base change over

$\mathfrak{X}$, which allows us to reduce the general case to the case for spectral schemes, and ultimately to showing the relevant vanishing of $H^n(\mathrm{cofib}(\phi^i_X)$ where X is an affine spectral scheme. Claim (1) now follows from [L4, Theorem 7.4.3.1] and Claim (2) follows from the identification of topological André-Quillen cohomology and classical André-Quillen cohomology of commutative rings in low homological degree (See [L3, Warning 1.0.7]). □

In the remainder of the article, we will not make use of the intermediate object $\mathbb{L}^{E_\infty}_{\mathfrak{X}^{\mathrm{cl}}/\mathfrak{Y}^{\mathrm{cl}}}$, so we can safely drop the superscript from our notation: for spectral stacks $\mathbb{L}_{\mathfrak{X}/\mathfrak{Y}}$ will always denote the cotangent complex in the spectral context, and for classical stacks $\mathbb{L}_{\mathfrak{X}/\mathfrak{Y}}$ will denote the cotangent complex in the classical context.

1.3. Some general properties of $\mathrm{Filt}^n(\mathfrak{X})$.

Proposition 1.18. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be stacks satisfying (†), and let $\phi : \mathfrak{Y} \to \mathfrak{X}$ be a morphism which is representable by algebraic spaces. Then so is the induced morphism $\mathrm{Filt}^n(\phi) : \mathrm{Filt}^n(\mathfrak{Y}) \to \mathrm{Filt}^n(\mathfrak{X})$. Furthermore:*

1. *If ϕ is a monomorphism, then so is $\mathrm{Filt}^n(\phi)$.*
2. *If ϕ is a closed immersion, then so is $\mathrm{Filt}^n(\phi)$, and $\mathrm{Filt}^n(\phi)$ identifies $\mathrm{Filt}^n \mathfrak{Y}$ with the closed substack $\mathrm{ev}_1^{-1} \mathfrak{Y} \subset \mathrm{Filt}^n(\mathfrak{X})$.*
3. *If ϕ is an open immersion, then so is $\mathrm{Filt}^n(\phi)$, and $\mathrm{Filt}^n(\phi)$ identifies $\mathrm{Filt}^n(\mathfrak{Y})$ with the preimage of $\mathfrak{Y} \subset \mathfrak{X}$ under the composition*
$$\mathrm{Filt}^n(\mathfrak{X}) \xrightarrow{\mathrm{ev}_0} \mathrm{Grad}^n(\mathfrak{X}) \to \mathfrak{X}.$$
4. *If ϕ is smooth (respectively étale), then so are $\mathrm{Filt}^n(\phi)$ and $\mathrm{Grad}^n(\phi) : \mathrm{Grad}^n(\mathfrak{Y}) \to \mathrm{Grad}^n(\mathfrak{X})$, and this holds even if ϕ is not representable.*

Proof. Let $S \to \mathrm{Filt}^n(\mathfrak{X}) = \underline{\mathrm{Map}}(\Theta^n, \mathfrak{X})$ be an S-point defined by a morphism $\Theta^n_S \to \mathfrak{X}$. Then the fiber product $\Theta^n_S \times_{\mathfrak{X}} \mathfrak{Y} \to \Theta^n_S$ is representable and is thus isomorphic to $E/\mathbb{G}^n_m$ for some algebraic space E with a $\mathbb{G}^n_m$-equivariant map $E \to \mathbb{A}^n_S$. The fiber of $\mathrm{Filt}^n(\mathfrak{Y}) \to \mathrm{Filt}^n(\mathfrak{X})$ over the given S-point of $\mathrm{Filt}^n(\mathfrak{X})$ corresponds to the groupoid of sections of $E/\mathbb{G}^n_m \to \mathbb{A}^n_S/\mathbb{G}^n_m$, which form a set. Thus $\mathrm{Filt}^n(\mathfrak{Y})$ is equivalent to a sheaf of sets as a category fibered in groupoids over $\mathrm{Filt}^n(\mathfrak{X})$. Both $\mathrm{Filt}^n(\mathfrak{X})$ and $\mathrm{Filt}^n(\mathfrak{Y})$ are algebraic by Proposition 1.2, so $\mathrm{Filt}^n(\phi)$ is representable by algebraic spaces.

Say ϕ is a monomorphism, meaning it induces a fully faithful embedding on groupoids of S-points for any B-scheme S. Then smooth descent implies implies that $\mathrm{Map}(\Theta^n_S, \mathfrak{X}) \to \mathrm{Map}(\Theta^n_S, \mathfrak{Y})$ is fully faithull as well, so $\mathrm{Filt}^n(\mathfrak{Y}) \to \mathrm{Filt}^n(\mathfrak{X})$ is a monomorphism. To identify the full subfunctor $\mathrm{Filt}^n(\mathfrak{Y}) \subset \mathrm{Filt}^n(\mathfrak{X})$, one need only identify for each B-scheme S which maps $f : \Theta^n_S \to \mathfrak{X}$ factor through $\mathfrak{Y}$.

The smallest closed substack of Θ^n_S containing the S-point determined by the point $(1, \ldots, 1) \in \mathbb{A}^n$ is Θ^n_S itself, so if ϕ is a closed immersion then f factors through $\mathfrak{Y}$ if and only if the composition $\{1\} \times S \to \Theta^n_S \to \mathfrak{X}$ factors through $\mathfrak{Y}$, which shows (2). Likewise any open substack of Θ^n_S containing $(\{0\}/\mathbb{G}^n_m)_S$ contains all of Θ^n_S, so if ϕ is an open immersion then f factors through $\mathfrak{Y}$ if and only if the induced map $(\{0\}/\mathbb{G}_m^n)_S \to \mathfrak{X}$ factors through $\mathfrak{Y}$. Finally the map $(\mathrm{pt}/\mathbb{G}_m^n)_S \to S$ induces a bijection between posets of open substacks, which shows (3).

It suffices to show (4) in the case $n = 1$, because $\mathrm{Filt}^n(-) \simeq \mathrm{Filt}(\mathrm{Filt}(\cdots))$ and likewise for $\mathrm{Grad}^n(-)$. The computation of the cotangent complex of the spectral mapping stack in Lemma 1.15 and Lemma 1.16, combined with the fact that the projection functors onto weight spaces $(-)^0$,$(-)^{\leq 0}$, and $(-)^{>0}$ are exact, implies that $\mathrm{Grad}(\mathfrak{Y}^{sp}) \to \mathrm{Grad}(\mathfrak{X}^{sp})$ is smooth, and $\mathrm{Filt}(\mathfrak{Y}^{sp}) \to \mathrm{Filt}(\mathfrak{X}^{sp})$ is smooth at every split filtration in $\mathrm{Filt}(\mathfrak{Y})$. The comparison result Lemma 1.17 shows that the same is true for the classical mapping stacks. Finally, we will see below in Lemma 1.24 that every point of $\mathrm{Filt}(\mathfrak{Y})$ specializes to split point, so because smoothness is an open condition on the source of a locally finite type map, it follows that $\mathrm{Filt}(\mathfrak{Y}) \to \mathrm{Filt}(\mathfrak{X})$ is smooth. $\square$

Another important property of the induced map $\mathrm{Filt}^n(\mathfrak{X}) \to \mathrm{Filt}^n(\mathfrak{Y})$ is the following:

Proposition 1.19. *Let $\pi : \mathfrak{X} \to \mathfrak{Y}$ be an affine (respectively finite) morphism of algebraic stacks. Then for any n the map $\mathrm{Filt}^n(\mathfrak{X}) \to \mathrm{Filt}^n(\mathfrak{Y})$ is an affine (respectively finite) morphism, and the canonical map $\mathrm{Filt}^n(\mathfrak{X}) \to \mathrm{Filt}^n(\mathfrak{Y}) \times_{\mathfrak{Y}} \mathfrak{X}$ is a closed immersion (respectively a surjective closed immersion).*

Lemma 1.20. *Let T be a scheme, and let $\mathcal{A} = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}_n$ be a graded $\mathcal{O}_T[t]$-algebra, where t has degree -1. Then a graded map of $\mathcal{O}_T[t^{\pm}]$-algebras $\psi : \mathcal{A}[t^{-1}] \to \mathcal{O}[t^{\pm}]$ is the localization of a map of graded $\mathcal{O}_T[t]$-algebras $\tilde{\psi} : \mathcal{A} \to \mathcal{O}_T[t]$ if and only if the composition*

$$\mathcal{A}_1 \xrightarrow{\times t} \mathcal{A}[t^{-1}] \xrightarrow{\psi} \mathcal{O}_T[t^{\pm}]$$

vanishes. If such a $\tilde{\psi}$ exists then it is unique.

Proof. ψ is uniquely determined by its restriction to $\mathcal{A}$, by the universal property of the localization. For degree reasons, graded maps $\mathcal{A} \to \mathcal{O}_T[t]$ factor uniquely through the quotient $\mathcal{A} \to \bigoplus_{n\leq 0} \mathcal{A}_n/t^{n+1} \cdot \mathcal{A}_1$, so it follows that the restriction of ψ to $\mathcal{A}$ factors through $\mathcal{O}_T[t] \subset \mathcal{O}_T[t^{\pm}]$ if and only if $\psi(t \cdot \mathcal{A}_1) = 0$.

If ψ is the localization of a map $\tilde{\psi} : \mathcal{A} \to \mathcal{O}_T[t]$, then the restriction of ψ to $\mathcal{A}$ factors through $\mathcal{O}_T[t]$ and thus annihilates $t \cdot \mathcal{A}_1$. Conversely if $\psi(t \cdot \mathcal{A}_1) = 0$ then we have a map of algebras $\tilde{\psi} : \mathcal{A} \to \mathcal{O}_T[t]$ such that $\tilde{\psi}[t^{-1}]$ agrees with ψ after restricting to $\mathcal{A}$, and thus $\psi = \tilde{\psi}[t^{-1}]$. $\square$

Proof of Proposition 1.19. We first prove the claim when $n = 1$. Consider a map $T \to \mathrm{Filt}(\mathfrak{Y}) \times_{\mathfrak{Y}} \mathfrak{X}$. This corresponds to a map $f : T \times \Theta \to \mathfrak{Y}$, and a cosection of the sheaf of algebras $f^*\pi_*\mathcal{O}_{\mathfrak{X}}$ over $T \times (\mathbb{A}^1 - \{0\}) \times T/\mathbb{G}_m$. Under the identification between quasicoherent sheaves of $T \times \Theta$ and graded

$\mathcal{O}_T[t]$-modules Proposition 1.1, we can identify $f^*\pi_*\mathcal{O}_{\mathfrak{X}}$ with a graded $\mathcal{O}_T[t]$-algebra, $\mathcal{A}$, and the section is the same as a map of $\mathcal{O}_T[t^{\pm}]$-algebras $\psi : \mathcal{A}[t^{-1}] \to \mathcal{O}_T[t^{\pm}]$.

On the other hand, the set of lifts of $T \to \mathrm{Filt}(\mathfrak{Y}) \times_{\mathfrak{Y}} \mathfrak{X}$ to $\mathfrak{X}$ corresponds bijectively to the set of maps $\tilde{\psi} : \mathcal{A}[t] \to \mathcal{O}_T[t]$ for which ψ is the localization. By the previous lemma this is unique and exists if and only if the ideal $\psi(t \cdot \mathcal{A}_1) \subset \mathcal{O}_T$ vanishes. Because the formation of this ideal is compatible with pulling back along a map $(g, \mathrm{id}_\Theta) : T' \times \Theta \to T \times \Theta$, it follows that the fiber product $T \times_{\mathrm{Filt}(\mathfrak{Y}) \times_{\mathfrak{Y}} \mathfrak{X}} \mathrm{Filt}(\mathfrak{X})$ is represented by the closed subscheme of T defined by this ideal.

We have shown the morphism $\mathrm{Filt}(\mathfrak{X}) \to \mathrm{Filt}(\mathfrak{Y}) \times_{\mathfrak{Y}} \mathfrak{X}$ is a closed immersion and hence affine. It follows that the composition

$$\mathrm{Filt}(\mathfrak{X}) \to \mathrm{Filt}(\mathfrak{Y}) \times_{\mathfrak{Y}} \mathfrak{X} \to \mathrm{Filt}(\mathfrak{Y})$$

is affine as well, and it is finite if the map π is finite. Furthermore if π is finite, then the valuative criterion for properness for the map π implies that $\mathrm{Filt}(\mathfrak{X}) \to \mathrm{Filt}(\mathfrak{Y}) \times_{\mathfrak{Y}} \mathfrak{X}$ is surjective, because the fiber for any k-point of $\mathrm{Filt}(\mathfrak{Y}) \times_{\mathfrak{Y}} \mathfrak{X}$ consists of the set of lifts of the map $\Theta_k \to \mathfrak{Y}$ to a map $\Theta_k \to \mathfrak{X}$ given a lift at the generic point. We can identify $\mathrm{Filt}^n(\mathfrak{X}) \simeq \mathrm{Filt}(\mathrm{Filt}^{n-1}(\mathfrak{X}))$, so the morphism $\mathrm{Filt}^n(\mathfrak{X}) \to \mathrm{Filt}^n(\mathfrak{Y})$ is affine (respectively finite) for any $n \geq 1$ by induction.

Finally, we prove that $\mathrm{Filt}^n(\mathfrak{X}) \to \mathrm{Filt}^n(\mathfrak{Y}) \times_{\mathfrak{Y}} \mathfrak{X}$ is a closed immersion for $n > 1$. Again using the identification of $\mathrm{Filt}^n(\mathfrak{X})$ as an iterated mapping stack, we can factor the map $\mathrm{ev}_1 : \mathrm{Filt}^n(\mathfrak{X}) \to \mathfrak{X}$ into a sequence of maps $\mathrm{Filt}^n(\mathfrak{X}) \to \mathrm{Filt}^{n-1}(\mathfrak{X}) \to \cdots \to \mathfrak{X}$. We consider the following commutative diagram in which every square is Cartesian

$$\begin{tikzcd}
\mathrm{Filt}^n(\mathfrak{X}) \arrow[r, dotted, "\text{cl.imm.}"'] & \mathfrak{Z}_{n,2} \arrow[r, "\text{cl.imm.}"'] \arrow[d, dotted] & \mathfrak{Z}_{n,1} \arrow[r, "\text{cl.imm.}"'] \arrow[d, dotted] & \mathrm{Filt}^n(\mathfrak{Y}) \times_{\mathfrak{Y}} \mathfrak{X} \arrow[r, "\text{affine}"'] \arrow[d, dotted] & \mathrm{Filt}^n(\mathfrak{Y}) \arrow[d, dotted] \\
& \mathrm{Filt}^2(\mathfrak{X}) \arrow[r, "\text{cl.imm.}"'] & \mathfrak{Z}_{2,1} \arrow[r, "\text{cl.imm.}"'] \arrow[d] & \mathrm{Filt}^2(\mathfrak{Y}) \times_{\mathfrak{Y}} \mathfrak{X} \arrow[r, "\text{affine}"'] \arrow[d] & \mathrm{Filt}^2(\mathfrak{Y}) \arrow[d] \\
& & \mathrm{Filt}(\mathfrak{X}) \arrow[r, "\text{cl.imm.}"'] & \mathrm{Filt}(\mathfrak{Y}) \times_{\mathfrak{Y}} \mathfrak{X} \arrow[r, "\text{affine}"'] \arrow[d] & \mathrm{Filt}(\mathfrak{Y}) \arrow[d] \\
& & & \mathfrak{X} \arrow[r, "\pi", "\text{affine}"'] & \mathfrak{Y}
\end{tikzcd}$$

In each row, we know that the left-most arrow is a closed immersion because the composition of the maps in the row below is affine and we have already shown the claim when $n = 1$. The claim thus follows for all n by induction. The same argument shows that $\mathrm{Filt}^n(\mathfrak{X}) \to \mathrm{Filt}^n(\mathfrak{Y}) \times_{\mathfrak{Y}} \mathfrak{X}$ is a surjective closed immersion if π is finite, by replacing the label "affine" with "finite" and replacing the label "closed immersion" with "surjective closed immersion" in the diagram above. □

Corollary 1.20.1. *Let $\pi : \mathfrak{Y} \to \mathfrak{X}$ be a finite map of stacks which both satisfy* (†)*. Then for any B-algebra R and $\xi : \mathrm{Spec}(R) \to \mathfrak{Y}$ the canonical map $\mathrm{Flag}^n(\xi) \to \mathrm{Flag}^n(\pi \circ \xi)$ is a surjective closed immersion (and hence a universal homeomorphism) of algebraic spaces and an isomorphism if π is a closed immersion.*

Proof. The case of a closed immersion follows immediately from part (2) of Proposition 1.18, which implies that in the diagram

$$\begin{tikzcd} \mathrm{Flag}^n(\xi) \arrow[r] \arrow[d] & \mathrm{Filt}^n(\mathfrak{Y}) \arrow[r] \arrow[d, "\mathrm{ev}_1"] & \mathrm{Filt}^n(\mathfrak{X}) \arrow[d, "\mathrm{ev}_1"] \\ \mathrm{Spec}(R) \arrow[r, "\xi"] & \mathfrak{Y} \arrow[r, "\pi"] & \mathfrak{X} \end{tikzcd},$$

the right square is cartesian and thus so is the outermost rectangle. For the more general case where π is finite, one argues not that the squares are cartesian, but that the canonical map from the left corner to the fiber product is a surjective closed immersion, using Proposition 1.19. A diagram chase shows that for this property of 2-commutative squares, if the rightmost square has the property, then the left square has the property if and only if the combined square does. □

1.3.1. *Deformation retract of* $\mathrm{Filt}^n(\mathfrak{X})$*.* We shall say that a Θ-*deformation retract of a stack $\mathfrak{Y}$ onto a stack $\mathfrak{Z}$* consists of morphisms $\pi : \mathfrak{Y} \to \mathfrak{Z}$, $\sigma : \mathfrak{Z} \to \mathfrak{Y}$, and $r : \Theta \times \mathfrak{Y} \to \mathfrak{Y}$ such that: 1) $\pi \circ \sigma \simeq \mathrm{id}_{\mathfrak{Z}}$, 2) $r|_{\{1\}\times\mathfrak{Y}} \simeq \mathrm{id}_{\mathfrak{Y}}$, 3) $r|_{\{0\}\times\mathfrak{Y}} \simeq \sigma \circ \pi$, and 4) the two compositions in the square

$$\begin{tikzcd} \Theta \times \mathfrak{Y} \arrow[r, "r"] \arrow[d, "\mathrm{project}_{\mathfrak{Y}}"'] & \mathfrak{Y} \arrow[d, "\pi"] \\ \mathfrak{Y} \arrow[r, "\pi"] & \mathfrak{Z} \end{tikzcd}$$

are isomorphic. We shall see in Theorem 1.37 that $\sigma : \mathfrak{Z} \to \mathfrak{Y}$ need not be a monomorphism. Note that restricting r to a map $\mathbb{A}^1 \times \mathfrak{Y} \to \mathfrak{Y}$ gives an $\mathbb{A}^1$-deformation retract, so a Θ-deformation retract is a stronger notion.

Lemma 1.21. *Given a Θ-deformation retract of an algebraic stack $\mathfrak{Y}$ onto $\mathfrak{Z}$, if $\mathfrak{U} \subset \mathfrak{Y}$ is an open substack such that $\sigma^{-1}(\mathfrak{U}) = \mathfrak{Z}$, then $\mathfrak{U} = \mathfrak{Y}$.*

Proof. Consider the open substack $r^{-1}(\mathfrak{U}) \subset \Theta \times \mathfrak{Y}$. Because the composition $\{0\} \times \mathfrak{Y} \to \Theta \times \mathfrak{Y} \to \mathfrak{Y}$ factors through σ and $\sigma^{-1}(\mathfrak{U}) = \mathfrak{Z}$, we have

$$(\{0\}/\mathbb{G}_m) \times \mathfrak{Y} \subset r^{-1}(\mathfrak{U}).$$

We choose an atlas $Y \to \mathfrak{Y}$ and note that the restriction of $r^{-1}(\mathfrak{U})$ along the map $\Theta \times Y \to \Theta \times \mathfrak{Y}$ is an open substack of $\Theta \times Y$ which contains $(\{0\}/\mathbb{G}_m) \times Y$. Such an open substack is all of $\Theta \times Y$ by necessity, so $r^{-1}(\mathfrak{U}) = \Theta \times \mathfrak{Y}$. This implies that the restriction $\mathrm{id}_{\mathfrak{Y}} \simeq r|_{\{1\}\times\mathfrak{Y}} : \mathfrak{Y} \to \mathfrak{Y}$ factors through $\mathfrak{U}$, hence $\mathfrak{U} = \mathfrak{Y}$. □

Lemma 1.22. *Given a Θ-deformation retract of an algebraic stack $\mathfrak{Y}$ onto $\mathfrak{Z}$, if $\pi : \mathfrak{Y} \to \mathfrak{Z}$ is locally finitely presented then it is quasi-compact.*

Proof. Let S be a quasi-compact B-scheme and consider a map $\xi : S \to \mathfrak{Z}$, then the stack $\mathfrak{Y}' := \mathfrak{Y} \times_{\pi,\mathfrak{Z},\xi} S$ is locally finitely presented over S. The definition of a Θ-deformation retract is preserved under base change along a map to $\mathfrak{Z}$, so $\mathfrak{Y}'$ admits a Θ-deformation retract onto S. We must show that $\mathfrak{Y}'$ is quasi-compact. Because S is quasi-compact one can find a smooth map from a quasi-compact scheme $Y \to \mathfrak{Y}'$ such that $\sigma : S \to \mathfrak{Y}'$ factors through the image of $Y \to \mathfrak{Y}'$. Lemma 1.21 now implies that the open substack $\mathrm{im}(Y \to \mathfrak{Y}') \subset \mathfrak{Y}'$ is all of $\mathfrak{Y}'$, so $Y \to \mathfrak{Y}'$ is surjective and hence $\mathfrak{Y}'$ is quasi-compact. $\square$

Lemma 1.23. *Given a Θ-deformation retract of an algebraic stack $\mathfrak{Y}$ onto $\mathfrak{Z}$, π induces a bijection on connected components with inverse given by σ.*

Proof. Consider a map of stacks $\varphi : \mathfrak{Y} \to \mathfrak{Y}$ which is $\mathbb{A}^1$-homotopic to the identity in the sense that there is a map $r : \mathbb{A}^1 \times \mathfrak{Y} \to \mathfrak{Y}$ with $\{1\} \times \mathfrak{Y} \to \mathfrak{Y}$ isomorphic to the identity and $\{0\} \times \mathfrak{Y} \to \mathfrak{Y}$ isomorphic to φ. Then resicting r to $\mathbb{A}^1 \times \mathrm{Spec}(k)$ for any k-point of $\mathfrak{Y}$ shows that any point of $\mathfrak{Y}$ lies on the same connected component as some point in the image of φ, so $\varphi : \pi_0(\mathfrak{Y}) \to \pi_0(\mathfrak{Y})$ is surjective. More precisely, any $p \in |\mathfrak{Y}|$ lies in the same connected component as $\varphi(p)$, which shows that if $p, q \in |\mathfrak{Y}|$ are such that $\varphi(p)$ and $\varphi(q)$ lie on the same connected component, then p and q lie on the same connected component as well. This shows that $\varphi : \pi_0(\mathfrak{Y}) \to \pi_0(\mathfrak{Y})$ is in fact bijective. The statement of the lemma now follows from this general fact applied to $\varphi = \sigma \circ \pi$ and from the fact that $\pi \circ \sigma \simeq \mathrm{id}$ and is thus also bijective on connected components. $\square$

We now apply these lemmas in our situation. Consider the map $\Theta \times \Theta^n \to \Theta^n$ induced by the scalar multiplication map $\mathbb{A}^1 \times \mathbb{A}^n \to \mathbb{A}^n$ taking $(t, v) \mapsto tv$, which is equivariant with respect to the group homomorphism $\mathbb{G}_m \times \mathbb{G}_m^n \to \mathbb{G}_m^n$ taking $(t, z_1, \ldots, z_n) \mapsto (tz_1, \ldots, tz_n)$. This defines a map

$$r : \Theta \times \underline{\mathrm{Map}}(\Theta^n, \mathfrak{X}) \to \underline{\mathrm{Map}}(\Theta^n, \mathfrak{X}) \tag{6}$$

which takes any S-point, corresponding to a pair $(a : S \to \Theta, b : \Theta^n_S \to \mathfrak{X})$, to the S-point of $\underline{\mathrm{Map}}(\Theta^n, \mathfrak{X})$ corresponding to the composition

$$\Theta^n_S \xrightarrow{(a,\mathrm{id})} \Theta \times \Theta^n_S \xrightarrow{\substack{\text{scalar} \\ \text{multiplication}}} \Theta^n_S \to \mathfrak{X}.$$

Lemma 1.24. *The maps $\pi = \mathrm{ev}_0$, σ, and the map r from (6) define a Θ-deformation retract of the stack $\mathrm{Filt}^n(\mathfrak{X})$ onto $\mathrm{Grad}^n(\mathfrak{X})$. Hence ev_0 is quasi-compact and induces a bijection on connected components whose inverse is given by σ.*

Proof. This is an immediate consequence of the definition of r on S-points. If $a = 1$ is the constant map $S \to \Theta$, then the composition $\Theta^n_S \xrightarrow{(a,\mathrm{id})} \Theta \times \Theta^n_S \to \Theta^n_S$ is isomorphic to the identity. If $a = 0$ is the constant map, then this composition is isomorphic to the map $\Theta^n_S \to \Theta^n_S$ induced by the $\mathbb{G}_m^n$-equivariant map $\mathbb{A}^n \mapsto \{0\} \subset \mathbb{A}^n$. The fact that the map r of (6) commutes with the projection $\mathrm{ev}_0 : \mathrm{Filt}^n(\mathfrak{X}) \to \mathrm{Grad}^n(\mathfrak{X})$ follows from the fact that for any $a : S \to \Theta$ the composition

$$(\{0\}/\mathbb{G}_m^n)_S \to \Theta_S^n \to \Theta \times \Theta_S^n \to \Theta_S^n$$

is canonically isomorphic to the inclusion $(\{0\}/\mathbb{G}_m^n)_S \hookrightarrow \Theta_S^n$, and this isomorphism is natural in S. ☐

1.3.2. *More on connected components of* $\mathrm{Filt}^n(\mathfrak{X})$. Given a k-point of $\mathrm{Filt}^n(\mathfrak{X})$, corresponding to a map $f : \Theta_k^n \to \mathfrak{X}$, one can consider the "inertia" k-subgroup

$$I_f := \ker((\mathbb{G}_m^n)_k \to \mathrm{Aut}(f(0))). \tag{7}$$

We use the notation $D_S(A)$ for the diagonalizable S-group scheme associated to a finitely generated abelian group A as in [C, Appendix B]. $I_f \subset (\mathbb{G}_m^n)_k$ is an fppf sub-group-scheme and is thus a diagonalizable group of the form $D_k(\mathbb{Z}^n/N_f)$ for some sub-group $N_f \subset \mathbb{Z}^n$ [C, Corollary B.3.3, Proposition B.3.4]. We can use this observation to separate connected components of $\mathrm{Filt}^n(\mathfrak{X})$.

Proposition 1.25. *The map $f \in \mathrm{Filt}^n(\mathfrak{X})(k) \mapsto N_f \subset \mathbb{Z}^n$ is locally constant on $\mathrm{Filt}^n(\mathfrak{X})$.*

Proof. By construction the map $f \mapsto N_f$ factors throught the projection $\mathrm{ev}_0 : \mathrm{Filt}^n(\mathfrak{X}) \to \mathrm{Grad}^n(\mathfrak{X})$, so it suffices to show that this function is locally constant on $\mathrm{Grad}^n(\mathfrak{X})$. It suffices to consider an integral B-scheme of finite type S along with a family of graded objects $f : (\mathrm{pt}/\mathbb{G}_m^n)_S \to \mathfrak{X}$. Consider the analogous subgroup

$$I_f := \ker((\mathbb{G}_m^n)_S \to \mathrm{Aut}(f_S)) \subset (\mathbb{G}_m^n)_S.$$

As a kernel of a homomorphism to a separated group scheme, it is closed, and its restriction to each point of S is the subgroup (7). Let $1 \in S(K)$ denote the generic point and $0 \in S(k)$ any other point, and let $N_0, N_1 \subset \mathbb{Z}^n$ be the corresponding subgroups such that $I_f|_{\mathrm{Spec}(K)} = D_K(\mathbb{Z}^n/N_1)$ and $I_f|_{\mathrm{Spec}(k)} = D_k(\mathbb{Z}^n/N_0)$. The proposition amounts to the claim that $N_0 = N_1$.

Proof that $N_0 \subset N_1$:

We can find an open subscheme $U \subset S$ for which $I_f|_U$ is flat and hence of multiplicative type [C, Corollary B.3.3], and in fact $I_f|_U \simeq D_U(\mathbb{Z}^n/N_1) \subset D_U(\mathbb{Z}^n) = (\mathbb{G}_m^n)_U$ by [C, Proposition B.3.4]. Because S is integral it is the scheme theoretic closure of U, and it follows that $D_S(\mathbb{Z}^n/N_1)$ is the scheme theoretic closure of the open subscheme $D_U(\mathbb{Z}^n/N_1) \subset D_S(\mathbb{Z}^n/N_1)$ because $D_S(\mathbb{Z}^n/N_1) \to S$ is flat and affine. As a consequence, $D_S(\mathbb{Z}^n/N_1)$ is the scheme theoretic closure of $D_U(\mathbb{Z}^n/N_1)$ in $(\mathbb{G}_m^n)_S$ and hence $D_S(\mathbb{Z}^n/N_1) \subset I_f$. In particular $D_k(\mathbb{Z}^n/N_1) \subset I_f|_{\mathrm{Spec}(k)}$, which establishes that $N_0 \subset N_1$.

Proof that $N_1 \subset N_0$:

The argument is tannakian. First note that for $N \subset \mathbb{Z}^n$, $D_k(N)$ is again a torus, and we have a map $(\mathrm{pt}/\mathbb{G}_m^n)_k \to BD_k(N)$ whose fiber is $BD_k(\mathbb{Z}^n/N)$.

Lemma 1.6 implies that a graded object $f : (\mathrm{pt}/\mathbb{G}_m^n)_k \to \mathfrak{X}$ factors through the map to $BD_k(N)$ if and ony if $D_k(\mathbb{Z}^n/N) \subset \ker((\mathbb{G}_m^n)_k \to \mathrm{Aut}(f))$, and the factorization is unique in this case. Thus N_f can be characterized as the largest subgroup of $\mathbb{Z}^n$ for which f factors through $(\mathrm{pt}/\mathbb{G}_m^n)_k \to BD_k(N)$.

Given a complex $F \in \mathrm{APerf}((\mathrm{pt}/\mathbb{G}_m^n)_k)$ we let F_χ denote the direct summand of F on which $\mathbb{G}_m^n$ acts with the character $\chi \in \mathbb{Z}^n$. For $N \subset \mathbb{Z}^n$, consider the map $p : (\mathrm{pt}/\mathbb{G}_m^n)_k \to BD_k(N)$. The pullback functor $p^* : \mathrm{APerf}(BD_k(N)) \to \mathrm{APerf}((\mathrm{pt}/\mathbb{G}_m^n)_k)$ is fully faithful, with essential image consisting of complexes F for which $F_\chi = 0$ for $\chi \notin N$. Tannaka duality implies that f factors through $(\mathrm{pt}/\mathbb{G}_m^n)_k \to BD_k(N)$ if and only if the pullback functor $f^* : \mathrm{APerf}(\mathfrak{X}) \to \mathrm{APerf}((\mathrm{pt}/\mathbb{G}_m^n)_k)$ lies in this subcategory, or equivalently

$$\bigcup_{E \in \mathrm{APerf}(\mathfrak{X})^{\leq 0}} \{\chi \in \mathbb{Z}^n \text{ s.t. } (f^*E)_\chi \neq 0\} \subset N.$$

Under the characterization of N_f above, we see that N_f is the subgroup generated by those $\chi \in \mathbb{Z}^n$ for which $\exists E \in \mathrm{APerf}(\mathfrak{X})^{\leq 0}$ with $(f^*E)_\chi \neq 0$.

Now let $f_0 : (\mathrm{pt}/\mathbb{G}_m^n)_k \to \mathfrak{X}$ and let $f_1 : (\mathrm{pt}/\mathbb{G}_m^n)_K \to \mathfrak{X}$ be the corresponding maps. Choose a finite generating set $\chi_1, \ldots, \chi_k \in N_1$ for which there exists $E_1, \ldots, E_k \in \mathrm{APerf}(\mathfrak{X})^{\leq 0}$ for which $(f_1^* E_i)_{\chi_i} \neq 0$. Nakayama's lemma applied to $(f^*E_i)_{\chi_i} \in \mathrm{APerf}(S)$ implies that $(f_0^*E_i)_{\chi_i} \neq 0$ for all i as well, and in particular $\chi_i \in N_0$. This shows that $N_1 \subset N_0$. □

1.3.3. *The action of* $\mathbb{N}^\times$ *on* $\mathrm{Filt}(\mathfrak{X})$. We define an action of $\mathbb{N}^\times$ on Θ canonically commuting with the inclusion of the point 1. For each $n \in \mathbb{N}$, the morphism $(\bullet)^n : \Theta \to \Theta$ is defined by the map $\mathbb{A}^1 \to \mathbb{A}^1$ given by $z \mapsto z^n$, which is equivariant with respect to the group homomorphism $\mathbb{G}_m \to \mathbb{G}_m$ given by the same formula. Given a morphism $f : \Theta_S \to \mathfrak{X}$, we let f^n denote the precomposition of f with $(\bullet)^n$, and we also use $(\bullet)^n$ to denote the pre-composition morphism $\mathrm{Filt}(\mathfrak{X}) \to \mathrm{Filt}(\mathfrak{X})$. This defines an action of the monoid $\mathbb{N}^\times$ on $\mathrm{Filt}(\mathfrak{X})$ for which the morphism $\mathrm{ev}_1 : \mathrm{Filt}(\mathfrak{X}) \to \mathfrak{X}$ is canonically invariant.

Remark 1.26. It follows from the fact that ev_1 is canonically $\mathbb{N}^\times$-invariant that for any $\xi : T \to \mathfrak{X}$ the $\mathbb{N}^\times$-action induces an action on $\mathrm{Flag}(\xi)$.

Proposition 1.27. *If $\mathfrak{X}$ is a quasi-geometric stack and $n > 0$, then the map $(\bullet)^n : \mathrm{Filt}(\mathfrak{X}) \to \mathrm{Filt}(\mathfrak{X})$ is a monomorphism of stacks. If furthermore $\mathfrak{X}$ satisfies (†), then $(\bullet)^n$ is both an open and a closed immersion whose image consists of maps $f : \Theta_k \to \mathfrak{X}$ for which the subgroup I_f of (7) contains $(\mu_n)_k$.*

Proof. Let us denote the morphism $(\bullet)^n : \Theta \to \Theta$ as p. Observe that $Rp_*\mathcal{O}_\Theta \simeq \mathcal{O}_\Theta$, as can be observed after base change to $\mathbb{A}^1$, where we can identify p with the coarse moduli space map $\mathbb{A}^1/\mu_n \to \mathbb{A}^1$. It follows from the projection formula and the adjunction between Lp^* and Rp_* that $p^* : \mathrm{D}_{qc}(\Theta) \to \mathrm{D}_{qc}(\Theta)$ is fully faithful. So the image of p^* is a full symmetric monoidal ∞-category, and the same can be said for $\mathrm{Perf}(\Theta)$, $\mathrm{APerf}(\Theta)$, and

$\mathrm{D}_{qc}(\Theta)^{\leq 0}$. By Tannaka duality [L2, Proposition 3.3.11], a map $f : \Theta \to \mathfrak{X}$ is uniquely determined by the corresponding functor on symmetric monoidal ∞-categories f^*, hence p is an epimorphism in the category of quasi-geometric stacks, and as a result the corresponding morphism $(\bullet)^n : \mathrm{Filt}(\mathfrak{X}) \to \mathrm{Filt}(\mathfrak{X})$ is a monomorphism in the category of quasi-geometric stacks.

Now let $\mathfrak{X}$ satisfy (†) and consider a finite type B-scheme S and a map $f : \Theta_S \to \mathfrak{X}$. For any $p \in S(k)$, let f_p denote the restriction $f|_{\Theta_k} : \Theta_k \to \mathfrak{X}$. Proposition 1.25 implies that the set of $p \in |S|$ for which $(\mu_n)_k \subset I_{f_p}$ is both open and closed. In particular there is a unique maximal open subscheme $U \subset S$ for which $f|_U : \Theta_U \to \mathfrak{X}$ factors through $(\bullet)^n : \Theta_U \to \Theta_U$ by Lemma 1.28 below, and the factorization of $f|_U$ is unique up to unique isomorphism by the previous paragraph. This implies that $(\bullet)^n : \mathrm{Filt}(\mathfrak{X}) \to \mathrm{Filt}(\mathfrak{X})$ is an open immersion. However we have seen that the image, which is the set of points for which $(\mu_n)_k \subset I_{f_p}$ is closed as well, so $(\bullet)^n$ is an open immersion of locally Noetherian stacks whose image is closed, and is thus a closed immersion as well. ☐

We have used the following lemma in the previous proof.

Lemma 1.28. *Let $\mathfrak{X}$ satisfy (†), and consider a map $f : \Theta_S \to \mathfrak{X}$, where S is a Noetherian B-scheme. Then the following are equivalent:*

(1) f factors through $(\bullet)^n : \Theta_S \to \Theta_S$,
(2) $(\mu_n)_S \subset I_f := \ker((\mathbb{G}_m)_S \to \mathrm{Aut}(f|_{S\times\{0\}}))$,
(3) for any $E \in \mathrm{APerf}(\mathfrak{X})$ and $\chi \in \mathbb{Z} \setminus n\mathbb{Z}$, we have $((f^*E)_{(\{0\}/\mathbb{G}_m)_S})_\chi = 0$, and
(4) for any $E \in \mathrm{APerf}(\mathfrak{X})$, $p \in S(k)$, and $\chi \notin n\mathbb{Z}$, we have

$$((f^*E)_{p\times\{0\}/\mathbb{G}_m})_\chi = 0.$$

In this case the factorization through $(\bullet)^n$ is unique up to unique isomorphism.

Proof. First, note that (3) and (4) are equivalent by Nakayama's lemma. The implication (1) ⇒ (2) follows from the fact that $(\mu_n)_S$ is the kernel of the map on automorphism groups induced by the restriction of $(\bullet)^n$ to $(\mathrm{pt}/\mathbb{G}_m)_S$. The implication (2) ⇒ (3) is a consequence of the effect that pullback along $(\bullet)^n : (\mathrm{pt}/\mathbb{G}_m)_S \to (\mathrm{pt}/\mathbb{G}_m)_S$ has on complexes $E \in \mathrm{APerf}((\mathrm{pt}/\mathbb{G}_m)_S)$, it simply rescales the non-vanishing weights the complex by n.

Finally, the implication (3) ⇒ (1) is a consequence of Tannaka duality, as has already been discussed in the proof above: Because $\mathfrak{X}$ is locally Noetherian, the ∞-category of maps $f : \Theta_S \to \mathfrak{X}$ is equivalent to the ∞-category of colimit preserving symmetric monoidal functors $\mathrm{APerf}(\mathfrak{X})^{\leq 0} \to \mathrm{APerf}(\Theta)^{\leq 0}$ [BHL, Theorem 5.1, Lemma 3.13]. The essential image of the fully faithful pullback functor $\mathrm{APerf}(\Theta_S)^{\leq 0} \to \mathrm{APerf}(\Theta_S)^{\leq 0}$ for the map $(\bullet)^n : \Theta_S \to \Theta_S$ consists exactly of almost perfect complexes F for which the homology of the restriction $F|_{S\times\{0\}/\mathbb{G}_m}$ vanishes in any weight which is *not* divisible by n. ☐

The following is an immediate corollary of Proposition 1.25 and Proposition 1.27, which we record for later use.

Corollary 1.28.1. *Let $\mathfrak{X}$ satisfy (†), and let $n > 0$. Then the morphism $(\bullet)^n : \mathrm{Filt}(\mathfrak{X}) \to \mathrm{Filt}(\mathfrak{X})$ induces an isomorphism between connected components, acts injectively on the set of connected components, and only fixes those components for which the subgroup (7) is all of $\mathbb{G}_m$.*

1.3.4. *Change of base lemmas.*

Lemma 1.29. *If $\mathfrak{X}$ is a stack satisfying (†) which has quasi-finite inertia relative to B, then $\mathrm{ev}_1 : \mathrm{Filt}^n(\mathfrak{X}) \to \mathfrak{X}$ is an equivalence, as are the maps $\mathrm{ev}_0 : \mathrm{Filt}^n(\mathfrak{X}) \to \mathrm{Grad}^n(\mathfrak{X})$ and the forgetful map $u : \mathrm{Grad}^n(\mathfrak{X}) \to \mathfrak{X}$.*

Proof. Consider a map $f : \Theta^n_S \to \mathfrak{X}_S$ relative to S. Then the induced group homomorphism $(\mathbb{G}_m^n)_S \to \underline{\mathrm{Aut}}_{\mathfrak{X}/B}(f|_{\{0\}\times S})$ is trivial on geometric fibers, because $\mathfrak{X}$ has quasi-finite inertia and $\mathbb{G}_m$ is geometrically connected. It follows that $f^* : \mathrm{APerf}(\mathfrak{X})^{\leq 0} \to \mathrm{APerf}(\Theta^n_S)^{\leq 0}$ factors through the full symmetric monoidal subcategory consisting of complexes for which $E|_{\{0\}\times S}$ is acyclic in non-zero weights with respect to $\mathbb{G}_m^n$. This subcategory is the essential image of the fully faithful pullback functor along $\Theta^n_S \to S$, so f factors uniquely through the projection $\Theta^n_S \to S$. Futhermore, the pullback along $\Theta^n_S \to S$ followed by restriction along $\{1\} \times S \to \Theta^n_S$ is equivalent to the identity functor on $\mathrm{APerf}(S)^{\leq 0}$, so the map f is uniquely determined up to unique isomorphism by its restiction to $\{1\} \times S$. The arguments for the maps u and ev_0 are similar. □

Remark 1.30. Not the embedding $I_{\mathfrak{X}/B} \hookrightarrow I_{\mathfrak{X}} := I_{\mathfrak{X}/\operatorname{Spec}(\mathbb{Z})}$, so if $\mathfrak{X}$ has quasi-finite absolute inertia, for instance if it is a scheme, then the condition in the lemma, that $I_{\mathfrak{X}/B} \to \mathfrak{X}$ is quasi-finite, holds as well.

Corollary 1.30.1. *Consider a cartesian diagram of stacks satisfying (†)*

$$\begin{tikzcd} \mathfrak{X}' \arrow[r] \arrow[d] & \mathfrak{Y}' \arrow[d] \\ \mathfrak{X} \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

in which $\mathfrak{Y}$ and $\mathfrak{Y}'$ have quasi-finite inertia relative to B. Then $\mathrm{Filt}^n(\mathfrak{X}') \simeq \mathrm{Filt}^n(\mathfrak{X}) \times_{\mathfrak{X}} \mathfrak{X}'$, where the fiber product is taken with respect to $\mathrm{ev}_1 : \mathrm{Filt}^n(\mathfrak{X}) \to \mathfrak{X}$, and $\mathrm{Grad}^n(\mathfrak{X}') \simeq \mathrm{Grad}^n(\mathfrak{X}) \times_{\mathfrak{X}} \mathfrak{X}'$, where the fiber product is take with respect to $u : \mathrm{Grad}^n(\mathfrak{X}) \to \mathfrak{X}$.

Proof. For any B-scheme S, the functor $\mathrm{Map}(\Theta^n_S, -)$ commutes with homotopy limits, so the diagram

$$\begin{tikzcd} \underline{\mathrm{Map}}(\Theta^n, \mathfrak{X}') \arrow[r] \arrow[d] & \underline{\mathrm{Map}}(\Theta^n, \mathfrak{Y}') \arrow[d] \\ \underline{\mathrm{Map}}(\Theta^n, \mathfrak{X}) \arrow[r] & \underline{\mathrm{Map}}(\Theta^n, \mathfrak{Y}) \end{tikzcd}$$

is automatically cartesian. Lemma 1.29 identifies $\underline{\mathrm{Map}}(\Theta^n, \mathfrak{Y}')$ and $\underline{\mathrm{Map}}(\Theta^n, \mathfrak{Y})$ with $\mathfrak{Y}'$ and $\mathfrak{Y}$ respectively, via the maps ev_1. Therefore the right square and outermost square in the commutative diagram

$$\begin{tikzcd} \mathrm{Filt}^n(\mathfrak{X}') \arrow[r, "\mathrm{ev}_1"] \arrow[d] & \mathfrak{X}' \arrow[r] \arrow[d] & \mathfrak{Y}' \arrow[d] \\ \mathrm{Filt}^n(\mathfrak{X}) \arrow[r, "\mathrm{ev}_1"] & \mathfrak{X} \arrow[r] & \mathfrak{Y} \end{tikzcd},$$

are cartesian, and it follows that the left square is cartesian as well. The argument for $\mathrm{Grad}^n(\mathfrak{X}')$ is similar. □

1.4. **Graded and filtered points in global quotient stacks.** Here we compute the stacks of filtered and graded objects first for stacks of the form X/GL_N in Theorem 1.36 and then for quotients by split reductive groups over a field in Theorem 1.37.

1.4.1. *Biaynicki-Birula type theorems.* First we recall some facts about concentration under the action of $\mathbb{G}_m^n$.

Proposition 1.31 ([H2],[DG])**.** *Let $X \to B$ be a quasi-separated locally finite type map of algebraic spaces, where B can be covered by G-rings, and let $\mathbb{G}_m^n$ act on X so that the structure map $X \to B$ is $\mathbb{G}_m^n$-invariant. Then the functor*

$$\Phi_X(T) = \{\mathbb{G}_m^n\text{-}\textit{equivariant maps } \mathbb{A}^n \times T \to X \textit{ over } B\}$$

is representable by a locally finite type algebraic space Y over B.

Proof. When X is a scheme admitting a $\mathbb{G}_m$-equivariant affine open cover, then this is a special case of the main theorem of section 4 of [H2] for which the "center" is $C = X$ and the "speed" is $m = 1$. The general statement of the existence of Y when B is a field and $n = 1$ is the main result of [DG].

The general case follows from Proposition 1.2 and Proposition 1.18. Let $B \to \mathrm{Filt}^n((\mathrm{pt}/\mathbb{G}_m^n)_B)$ be the point classifying the projection map $\Theta^n \to \mathrm{pt}/\mathbb{G}_m^n$. We claim that we have a cartesian diagram of functors

$$\begin{tikzcd} \Phi_X \arrow[r] \arrow[d] & \mathrm{Filt}^n(X/\mathbb{G}_m^n) \arrow[d] \\ B \arrow[r] & \mathrm{Filt}^n((\mathrm{pt}/\mathbb{G}_m^n)_B) \end{tikzcd}.$$

Indeed, S points of this fiber product correspond to sections of the pullback of the map $X/\mathbb{G}_m^n \to (\mathrm{pt}/\mathbb{G}_m^n)_B$ along the canonical projection map $\Theta^n \times S \to \mathrm{pt}/\mathbb{G}_m^n$, which is the same as $\mathbb{G}_m^n$-equivariant sections of the map $X \times \mathbb{A}^n \to \mathbb{A}^n$. Proposition 1.2 implies that $\mathrm{Filt}^n(X/\mathbb{G}_m^n)$ is an algebraic stack, and Proposition 1.18 implies that the map $\mathrm{Filt}^n(X/\mathbb{G}_m^n) \to \mathrm{Filt}^n(\mathrm{pt}/\mathbb{G}_m^n)$ is representable by an algebraic space, hence Φ_X is representable by an algebraic space. □

Remark 1.32. If X admits a $\mathbb{G}_m$-equivariant Zariski open cover by affine $\mathbb{G}_m$-schemes, which happens if X is a normal scheme by Sumihiro's theorem, then the explicit construction of [H2] shows that restriction to $\{0\} \times T \subset \mathbb{A}^1 \times T$ defines a morphism $\pi : Y \to X^{\mathbb{G}_m}$ which is affine. Furthermore, $\pi : Y \to X^{\mathbb{G}_m}$ has connected geometric fibers.

Corollary 1.32.1. *If X is separated, then restriction of a map $\mathbb{A}^1 \times T \to X$ to $\{1\} \times T \subset \mathbb{A}^1 \times T$ defines a monomorphism of functors, identifying $\Phi_X(T)$ with*

$$\left\{ f : T \to X \text{ s.t. } \mathbb{G}_m \times T \xrightarrow{t \cdot f(x)} X \text{ extends to } \mathbb{A}^1 \times T \right\} \subset \operatorname{Map}_B(T, X)$$

Hence the inclusion of functors defines a locally finite type monomorphism $j : Y \to X$, which is also unramified.

Proof. Restriction to $\{1\} \times T$ identifies the set of equivariant maps $\mathbb{G}_m \times T \to X$ with $\operatorname{Hom}(T, X)$. If the corresponding map extends to $\mathbb{A}^1 \times T$ it will be unique because X is separated. Likewise the uniqueness of the extension of $\mathbb{G}_m \times \mathbb{G}_m \times T \to X$ to $\mathbb{G}_m \times \mathbb{A}^1 \times T \to X$ guarantees the $\mathbb{G}_m$ equivariance of the extension $\mathbb{A}^1 \times T \to X$. The map is locally finite type because both Y and Y are, and unramified by [S2, Tag 05VH]. □

Remark 1.33. Y typically has several connected components, and if X admits an equivariant immersion $X \hookrightarrow \mathbb{P}^n$ for some linear action of $\mathbb{G}_m$ on $\mathbb{P}^n$, then Y is the disjoint union of the Biaynicki-Birula strata for the action of $\mathbb{G}_m$ on X and the map j is a local immersion when restricted to each connected component. In general this need not be the case: Let $\mathbb{G}_m$ act on $\mathbb{P}^1$ fixing two points $\{0\}$ and $\{\infty\}$. If X is the nodal curve obtained by identifying these two points, then $\mathbb{G}_m$ acts on X as well. In this case $Y = \mathbb{A}^1$ and j is the composition $\mathbb{A}^1 \to \mathbb{P}^1 \to X$. There is no neighborhood of $\{0\}$ in which j is a local immersion.

We observe the following strengthened version of the Biaynicki-Birula theorem, which follows from the discussion above and the results of Section 1.2:

Proposition 1.34. *Let $X \to X'$ be a smooth $\mathbb{G}_m^n$-equivariant map of B-spaces which both satisfy the conditions of Proposition 1.31. Then*

1. *At any point $p \in X^{\mathbb{G}_m^n}(k)$, the fiber of the relative cotangent complex $\mathbb{L}_{X/X',p}$ is canonically a representation of $(\mathbb{G}_m^n)_k$, and there are canonical isomorphisms*

$$\mathbb{L}_{X^{\mathbb{G}_m}/(X')^{\mathbb{G}_m},p} \simeq \mathbb{L}^{\text{weight}0}_{X/X',p} \quad \text{and} \quad \mathbb{L}_{Y/Y',p} \simeq \mathbb{L}^{\text{weight}\leq 0}_{X/X',p}.$$

   In particular $X^{\mathbb{G}_m} \to (X')^{\mathbb{G}_m}$ and $Y \to Y'$ are smooth.
2. *If $X \to B$ is furthermore a smooth schematic map, then the projection $Y \to X^{\mathbb{G}_m^n}$ is an étale locally trivial bundle of affine spaces with linear $\mathbb{G}_m^n$-action on the fibers.*

1.4.2. *Stacks of the form X/GL_N, where X is an algebraic space.* We must first establish some notational book keeping. We regard $\mathbb{G}_m^N \subset \mathrm{GL}_N$ as the subgroup of diagonal matrices. For any sequence of integers $(w_1, \ldots, w_N)$ we define a one-parameter subgroup $\lambda_w : \mathbb{G}_m \to \mathrm{GL}_N$ given by $\lambda_w(t) = \mathrm{diag}(t^{w_1}, t^{w_2}, \ldots, t^{w_N})$. More generally, homomorphisms $\mathrm{Hom}(\mathbb{G}_m^q, \mathbb{G}_m^N)$ correspond bijectively to q-tuples of one-parameter subgroups $(\lambda_1, \ldots, \lambda_q)$, or $N \times q$ matrices.

Given a $\psi \in \mathrm{Hom}(\mathbb{G}_m^n, \mathbb{G}_m^N)$, let X^ψ denote the fixed locus for the $\mathbb{G}_m^n$-action induced via the GL_N action on X and the homomorphism $\psi : \mathbb{G}_m^n \to \mathrm{GL}_N$. Let Y_ψ denote the *blade* corresponding to ψ, which we define to be the algebraic space represented by the functor in Proposition 1.31.

Remark 1.35. Note that X^ψ and Y_ψ typically have several connected components, and that the projection $Y_\psi \to X^\psi$ is a bijection on connected components. The term "blade" is sometimes used instead to refer to the connected components of Y_ψ.

Given $\psi \in \mathrm{Hom}(\mathbb{G}_m^n, \mathbb{G}_m^N)$, we let P_ψ denote the blade for the action of GL_n on itself by conjugation. Concretely, $P_\psi \subset G$ is the closed subgroup of block matrices whose entries have nonnegative weight under the action of $\mathbb{G}_m$ on $N \times N$ matrices

$$M \mapsto \psi(1, \ldots, 1, t, 1, \ldots, 1) M \psi(1, \ldots, 1, t, 1, \ldots, 1)^{-1}$$

where the position of t ranges over all N possible positions. When $n = 1$ this is a standard parabolic subgroup, and for $n > 1$ it is an intersection of n such parabolic subgroups. Likewise we define $L_\psi := (\mathrm{GL}_N)^\psi$ with respect to the conjugation action, i.e. the centralizer of ψ. It a closed subgroup of block diagonal matrices of a shape determined by ψ, and from the universal properties of the blade and fixed locus one has a canonical split surjective group homomorphisms $P_\psi \to L_\psi$.

Applying the blade construction to the group action map $G \times X \to X$, gives a map

$$P_\psi \times Y_\psi \simeq (G \times X)_\psi \to Y_\psi$$

which satisfies the axioms for a group action of P_ψ on Y_ψ. Finally, note that the symmetric group S_N acts on $\mathrm{Hom}(\mathbb{G}_m^n, \mathbb{G}_m^N)$ by conjugation by permutation matrices, and for $w \in S_N \subset \mathrm{GL}_N$, $w \cdot Y_\psi = Y_{w\psi w^{-1}}$ and $w P_\psi w^{-1} = P_{w\psi w^{-1}}$.

Theorem 1.36. *Let X be an algebraic space with an action of GL_n, and let $X \to B$ be a quasi-separated locally finite type GL_n-invariant map to an algebraic space B which can be covered by G-rings. There are canonical* *isomorphisms*

$$\mathrm{Filt}^n(\mathfrak{X}) \simeq \bigsqcup_{\psi \in \mathrm{Hom}(\mathbb{G}_m^n, \mathbb{G}_m^N)/S_N} Y_\psi / P_\psi \quad \textit{and}$$

$$\mathrm{Grad}^n(\mathfrak{X}) \simeq \bigsqcup_{\psi \in \mathrm{Hom}(\mathbb{G}_m^n, \mathbb{G}_m^N)/S_N} X^\psi / L_\psi$$

where the notation $\psi \in \mathrm{Hom}(\mathbb{G}_m^n, \mathbb{G}_m^N)/S_N$ *means that we choose a single representative for each* S_N*-orbit on the set of homomorphisms.*

We have the following description of the universal maps (1) in this case: ev_0 *corresponds to the projection* $Y_\psi \to X^\psi$*, which is equivariant with respect to the group homomorphism* $P_\psi \to L_\psi$*, and* ev_1 *corresponds to the map* $Y_\psi \to X$*, which is equivariant with respect to the inclusion of groups* $P_\psi \subset \mathrm{GL}_N$*.*

Proof. The case where $X = \mathrm{pt}$*:* The stack $\mathrm{Filt}^n(\mathrm{pt}/\mathrm{GL}_N)$ is the stack of equivariant vector bundles on the toric variety $\mathbb{A}^n$, and our description agrees with that Payne's description [P2]. The result states that for any $\psi \in \mathrm{Hom}(\mathbb{G}_m^n, \mathbb{G}_m^N)$, we associate the vector bundle $\mathcal{E}_\psi = \mathcal{O}_{\mathbb{A}^n}^{\oplus N}$ with an equivariant structure in which $\mathbb{G}_m^n$ acts in the fiber directions via the homomorphism ψ. The Rees construction, Proposition 1.1, identifies the category of equivariant vector bundles on $\mathbb{A}^n$ with $\mathbb{Z}^n$-weighted filtrations of the fiber at $1^n \in \mathbb{A}^n$. Under this identification, the automorphisms of $\mathcal{E}_\psi$ are precisely the automorphisms of the fiber which preserve this filtration, which is precisely the group P_ψ. Thus we have maps $\mathrm{pt}/P_\psi \to \mathrm{Filt}^n(\mathrm{pt}/\mathrm{GL}_N)$, and Payne's result says that each of these maps is an open and closed immersion. For completeness, we recall his argument at the end of this proof. The computation of $\mathrm{Grad}^n(\mathrm{pt}/\mathrm{GL}_N)$ is similar, but uses the identification of vector bundles on $\mathrm{pt}/\mathbb{G}_m^n$ with $\mathbb{Z}^n$-graded vector spaces, rather than the Rees construction.

The general case:

Consider the cartesian square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \mathcal{Y} \arrow[r] \arrow[d] & \mathrm{Filt}^n(\mathfrak{X}) \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathrm{pt}/P_\psi \arrow[r, "\mathcal{E}_\psi"] & \mathrm{Filt}^n(\mathrm{pt}/\mathrm{GL}_N) \end{tikzcd} .$$

We know from Proposition 1.18 that Y is actually representable by an algebraic space, and therefore $\mathcal{Y} = Y/P_\psi$ for some action of P_ψ defined by the existence of this cartesian square. Note that if $\mathrm{Frame}(\mathcal{E}_\psi)$ denotes the frame bundle, then $\mathrm{Frame}(\mathcal{E}_\psi) \times_{\mathrm{GL}_N} X = \mathbb{A}^n \times X$ with $\mathbb{G}_m^n$ acting simultaneously on the left via its standard action, and on the right via $\psi : \mathbb{G}_m^n \to \mathrm{GL}_N$. Unravelling the definitions, one can compute for any B-scheme S

$$Y(S) = \{\mathbb{G}_m^n\text{-invariant sections of the bundle } \mathbb{A}^n \times X_S \to \mathbb{A}^n\}$$

which is equivalent to the functor represented by Y_ψ in Proposition 1.31. Under this identification $Y \simeq Y_\psi$, the P_ψ action on Y agrees with the action on Y_ψ obtained from applying the blade construction to the G action on X.

The argument which reduces the computation of $\mathrm{Grad}^n(X/\mathrm{GL}_N)$ to the computation of $\mathrm{Grad}^n(\mathrm{pt}/\mathrm{GL}_N)$ is identical, but with the $\mathbb{G}_m^n$-invariants construction replacing the blade construction.

Recalling the computation in [P2] *of* $\mathrm{Filt}^n(\mathrm{pt}/\mathrm{GL}_N)$.

Given a scheme S and an equivariant locally free sheaf $\mathcal{V}$ on Θ_S^n, the restriction to $\{(0,\ldots,0)\} \times S/\mathbb{G}_m^n$ is a $\mathbb{Z}^n$-graded locally free sheaf which splits into a direct sum of sub-bundles of constant weight with respect to $\mathbb{G}_m^n$. The ranks of each weight bundle are locally constant, and this data is encoded by a homomorphism $\psi : \mathbb{G}_m^n \to \mathrm{GL}_N$ up to conjugation, which we can assume comes from a homomorphism $\psi : \mathbb{G}_m^n \to \mathbb{G}_m^N$. Assume that S is connected, so that ψ is constant on S. Now for any point $s \in S$ one can choose an affine open neighborhood $s \in U \subset S$ and sections $v_1, \ldots, v_N \in \Gamma(\mathbb{A}_U^n, \mathcal{V})$ which are eigenvectors for the $\mathbb{G}_m^n$ action and restrict to a basis in the fiber of $\mathcal{V}$ over $s \times \{(0,\ldots,0)\}$. These sections must give a framing of the locally free sheaf in a $\mathbb{G}_m^n$-equivariant neighborhood of this point in $\mathbb{A}_U^n$, and they define an isomorphism $\mathcal{E}_\psi \simeq \mathcal{V}|_{\Theta_{U'}^n}$ for some smaller open subset $s \in U' \subset U$. Because we get such an isomorphism in an open neighborhood of any point, we see that $\mathcal{V}$ is the locally free sheaf on Θ_S^n associated to a principal $\mathrm{Aut}(\mathcal{E}_\psi) \simeq P_\psi$-bundle over S.

□

1.4.3. *Quotients by split algebraic groups over a field.* A result of Totaro [T5] implies that any finite normal Noetherian stack with the resolution property is equivalent to a stack of the form X/GL_N for some quasi-affine scheme X. In particular this includes stacks of the form X/G, where G is an algebraic group over a field k with split maximal torus, and X is a G-quasi-projective scheme. It will nevertheless be useful to have explicit descriptions of the stack of filtered and graded objects which more closely reflects the representation theory of G.

For any homomorphism $\psi : (\mathbb{G}_m^n)_k \to G$ we consider the blade Y_ψ for the action of $(\mathbb{G}_m^n)_k$ on X via ψ as in Proposition 1.31. Likewise we define P_ψ to be the blade for the conjugation action of $(\mathbb{G}_m^n)_k$ on G via ψ. If G is reductive and $n = 1$, then $P_\psi \subset G$ is a standard parabolic, but P_ψ need not be parabolic generally. As before, P_ψ acts naturally on Y_ψ because the blade construction commutes with products. Similarly the closed subgroup $L_\psi \subset G$, the centralizer of ψ in G, acts naturally on X^ψ.

We wish to define a map $Y_\psi/P_\psi \to \mathrm{Filt}^n(X/G)$ by constructing a map $\Theta^n \times (Y_\psi/P_\psi) \to X/G$. The latter map classifies a $(\mathbb{G}_m^n)_k \times P_\psi$-equivariant G-bundle over $\mathbb{A}_k^n \times Y_\psi$ along with a G-equivariant and $(\mathbb{G}_m)_k \times P_\psi$ invariant map to X. We use the trivial G-bundle $\mathbb{A}_k^n \times Y_\psi \times G$ equipped with a

$(\mathbb{G}_m^n)_k \times P_\psi$-equivariant structure via the left action

$$(t,p)\cdot(z,x,g) = (tz, p\cdot x, \psi(tz)p\psi(z)^{-1}g)$$

This expression is only well defined when $z \neq 0$, but it extends to a regular morphism because $\lim_{z\to 0}\psi(z)p\psi(z)^{-1} = l$ exists. It is straightforward to check that this defines an action of $(\mathbb{G}_m^n)_k \times P_\psi$, that the action commutes with right multiplication by G, and that the map $\mathbb{A}_k^n \times Y_\psi \times G \to X$ defined by

$$(z,x,g) \mapsto g^{-1}\psi(z)\cdot x$$

is $(\mathbb{G}_m^n)_k \times P_\psi$-invariant.

It is simpler to construct a map $\mathrm{pt}/(\mathbb{G}_m^n)_k \times X^\psi/L_\psi \to X/G$ and thus a map $X^\psi/L_\psi \to \mathrm{Grad}^n(X/G)$. We simply use the inclusion of schemes $X^\psi \hookrightarrow X$, which is equivariant with respect to the group homomorphism $\mathbb{G}_m \times L_\psi \to G$ given by $(t,l)\mapsto \psi(t)l \in G$.

Theorem 1.37. *Let $\mathfrak{X} = X/G$ be a quotient of a k-scheme X by a smooth affine k-group G which contains a split maximal torus $T \subset G$ and Weyl group W. The natural maps $Y_\psi/P_\psi \to \mathrm{Filt}^n(\mathfrak{X})$ and $X^\psi/L_\psi \to \mathrm{Grad}^n(\mathfrak{X})$ induce isomorphisms*

$$\mathrm{Filt}^n(\mathfrak{X}) \simeq \bigsqcup_{\psi\in \mathrm{Hom}(\mathbb{G}_m^n,T)/W} Y_\psi/P_\psi, \quad \textit{and}$$
$$\mathrm{Grad}^n(\mathfrak{X}) \simeq \bigsqcup_{\psi\in \mathrm{Hom}(\mathbb{G}_m^n,T)/W} X^\psi/L_\psi.$$

Furthermore, ev_0 *corresponds to the projection $Y_\psi \to X^\psi$, which is equivariant with respect to the group homomorphism $P_\psi \to L_\psi$, and* ev_1 *corresponds to the canonical map $Y_\psi \to X$, which is equivariant with respect to the inclusion of groups $P_\psi \subset G$.*

The statement is essentially the same as Theorem 1.36 and in principle can be reduced to it by choosing a linear embedding $G \hookrightarrow \mathrm{GL}_N$ and identifying $X/G \simeq \mathrm{GL}_N \times_G X/\mathrm{GL}_N$. We give a different, more direct proof in Appendix A which does not make use of the Rees construction.

One application of Theorem 1.37 is a concrete description of points of the flag scheme $\mathrm{Flag}(p)$ for $p \in X(k)$. Its k-points are specified by the three pieces of data:

- a one parameter subgroup $\lambda \in \mathrm{Hom}(\mathbb{G}_m, T)/W$;
- a point $q \in X$ such that $\lim_{t\to 0}\lambda(t)\cdot q$ exists; and
- a $g \in G$ such that $g\cdot q = p$.

Where two sets of such data specify the same point of the fiber if and only if $(g', q') = (gh^{-1}, h\cdot q)$ for some $h \in P_\lambda$.

Remark 1.38. Alternatively, given such a datum we define the one parameter subgroup $\lambda'(t) := g\lambda(t)g^{-1}$, and $\lim_{t\to 0}\lambda'(t)\cdot p$ exists. The point in $\mathrm{ev}_1^{-1}(p)$ is uniquely determined by this data, thus we can specify a point in the fiber by one parameter subgroup λ, not necessarily in T, for which $\lim_{t \to 0} \lambda(t) \cdot p$ exists. Two one parameter subgroups specify the same point in the fiber if and only if $\lambda' = h\lambda h^{-1}$ for some $h \in P_\lambda$.

1.5. **Geometric stacks have separated flag spaces.** We first collect some facts about flat quasi-coherent sheaves on Θ_R, where R is a discrete valuation ring. We denote by $\mathcal{U} = \Theta_R \setminus \{(0,0)\}$ the open compliment of the special point of codimension 2, and we let $j : \mathcal{U} \hookrightarrow \Theta_R$ denote the inclusion.

Lemma 1.39. *Given a flat quasi-coherent sheaf $E \in \mathrm{QCoh}(\mathcal{U})$, the (non-derived) pushforward $j_*(E) \in \mathrm{QCoh}(\Theta_R)$ is also flat.*

Proof. First write E as a filtered colimit of coherent sheaves $E = \mathrm{colim}_\alpha E_\alpha$. For each α we consider the short exact sequence $0 \to E_\alpha^{\mathrm{tor}} \to E_\alpha \to E^{\mathrm{fr}} \to 0$, where E_α^{tor} denotes the maximal torsion subsheaf. Because the colimit $\mathrm{colim}_\alpha E_\alpha^{\mathrm{tor}}$ is torsion and E is flat, it follows that $E \to \mathrm{colim}_\alpha E_\alpha^{\mathrm{fr}}$ is an isomorphism, so we may assume that each E_α is torsion-free. Note that $\mathcal{U}$ is a union of two open substacks, one isomorphic to $\mathrm{Spec}(R)$ and one isomorphic to Θ_K, where K is the fraction field of R. It follows that every torsion-free coherent sheaf on $\mathcal{U}$ is locally free, so we have written $E = \mathrm{colim}_\alpha E_\alpha$ as a filtered colimit of locally free sheaves. $j_* : \mathrm{QCoh}(\mathcal{U}) \to \mathrm{QCoh}(\Theta_R)$ commutes with filtered colimits, so it suffices to show that $j_*(F)$ is locally free whenever F is locally free. This follows from the fact that $\mathbb{A}^1_R$ is a regular scheme of dimension 2 and the special point $(0,0) \in \mathbb{A}^1_R$ is a complete intersection of codimension 2. $\square$

Lemma 1.40. *The restriction functor $j^* : \mathrm{QCoh}(\Theta_R) \to \mathrm{QCoh}(\mathcal{U})$ induces an equivalence between the category of flat quasi-coherent sheaves on Θ_R and the category of flat quasi-coherent sheaves on $\mathcal{U}$. The inverse is given by the pushforward $j_* : \mathrm{QCoh}(\mathcal{U}) \to \mathrm{QCoh}(\Theta_R)$.*

Proof. We first show that for any $E \in \mathrm{QCoh}(\Theta_R)$ which is flat, the canonical map $E \to j_*(E|_{\mathcal{U}})$ is an isomorphism. Indeed, there is an exact triangle in $\mathrm{D}_{qc}(\Theta_R)$

$$R\Gamma_{\{(0,0)\}} E \to E \to Rj_*(E|_{\mathcal{U}}) \to$$

Because $\{(0,0)\} \hookrightarrow \Theta_R$ is a regular embedding of codimension 2, the complex $R\Gamma_{\{(0,0)\}} \mathcal{O}_{\mathfrak{X}}$ is concentrated in cohomological degree 2. E is flat, so $R\Gamma_{\{(0,0)\}} E \simeq R\Gamma_{\{(0,0)\}} \mathcal{O}_{\mathfrak{X}} \otimes^L E$ is concentrated in cohomological degree 2 as well. Passing to the long exact sequence in homology we have an exact sequence

$$0 = H^0(R\Gamma_{\{(0,0)\}} E) \to E \to H^0(Rj_*(E|_{\mathcal{U}})) \to H^1(R\Gamma_{\{(0,0)\}} E) = 0.$$

Hence $E \to j_*(E|_{\mathcal{U}})$ is an isomorphism. It follows that the restriction functor j^* is fully faithful on the category of flat sheaves because

$$\mathrm{Hom}_{\Theta_R}(E, F) \simeq \mathrm{Hom}_{\Theta_R}(j_*(E|_{\mathcal{U}}), j_*(F|_{\mathcal{U}})) \simeq \mathrm{Hom}_{\mathcal{U}}(E|_{\mathcal{U}}, F|_{\mathcal{U}}),$$

where the last isomorphism comes from the adjunction between j^* and j_*. The functor j^* is essentially surjective on categories of flat sheaves by Lemma 1.39. □

These lemmas allow us to prove the following:

Proposition 1.41. *Let $\mathfrak{X}$ be an algebraic stack with affine diagonal. Then $\mathrm{ev}_1 : \mathrm{Filt}(\mathfrak{X}) \to \mathfrak{X}$ satisfies the valuative criterion for separatedness. In particular, if $\mathfrak{X}$ satisfies (†), then for any family $\xi : T \to \mathfrak{X}$ and any $n \geq 1$, the algebraic space $\mathrm{Flag}^n(\xi)$ is separated.*

Proof. The claim amounts to showing that for any discrete valuation ring R and two maps $f_1, f_2 : \Theta_R \to \mathfrak{X}$ along with an isomorphism $f_1|_{\mathcal{U}} \simeq f_2|_{\mathcal{U}}$, where $\mathcal{U} \subset \Theta_R$ is the open compliment of the codimension 2 special point, there is a unique isomorphism $f_1 \simeq f_2$ extending the given isomorphism over $\mathcal{U}$. f_1 and f_2 factor through some quasi-compact open substack, so we will assume that $\mathfrak{X}$ is quasi-compact.

Consider a smooth surjective map $\mathrm{Spec}(A^0) \to \mathfrak{X}$. The resulting simplicial presentation for $\mathfrak{X}$ is a simplicial affine scheme $\mathrm{Spec}(A^\bullet)$, because $\mathfrak{X}$ has affine diagonal. We may pull this presentation back along either f_1 or f_2 to a presentation for Θ_R. The result is, for $i = 1, 2$, a simplicial affine scheme $\mathrm{Spec}(B_i^\bullet)$ along with a map $f_i^\bullet : \mathrm{Spec}(B_i^\bullet) \to \mathrm{Spec}(A^\bullet)$ which descends to the map $f_i : \Theta_R \to \mathfrak{X}$. The isomorphism $f_1|_{\mathcal{U}} \simeq f_2|_{\mathcal{U}}$ induces a commutative diagram of simplicial schemes

$$\begin{tikzcd}
\mathrm{Spec}(B_1^\bullet)|_{\mathcal{U}} \arrow[r, hook] \arrow[d, "\simeq"] & \mathrm{Spec}(B_1^\bullet) \arrow[r, "f_1^\bullet"] \arrow[d, dashed, "\simeq"] & \mathrm{Spec}(A^\bullet) \\
\mathrm{Spec}(B_2^\bullet)|_{\mathcal{U}} \arrow[r, hook] & \mathrm{Spec}(B_2^\bullet) \arrow[ur, "f_2^\bullet"'] &
\end{tikzcd}$$

Because $B_i^\bullet$ for $i = 1, 2$ are cosimplicial objects in the category of flat quasi-coherent algebras over Θ_R, Lemma 1.40 implies that there is a unique isomorphism $\phi : B_1^\bullet \simeq B_2^\bullet$ of cosimplicial algebras extending the given isomorphism over $\mathcal{U}$, as indicated by the dotted arrow in the previous diagram. The fact that $f_2^\bullet \circ \phi = f_1^\bullet$ follows from the same fact for the restrictions $f_i|_{\mathcal{U}}$ and the fact that at every level in the simplicial scheme, $\mathrm{Spec}(B_i^n)$ is the affinization of the quasi-affine scheme $\mathrm{Spec}(B_i^n)|_{\mathcal{U}}$. The result is a unique isomorphism $f_1 \simeq f_2$ extending the isomorphism $f_1|_{\mathcal{U}} \simeq f_2|_{\mathcal{U}}$.

The final claim results from the fact that if $\mathfrak{X}$ satisfies (†), then $\mathrm{ev}_1 : \mathrm{Filt}(\mathfrak{X}) \to \mathfrak{X}$ is representable by locally finitely presented algebraic spaces, which are thus separated by the valuative criterion. For any $n \geq 1$, we use the identity $\mathrm{Filt}^n(\mathfrak{X}) \simeq \mathrm{Filt}(\mathrm{Filt}^{n-1}(\mathfrak{X}))$ to obtain a sequence of maps

$$\mathrm{Filt}^n(\mathfrak{X}) \xrightarrow{\mathrm{ev}_1} \mathrm{Filt}^{n-1}(\mathfrak{X}) \xrightarrow{\mathrm{ev}_1} \cdots \xrightarrow{\mathrm{ev}_1} \mathrm{Filt}(\mathfrak{X}) \xrightarrow{\mathrm{ev}_1} \mathfrak{X},$$

each of which is representable by separated algebraic spaces by the argument above. This implies that $\mathrm{Flag}^n(\xi)$ is separated over T for any $\xi : T \to \mathfrak{X}$. □

2. Θ-STRATIFICATIONS

In this section we introduce the notion of a Θ-stratification of a quasi-geometric algebraic stack $\mathfrak{X}$, and we establish some general theorems for constructing such stratifications. This generalizes the Kempf-Ness stratification in geometric invariant theory (see Example 4.13) as well as the Harder-Narasimhan stratification of the stack of vector bundles on a curve (see Section 5). The strata in $\mathfrak{X}$ parameterize points along with their "Harder-Narasimhan filtration," a canonical map $f : \Theta_k \to \mathfrak{X}$ chosen to maximize a certain "numerical invariant" $\mu(f)$. The main theorem of this section, Theorem 2.7 gives necessary and sufficient conditions for a numerical invariant to define a Θ-stratification. In later sections, we will establish stronger theorems along these lines (see Theorem 4.38).

2.1. **Definition and first properties.**

Definition 2.1. Let $\mathfrak{X}$ be a stack satisfying (†). A Θ-*stratum* in $\mathfrak{X}$ is a union of connected components $\mathfrak{S} \subset \mathrm{Filt}(\mathfrak{X})$ such that the restriction $\mathrm{ev}_1 : \mathfrak{S} \to \mathfrak{X}$ is a closed immersion. We call $\mathfrak{S}$ a *weak* Θ-*stratum* if ev_1 is finite and radicial.

Definition 2.2. Under the same hypotheses, a *(weak)* Θ-*stratification* of $\mathfrak{X}$ consists of:

(1) a totally ordered set Γ and a collection of open substacks $\mathfrak{X}_{\leq c}$ for $c \in \Gamma$ such that $\mathfrak{X}_{\leq c} \subset \mathfrak{X}_{\leq c'}$ for $c < c'$ and $\mathfrak{X} = \bigcup_c \mathfrak{X}_{\leq c}$;
(2) a (weak) Θ-stratum in each $\mathfrak{X}_{\leq c}$, $\mathrm{ev}_1 : \mathfrak{S}_c \hookrightarrow \mathfrak{X}_{\leq c}$, such that
$$\mathfrak{X}_{\leq c} \setminus \mathrm{ev}_1(\mathfrak{S}_c) = \mathfrak{X}_{<c} := \bigcup_{c'<c} \mathfrak{X}_{\leq c'}; \text{ and}$$
(3) for every point $x \in |\mathfrak{X}|$, the set $\{c \in \Gamma | x \in |\mathfrak{X}_{\leq c}|\}$ has a minimal element.

We assume there is a minimal element $0 \in \Gamma$. We refer to the open substack $\mathfrak{X}_{\leq 0}$ as the *semistable locus* $\mathfrak{X}^{\mathrm{ss}}$ and its complement in $|\mathfrak{X}|$ as the *unstable locus* $|\mathfrak{X}|^{\mathrm{us}}$. We allow the situation $\mathfrak{X}^{\mathrm{ss}} = \emptyset$.

Note that (3) holds automatically if the index set Γ is well-ordered.

Lemma 2.3 (HN filtrations)**.** *Let $\mathfrak{X}$ be a stack satisfying (†), and let $\{\mathfrak{X}_{\leq c}\}_{c\in\Gamma}$ be a weak Θ-stratification of $\mathfrak{X}$. Then for every unstable point $p \in \mathfrak{X}(k)$, there is a unique $c \in \Gamma$ and point $f \in |\mathfrak{S}_c|$ such that $p \in |\mathfrak{X}_{\leq c}|$ and $\mathrm{ev}_1(f) = p \in |\mathfrak{X}|$. f can be defined over a finite purely inseparable extension of k.*

We refer to f as the *Harder-Narasimhan (HN) filtration* of x. Note the consequence that if k is perfect, the HN filtration of $p \in \mathfrak{X}(k)$ will be defined over k.

Proof. Note that the composition $|\mathfrak{S}_c| \to |\mathfrak{X}_{\leq c}| \to |\mathfrak{X}|$ is a locally closed immersion, so we regard the former as a subset of the latter. Property (2) in

Definition 2.2 implies that $|\mathfrak{S}_c|$ are disjoint for different c, and property (3) implies that every unstable point lies in $\mathfrak{S}_{c^*}$, where $c^* = \min\{c \in \Gamma | x \in |\mathfrak{X}_{\leq c}\}$. Thus we have existence and uniqueness of HN filtrations.

We therefore have some field extension k'/k and a k'-point of $\mathrm{Flag}(p)$ representing the unique point of $|\mathrm{Flag}(p)|$ lying over the union of connected components $\mathfrak{S}_c \subset \mathrm{Filt}(\mathfrak{X}_{\leq c})$, one can take k' to be a finite extension because $\mathrm{Flag}(p)$ is locally finite type over $\mathrm{Spec}(k)$ and thus any irreducible component of $\mathrm{Flag}(p)$ contains some finite type point. Furthermore we may assume that k'/k is normal by replacing k' with its normal closure. If $f : \Theta_{k'} \to \mathfrak{X}$ is the HN filtration for $p \in \mathfrak{X}(k)$, then f is also the HN filtration for $f(1) \in \mathfrak{X}(k')$. Uniqueness of the HN filtration implies that f descends to a k''-point of $\mathrm{Flag}(p)$ for the purely inseparable extension $k \subset k'' := (k')^{\mathrm{Gal}(k'/k)}$. $\square$

Recall the map $\sigma : \mathrm{Grad}(\mathfrak{X}) \to \mathrm{Filt}(\mathfrak{X})$ of (1), and that a point of $\mathrm{Filt}(\mathfrak{X})$ is said to be *split* if it lies in the image of σ.

Definition 2.4. Let $\mathfrak{S}$ be a (weak) Θ-stratum in $\mathfrak{X}$. Then we define its *center* $\mathfrak{Z}^{\mathrm{ss}}$ to be the union of connected components $\sigma^{-1}(\mathfrak{S}) \subset \mathrm{Grad}(\mathfrak{X})$.

Lemma 1.24 guarantees that ev_0 maps $\mathfrak{S}$ to $\mathfrak{Z}^{\mathrm{ss}}$ and hence the canonical morphisms in (1) induce canonical morphisms for any weak Θ-stratum:

$$\begin{tikzcd} \mathfrak{Z}^{\mathrm{ss}} \arrow[r, bend right, "\sigma"'] & \mathfrak{S} \arrow[l, "\mathrm{ev}_0"'] \arrow[r, "\mathrm{ev}_1"] & \mathfrak{X} \end{tikzcd}.$$

In fact, the bijection of Lemma 1.24 implies that $\mathfrak{S} = \mathrm{ev}_0^{-1}(\mathfrak{Z}^{\mathrm{ss}}) \subset \mathrm{Filt}(\mathfrak{X})$, so a Θ-stratum can be equivalently specified by the collection of connected components $\mathfrak{Z}^{\mathrm{ss}} \subset \mathrm{Grad}(\mathfrak{X})$. In a Θ-stratification $\mathfrak{X} = \bigcup_c \mathfrak{X}_{\leq c}$, each center $\mathfrak{Z}^{\mathrm{ss}}_c$ is an open substack of $\mathrm{Grad}(\mathfrak{X})$ by Proposition 1.18, and the data of the Θ-stratification is uniquely encoded by these open substacks.

Lemma 2.5. *A weak Θ-stratification is a Θ-stratification if and only if at every finite type point $f \in \mathfrak{S}_c(k) \subset \mathrm{Filt}(\mathfrak{X})(k)$ which is split, either of the following equivalent conditions holds*

(1) *the fiber of the relative cotangent complex $(\mathbb{L}_{\mathfrak{S}_c/\mathfrak{X}})_f \in \mathrm{APerf}(\mathrm{Spec}(k))$ is 0-connective, i.e. $H^0((\mathbb{L}_{\mathfrak{S}_c/\mathfrak{X}})_f) = H^1((\mathbb{L}_{\mathfrak{S}_c/\mathfrak{X}})_f) = 0$, or*
(2) *the canonical map on Lie algberas $\mathrm{Lie}(\mathrm{Aut}_{\mathfrak{S}_c}(f)) \to \mathrm{Lie}(\mathrm{Aut}_{\mathfrak{X}}(f(1)))$ is surjective.*

Proof. By replacing $\mathfrak{X}$ with $\mathfrak{X}_{\leq c}$ it suffices to prove the claim for a single Θ stratum $\mathfrak{S} \hookrightarrow \mathfrak{X}$. Note that $\mathbb{L}_{\mathfrak{S}/\mathfrak{X}} \simeq \mathbb{L}_{\mathfrak{S}/\mathfrak{X}}$, and the proper and radicial map $\mathrm{ev}_1 : \mathfrak{S} \to \mathfrak{X}$ is a closed immersion if and only if $\mathbb{L}_{\mathfrak{S}/\mathfrak{X}}$ is acyclic in non-negative cohomological degree. This is an open condition on $\mathfrak{S}$, and can be checked at finite type points, by Nakayama's lemma. Thus $\mathfrak{S}$ is a Θ-stratum if and only if (1) holds at all finite type points of $\mathfrak{S}$, and the content of the claim is that it suffices to check only at *split* points of $\mathfrak{S}$.

The stratum $\mathfrak{S}$ is identified with a union of connected components of $\mathrm{Filt}(\mathfrak{X})$, which corresponds under the bijection of Lemma 1.23 to the union of connected components $\mathfrak{Z}^{\mathrm{ss}} \subset \mathrm{Grad}(\mathfrak{X})$. Lemma 1.24 implies that $\mathfrak{S}$ admits a Θ-deformation retract onto $\mathfrak{Z}^{\mathrm{ss}}$. The split points of $\mathfrak{S}$ are by definition the image of $\sigma : \mathfrak{Z}^{\mathrm{ss}} \to \mathfrak{S}$, so Lemma 1.21 implies that any open substack of $\mathfrak{S}$ containing all split finite type points is all of $\mathfrak{S}$. The first part of the claim follows.

Proof that (1) $\Leftrightarrow$ (2):

We can identify the map of Lie algebras in (2) with the k-linear dual of the map

$$H^1((\mathbb{L}_{\mathfrak{X}})_f) \to H^1((\mathbb{L}_{\mathfrak{S}})_f),$$

so we can reformulate the condition (2) as the property that this map is injective. We will consider the spectral stack $\mathfrak{X}^{\mathrm{sp}}$ associated to $\mathfrak{X}$, and we will denote by $\tilde{\mathfrak{S}} \subset \mathrm{Filt}(\mathfrak{X}^{\mathrm{sp}})$ the collection of connected components whose underlying classical stack is $\mathfrak{S}$ under the equivalence of Lemma 1.14. Note that $\tilde{\mathfrak{S}}$ is a spectral Θ-stratum in the sense that $\mathrm{ev}_1 : \tilde{\mathfrak{S}} \to \mathfrak{X}^{\mathrm{sp}}$ is a closed immersion, because a map of spectral algebraic stacks is a closed immersion if and only if the underlying map of classical stacks is a closed immersion. Lemma 1.17 shows that conditions (1) and (2) are equivalent to the analogous conditions formulated for the spectral Θ-stratum $\tilde{\mathfrak{S}} \to \mathfrak{X}^{\mathrm{sp}}$, and we prove it in this context.

Consider the long exact sequence coming from the exact triangle

$$H^0((\mathbb{L}_{\tilde{\mathfrak{S}}})_f) \xrightarrow{\alpha} H^0((\mathbb{L}_{\tilde{\mathfrak{S}}/\mathfrak{X}^{\mathrm{sp}}})_f) \longrightarrow H^1((\mathbb{L}_{\mathfrak{X}^{\mathrm{sp}}})_f) \longrightarrow H^1((\mathbb{L}_{\tilde{\mathfrak{S}}})_f) \longrightarrow 0\,.$$

Note that because f is split, there is a canonical non-trivial homomorphism $(\mathbb{G}_m)_k \to \mathrm{Aut}_{\tilde{\mathcal{S}}}(f)$, and this sequence canonically descends to a sequence of representations of $(\mathbb{G}_m)_k$. By Lemma 1.16 the complex $(\mathbb{L}_{\tilde{\mathfrak{S}}})_f$ has non-positive weights with respect to $(\mathbb{G}_m)_k$, and the complex $(\mathbb{L}_{\tilde{\mathfrak{S}}/\mathfrak{X}^{\mathrm{sp}}})_f$ has strictly positive weights with respect to $(\mathbb{G}_m)_k$. It follows that $\alpha = 0$ automatically, so $H^0((\mathbb{L}_{\tilde{\mathfrak{S}}/\mathfrak{X}^{\mathrm{sp}}})_f) = 0$ if and only if the map $H^1((\mathbb{L}_{\mathfrak{X}^{\mathrm{sp}}})_f) \to H^1((\mathbb{L}_{\tilde{\mathfrak{S}}})_f)$ is injective, which shows that (1) and (2) are equivalent. □

Example 2.6. The relevant modular example for the failure of the map on tangent spaces to be injective is the failure of Behrend's conjecture for the moduli of G-bundles on a curve in finite characteristic [H1]. In that example, the moduli of G bundles on a smooth projective curve C is stratified by the type of the canonical parabolic reduction of a given unstable G-bundle. In finite characteristic there are examples where the map $H^1(C, \mathfrak{p}) \to H^1(C, \mathfrak{g})$ is not injective, where $\mathfrak{g}$ is the adjoint bundle of a principle G-bundle and $\mathfrak{p}$ the adjoint bundle of its canonical parabolic reduction.

Corollary 2.6.1. *Let $\mathfrak{X}$ be a stack defined over a field of characteristic* 0 *and satisfying* (†). *Then any weak Θ-stratification of $\mathfrak{X}$ is a Θ-stratification.*

Proof. We use the characterization (2) of Lemma 2.5. The map $\mathrm{ev}_1 : \mathfrak{S}_c \to \mathfrak{X}_{\leq c}$ is by hypothesis, representable, proper, and radicial. It follows that for any $f \in \mathfrak{S}_c(k)$, the canonical homomorphism of algebraic k-groups

$\mathrm{Aut}_{\mathfrak{S}_c}(f) \to \mathrm{Aut}_{\mathfrak{X}}(f(1))$ is bijective on k' points for any field extension k'/k. If k has characteristic 0, then this implies the map $\mathrm{Aut}_{\mathfrak{S}_c}(f) \to \mathrm{Aut}_{\mathfrak{X}}(f(1))$ is an isomorphism of group schemes and hence induces an isomorphism of Lie algebras. □

2.2. **The Harder-Narasimhan problem and Θ-stratifications.** Say that $\mathfrak{X}$ is a stack with a Θ-stratification. We regard the given substack $\mathfrak{S}_c \subset \mathrm{Filt}(\mathfrak{X}_{\leq c})$ as an open substack of $\mathrm{Filt}(\mathfrak{X})$ as well under the open immersion $\mathrm{Filt}(\mathfrak{X}_{\leq c}) \subset \mathrm{Filt}(\mathfrak{X})$ induced by the open immersion $\mathfrak{X}_{\leq c} \subset \mathfrak{X}$. Letting $\mathrm{Irred}(\mathfrak{Y})$ denote the set of irreducible components of $|\mathfrak{Y}|$ for any locally finite type algebraic stack, we note that the open immersion induces inclusions of sets

$$\mathrm{Irred}(\mathfrak{S}_c) \subset \mathrm{Irred}(\mathrm{Filt}(\mathfrak{X}_{\leq c})) \subset \mathrm{Irred}(\mathrm{Filt}(\mathfrak{X})).$$

Thus the information of the Θ-stratification is entirely encoded by the set of irreducible components of $\mathrm{Filt}(\mathfrak{X})$ coming from $\mathfrak{S}_c \subset \mathrm{Filt}(\mathfrak{X})$, as well as the labelling of these components by $c \in \Gamma$. More formally, the data of a Θ-stratification is completely specified by the following data:

$$\begin{aligned} &1) \text{ a set of irreducible components } S \subset \mathrm{Irred}(\mathrm{Filt}(\mathfrak{X})), \text{ and} \\ &2) \text{ a map to a totally ordered set } \mu : S \to \Gamma. \end{aligned} \tag{8}$$

Given such data on an arbitrary stack, one can extend μ to a function $|\mathrm{Filt}(\mathfrak{X})| \to \Gamma \cup \{-\infty\}$, where we have adjoined a formal minimal element $-\infty$, by defining

$$\mu(f) = \max\left(\{-\infty\} \cup \{\mu(s) | f \text{ lies in irreducible component } s \in S\}\right)$$

We may assume without loss of generality that suprema exist in Γ, and one can define a *stability function* on $|\mathfrak{X}|$ by

$$M^\mu(p) = \sup\left\{\mu(f)\,|f \in |\mathrm{Filt}(\mathfrak{X})| \text{ s.t. } f(1) = p\right\} \in \Gamma \cup \{-\infty\} \tag{9}$$

One can then specify the data of a putative Θ-stratification by identifying subsets

$$\begin{aligned} &|\mathfrak{X}|_{\leq c} := \{p \in |\mathfrak{X}| \text{ s.t. } M^\mu(p) \leq c\} \subset |\mathfrak{X}| \\ &|\mathrm{Filt}(\mathfrak{X})|_c := \{f \in |\mathrm{Filt}(\mathfrak{X})| \text{ s.t. } f \text{ lies in } S \text{ and } \mu(f) = M^\mu(f)\} \end{aligned} \tag{10}$$

If the data (8) actually comes from a Θ-stratification, then $|\mathfrak{X}|_{\leq c}$ and $|\mathrm{Filt}(\mathfrak{X})|_c$ will be open and hence will define open substacks $\mathfrak{X}_{\leq c} \subset \mathfrak{X}$ and $\mathfrak{S}_c \subset \mathrm{Filt}(\mathfrak{X})$ respectively, and $p \in |\mathfrak{X}|$ is unstable if and only if $M^\mu(f) > -\infty$. This recovers all of the original data of the Θ-stratification. In fact, it suffices to consider only the corresponding subsets of *finite type* points. Of course, different data (8) can define the same Θ-stratification, and not every datum (8) defines a Θ-stratification in this way.

Theorem 2.7. *Let $\mathfrak{X}$ be an algebraic stack satisfying (†) and equipped with data as in (8). Then the subsets (10) define a weak Θ-stratification of $\mathfrak{X}$ if and only if the following conditions are satisfied:*

(1) ***HN-property:*** *For all* finite type *unstable points* $p \in \mathfrak{X}(k)$, $|\operatorname{Flag}(p)|$ *contains a unique point* f *lying over an irreducible component in* S *with* $\mu(f) = M^\mu(p)$. *This is the* Harder-Narasimhan (HN) filtration *of* p.

(2) ***HN-specialization:*** *For any valuation ring* R *with fraction field* K *and residue field* k *and any map* $\xi : \operatorname{Spec}(R) \to \mathfrak{X}$ *whose generic point is unstable and a HN filtration* $f_K \in \operatorname{Flag}(\xi)(K)$ *of* ξ_K, *one has*
$$\mu(f_K) \leq M^\mu(\xi|_{\operatorname{Spec}(k)}),$$
and when equality holds there is a unique extension of f_K *to a filtration* $f_R \in \operatorname{Flag}(\xi)(R)$.

(3) ***Open strata:*** *If* $\operatorname{Spec}(R) \to \operatorname{Filt}(\mathfrak{X})$ *is a map from a discrete valuation ring essentially of finite type over the base* B *whose special point is an HN filtration, then its generic point is an HN filtration as well.*

(4) ***Local finiteness:*** *For any map* $\varphi : T \to \mathfrak{X}$, *with* T *a finite type affine scheme, there is a finite subset of* S *such that every unstable finite type point in* T *has an HN filtration lying on one of these irreducible components.*

(5) ***Semi-continuity:*** *If* $f : \Theta_k \to \mathfrak{X}$ *is a HN filtration for* $f(1)$, *then* $M^\mu(f(0)) \leq \mu(f)$.

Alternatively, one can replace conditions (2) and (4) with the following:

(2') ***Simplified HN-specialization:*** *For any* discrete *valuation ring* R essentially of finite type over the base B *with fraction field* K *and residue field* k, *and for any map* $\xi : \operatorname{Spec}(R) \to \mathfrak{X}$ *whose generic point is unstable and a HN filtration* $f_K \in \operatorname{Flag}(\xi)(K)$ *of* ξ_K, *one has*
$$\mu(f_K) \leq M^\mu(\xi|_{\operatorname{Spec}(k)}),$$
and when equality holds there is an extension of discrete valuation rings $R' \supset R$ *with fraction field* K' *such that there is a unique extension of* $f_K|_{K'}$ *to a filtration* $f_{R'} \in \operatorname{Flag}(\xi)(R')$.

(4') ***HN-boundedness:*** *For any map* $\xi : T \to \mathfrak{X}$, *with* T *a finite type affine scheme, there is a quasi-compact subspace of* $\operatorname{Flag}(\xi)$ *which contains an HN filtration for every unstable finite type point of* T.

The proof of this theorem is conceptually straightforward, but the theorem itself is a useful way of organizing the construction of Θ-stratifications. In the rest of the paper we will restrict our attention to functions μ and stacks $\mathfrak{X}$ for which many of these conditions hold *automatically*. We can make some immediate simplifications before developing further theory:

Simplification 2.8 (Locally constant μ)**.** *In the context of this paper, our function* μ *will always be* locally constant, *in which case either "HN-specialization" condition* (2) *or* (2′) *immediately implies the "open strata" condition (3), making the latter redundant. We say that* μ *is locally constant if it is induced by a pair*
$$S' \subset \pi_0(\operatorname{Filt}(\mathfrak{X})) \text{ and } \mu' : S' \to \Gamma,$$

in the sense that S is the preimage of S' under the canonical surjective map $\mathrm{Irred}(\mathrm{Filt}(\mathfrak{X})) \to \pi_0(\mathrm{Filt}(\mathfrak{X}))$*, and μ is the restriction of μ' along this map.*

Proof. The HN-specialization condition implies that the open strata condition only fails if for some map $\mathrm{Spec}(R) \to \mathrm{Filt}(\mathfrak{X})$ whose special point is a HN filtration, the image of $\mathrm{Spec}(K) \to \mathrm{Filt}(\mathfrak{X})$ does not lie on an irreducible component of S or the value of μ is smaller at the generic point than at the special point of $\mathrm{Spec}(R)$. Neither of these things can happen if μ is locally constant. □

We say that $\mathfrak{X}$ has *quasi-compact flag spaces* if for any map $\xi : T \to \mathfrak{X}$ from a finite type affine scheme and any connected component $\mathfrak{Y} \subset \mathrm{Filt}(\mathfrak{X})$ the fiber product $\mathfrak{Y} \times_{\mathrm{ev}_1, \mathfrak{X}, \xi} T \subset \mathrm{Flag}(\xi)$ is quasi-compact. This property holds, for instance, for the stack of coherent sheaves on a projective scheme X (see Example 1.9), and we give a criterion which guarantees this property for a quasi-compact stack $\mathfrak{X}$ in Proposition 3.89.

Simplification 2.9 (Quasi-compact flag spaces)**.** *If $\mathfrak{X}$ has quasi-compact flag spaces, then the local finiteness condition (4) of Theorem 2.7 is equivalent to the HN boundedness condition. It is also equivalent to requiring a finite set of* connected components *of* $\mathrm{Filt}(\mathfrak{X})$*, rather than a finite set of irreducible components, such that every unstable point of T has a HN filtration in* $\mathrm{Flag}(\xi)$ *lying over one of these connected components.*

Remark 2.10. Note that a stronger condition which immediately implies the "HN-specialization" condition is the property that for any map $\xi : \mathrm{Spec}(R) \to \mathfrak{X}$ with $M^\mu(\xi|_K) > -\infty$ and a HN filtration $f_K : \Theta_K \to \mathfrak{X}$ of ξ_K, there is unique extension of f_K to a filtration $f_R : \mathrm{Spec}(R) \to \mathrm{Filt}(\mathfrak{X})$ of ξ. This motivates the definition of Θ-reductive stacks, Definition 4.16 below, for which HN-specialization holds automatically for discrete valuation rings. In Definition 4.1 below we provide a method for constructing locally constant μ, and we will show in Theorem 4.39 that a simpler version of the HN-boundedness condition, (B2), is equivalent to conditions (1)-(5) for such μ when $\mathfrak{X}$ is Θ-reductive.

Proof of Theorem 2.7. Let T be a finite type affine scheme and let $\xi : T \to \mathfrak{X}$ be a smooth map, and let $s_1, \ldots, s_n \in \mathrm{Irred}(\mathrm{Filt}(\mathfrak{X}))$ be the irreducible components capturing all of the HN filtrations of finite type points in T, whose existence is guaranteed by either condition (4) or (4′). The fact that ξ is smooth implies that $\mathrm{Flag}(\xi) \to \mathrm{Filt}(\mathfrak{X})$ is smooth, so there are finitely many irreducible components $Y_1, \ldots, Y_N \subset \mathrm{Flag}(\xi)$ in the preimage of some s_i, and we regard them as subspaces with their reduced structure. By relabelling and combining components with the same value of μ, we will assume that each Y_i is a finite union of irreducible components of $\mathrm{Flag}(\xi)$ such that $\mu = \mu_i$ on each irreducible component, and we will assume that $\mu_i < \mu_j$ for $i < j$.

Any point $p \in |T|$, not necessarily of finite type, must have $M^\mu(\xi(p)) \leq \mu_N$. Indeed if there were some point $f' \in |\mathrm{Flag}(\xi)|$ with $\mu(f') > \mu_N$, then the irreducible component containing f' would have to contain finite type points as well with $\mu > \mu_N$, which would contradict the hypothesis that $Y_1, \ldots, Y_N$ contain HN filtrations for all finite type points of ξ. This implies that any point $f \in Y_N$ must be an HN filtration for $f(1)$.

The HN-specialization property (2) now implies that for any valuation ring R with fraction field K and any map $\mathrm{Spec}(R) \to T$ with a lift $\mathrm{Spec}(K) \to Y_N$, there is a unique lift to a map $\mathrm{Spec}(R) \to \mathrm{Flag}(\xi)$. Because Y_N is closed the lift is actually a map $\mathrm{Spec}(R) \to Y_N$. Thus $Y_N \to T$ satisfies the valuative criterion for properness, and Lemma 2.11 below shows that Y_N is quasi-compact and hence proper over T.

At this point we modify the argument if the HN-boundedness condition ($4'$) holds: we let $U \subset \mathrm{Flag}(\xi)$ be a quasi-compact open subspace containing a HN filtration for every unstable finite type point of T. Then $Y_N \subset U$, because every finite type point of Y_N is a HN filtration. Hence $Y_N \to T$ is a finite type map of algebraic spaces, where T is noetherian and affine, and it suffices to establish any variant of the valuative criterion which works in this context. The simplified HN-specialization property ($2'$) corresponds to one such variant, which we will discuss at the end of the proof.

Under either sets of hypotheses we have now shown that $S_N := \mathrm{im}(Y_N \to T)$ is closed, and an inductive argument shows that $S_i := \bigcup_{j \geq i} \mathrm{im}(Y_j \to T)$ is closed and the induced map

$$\mathrm{ev}_1 : Y_i \setminus \mathrm{ev}_1^{-1}(S_{i+1}) \to T \setminus S_{i+1}$$

is proper with image S_i. Note also that every point $p \in |T \setminus S_i|$, not necessarily of finite type, has $M^\mu(p) \leq \mu_{i-1} < \mu_i$, so every point with $M^\mu(p) = \mu_i$ has an HN filtration in $Y_i \setminus \mathrm{ev}_1^{-1}(S_{i+1})$. Applying this observation inductively shows that every unstable point of T lies in the closed subscheme $S_1 \hookrightarrow T$, that all points of the quasi-compact locally closed subspace

$$Y := \bigcup_i Y_i \setminus \mathrm{ev}_1^{-1}(S_{i+1}) \subset \mathrm{Flag}(\xi)$$

are HN filtrations, and that Y contains an HN filtration of every unstable point of T, not just the finite type points.

This shows that the set $|X_{\leq c}|$ is open as claimed, and thus defines an open substack $\mathfrak{X}_{\leq c} \subset \mathfrak{X}$. Now let $\mathfrak{S}_c^{red} \subset \mathrm{Filt}(\mathfrak{X})$ denote the union of irreducible components corresponding to those s_i for which $\mu(s_i) = c$, with reduced structure. Proposition 1.18 implies that we have inclusions of open substacks $\mathrm{Filt}(\mathfrak{X}_{\leq c}) = \mathrm{ev}_0^{-1}(\mathfrak{X}_{\leq c}) \subset \mathrm{ev}_1^{-1}(\mathfrak{X}_{\leq c}) \subset \mathrm{Filt}(\mathfrak{X})$, and the semi-continuity property (5) implies that these inclusions become equalities after intersecting with $\mathfrak{S}_c^{red}$. Thus we have

$$\mathrm{ev}_1^{-1}(\mathfrak{X}_{\leq c}) \cap \mathfrak{S}_c^{red} = \mathrm{Filt}(\mathfrak{X}_{\leq c}) \cap \mathfrak{S}_c^{red},$$

which implies that the set of points of $|\mathrm{Filt}(\mathfrak{X})|$ corresponding to HN filtrations of points in $\mathfrak{X}$ with $\mu = c$ is actually a subset of $|\mathrm{Filt}(\mathfrak{X}_{\leq c})|$ which is closed in the subspace topology. Furthermore, this subset $|\mathrm{ev}_1^{-1}(\mathfrak{X}_{\leq c}) \cap \mathfrak{S}_c^{red}| \subset |\mathrm{Filt}(\mathfrak{X})|$ is a constructible subset of the locally finite type B-stack

$\mathrm{Filt}(\mathfrak{X}_{\leq c})$, so the open strata condition (3) implies that this set is open as well. Thus there is a union of connected components $\mathfrak{S}_c \subset \mathrm{Filt}(\mathfrak{X}_{\leq c})$ realizing all HN filtrations of points in $\mathfrak{X}$ for which $\mu = c$. The analyis of the previous paragraph shows shows that the map $\mathrm{ev}_1 : \mathfrak{S}_c \to \mathfrak{X}_{\leq c}$ is proper, and the uniqueness of the HN filtration implies that it is radicial, hence a weak Θ-stratum.

A variant on the valuative criterion for properness:

Under the HN-boundedness hypothesis, we used the following variant of the valuative criterion for properness which applies to a finite type map $Y \to T$ of noetherian algebraic spaces:

> For any discrete valuation ring R essentially of finite type over T and a section of the map $Y \to T$ over the generic point $\mathrm{Spec}(K) \subset \mathrm{Spec}(R)$, there is an extension of discrete valuation rings $R' \supset R$ with fraction field $K' \supset K$ such that after restricting the section to $\mathrm{Spec}(K')$ it can be extended uniquely to a section over $\mathrm{Spec}(R')$.

This is basically [S2, Tag 0ARK, Tag0ARL]. There it is not explicitly stated that it suffices to consider R which are essentially finite type over T, but it follows from the proof: the proof of separatedness in [S2, Tag 0ARJ] amounts to checking the valuative criterion for properness for the diagonal, representable by schemes, and it suffices to consider DVR's essentially of finite type by [S2, Tag 0207, part (4)]. It also suffices to show the uniqueness of the filling after passing to an extension of R by fpqc descent [S2, Tag 0APL]. The proof of properness in [S2, Tag 0ARK] uses Chow's lemma [S2, Tag 089J] for the separated algebraic space $Y \to T$ to reduce to a quasi-projective T-scheme, hence it suffices to check essentially finite type DVR's by [S2, Tag 0207, part (4)], and it suffices to pass to an extension of DVR's by [S2, Tag 0ARH]. ☐

Lemma 2.11. *Let $f : X \to Y$ be a map of algebraic spaces which is locally of finite presentation and satisfies the valuative criterion of properness. If X has finitely many irreducible components, then f is quasi-compact.*

Proof. If X has finitely many irreducible components and $X' \to X$ is a smooth map, then X' has finitely many irreducible components. This allows us to reduce to showing that if $Y = \mathrm{Spec}(A)$ is affine then X is quasi-compact if it has finitely many irreducible components. It suffices to assume furthermore that X is reduced and irreducible, and A is an integral domain. Let K denote the function field of X, and let $RZ(K, A)$ denote the Riemann-Zariski space, parameterizing valuations rings of K which contain A. Then $RZ(K, A)$ is quasi-compact with its Zariski topology. The valuative criterion for properness for the map $X \to \mathrm{Spec}(A)$ implies that for any valuation ring of K containing A, there is a unique map $\mathrm{Spec}(R) \to X$ over A. Assigning every valuation ring to the image of its special point defines a map $RZ(K, A) \to |X|$ which is continuous: the preimage of an affine open $\mathrm{Spec}(B) \subset X$ is the open subspace $RZ(K, B) \subset RZ(X, A)$. Furthermore, the map $RZ(K, A) \to |X|$

is surjective, so quasi-compactness of $|X|$ follows from quasi-compactness of $RZ(K, A)$. $\square$

2.3. **Induced stratifications.** Let $\pi : \mathfrak{Y} \to \mathfrak{X}$ be a map of quasi-geometric algebraic stacks, and let $\mathrm{Filt}(\pi) : \mathrm{Filt}(\mathfrak{Y}) \to \mathrm{Filt}(\mathfrak{X})$ denote the induced map.

Definition 2.12. We say that a (weak) Θ-stratum $\mathfrak{S} \subset \mathrm{Filt}(\mathfrak{X}) \to \mathfrak{X}$ *induces* a (weak) Θ-stratum in $\mathfrak{Y}$ if

$$\mathrm{Filt}(\pi)^{-1}(\mathfrak{S}) \subset \mathrm{Filt}(\mathfrak{Y}) \xrightarrow{\mathrm{ev}_1} \mathfrak{Y}$$

is a (weak) Θ-stratum whose image is $\pi^{-1}(\mathrm{ev}_1(\mathfrak{S})) \subset \mathfrak{Y}$.

Likewise we say a (weak) Θ-stratification $\{\mathfrak{X}_{\leq c}\}_{c\in\Gamma}$ of $\mathfrak{X}$ induces a (weak) Θ-stratification of $\mathfrak{Y}$ if each Θ-stratum in $\mathfrak{X}_{\leq c}$ induces a Θ-stratum in $\mathfrak{Y}_{\leq c} := \pi^{-1}(\mathfrak{X}_{\leq c})$ for all $c \in \Gamma$.

Note that in the previous definition if $\mathfrak{Y}_{\leq c} := \pi^{-1}(\mathfrak{X}_{\leq c})$ is an induced (weak) Θ-stratification, then the stability function $M^\mu : |\mathfrak{Y}| \to \Gamma$ defined as $M^\mu(p) = \min\{c \in \Gamma | p \in |\mathfrak{Y}_{\leq c}|\}$ is the restriction of $M^\mu : |\mathfrak{X}| \to \Gamma$ along the map $|\mathfrak{Y}| \to |\mathfrak{X}|$.

Lemma 2.13. *Let $\mathfrak{X}$ be a stack satisfying (†) with a (weak) Θ-stratification $\{\mathfrak{X}_{\leq c}\}_{c\in\Gamma}$, and let $\mathfrak{Y} \to \mathfrak{X}$ be either an open union of strata or a closed immersion. Then $\{\mathfrak{X}_{\leq c}\}_{c\in\Gamma}$ induces a (weak) Θ-stratification of $\mathfrak{Y}$.*

Proof. The case of an open union of strata is a straightforward consequence of the definition, so we focus on the case of a closed immersion. In this case Proposition 1.18 implies that the canonical map induces an equivalence $\mathrm{Filt}(\mathfrak{Y}) \xrightarrow{\simeq} \mathrm{Filt}(\mathfrak{X}) \times_{\mathfrak{X}} \mathfrak{Y}$, so we have a diagram in which each square is Cartesian

$$\begin{tikzcd} \mathrm{Filt}(\pi)^{-1}(\mathfrak{S}) \arrow[r] \arrow[d] & \mathrm{Filt}(\mathfrak{Y}) \arrow[r, "\mathrm{ev}_1"] \arrow[d, "\mathrm{Filt}(\pi)"] & \mathfrak{Y} \arrow[d, "\pi"] \\ \mathfrak{S} \arrow[r] & \mathrm{Filt}(\mathfrak{X}) \arrow[r, "\mathrm{ev}_1"] & \mathfrak{X} \end{tikzcd}$$

It follows that if $\mathfrak{S} \to \mathfrak{X}$ is a (weak) Θ-stratum then so is $\mathrm{Filt}(\pi)^{-1}(\mathfrak{S}) \to \mathfrak{Y}$, and the image of this map is the preimage of $\mathrm{ev}_1(\mathfrak{S})$. $\square$

Lemma 2.14. *Consider a cartesian diagram of stacks satisfying (†)*

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, "\pi'"] \arrow[d] & \mathfrak{X} \arrow[d] \\ \mathfrak{Y}' \arrow[r, "\pi"] & \mathfrak{Y} \end{tikzcd}$$

in which $\mathfrak{Y}'$ and $\mathfrak{Y}$ have quasi-finite inertia. Then any (weak) Θ-stratification in $\mathfrak{X}$ induces a (weak) Θ-stratification in $\mathfrak{X}'$. Conversely if $\mathfrak{Y}' \to \mathfrak{Y}$ is faithfully flat and $\mathfrak{S} \subset \mathrm{Filt}(\mathfrak{X})$ is a union of connected components for which $\mathrm{Filt}(\pi')^{-1}(\mathfrak{S}) \subset \mathrm{Filt}(\mathfrak{X}')$ is a (weak) Θ-stratum, then $\mathfrak{S}$ is a (weak) Θ-stratum.

Proof. Corollary 1.30.1 implies that $\mathrm{Filt}(\mathfrak{X}') \simeq \mathrm{Filt}(\mathfrak{X}) \times_{\mathfrak{X}} \mathfrak{X}'$. As in the proof of Lemma 2.13, this implies that for any Θ-stratum $\mathfrak{S} \subset \mathrm{Filt}(\mathfrak{X})$ the substack $\mathrm{Filt}(\pi')^{-1}(\mathfrak{S}) \subset \mathrm{Filt}(\mathfrak{X}')$ is a Θ-stratum as well. Conversely, if $\mathfrak{X}' \to \mathfrak{X}$ is faithfully flat then $\mathrm{Filt}(\pi') : \mathrm{Filt}(\mathfrak{X}') \to \mathrm{Filt}(\mathfrak{X})$ is as well, because it is the base change of the map $\mathfrak{X}' \to \mathfrak{X}$. We know that $\mathrm{Filt}(\pi')^{-1}(\mathfrak{S}) \simeq \mathfrak{S} \times_{\mathfrak{X}} \mathfrak{X}'$, so faithfully flat descent implies that if $\mathrm{ev}_1 : \mathrm{Filt}(\pi')^{-1}(\mathfrak{S}) \simeq \mathfrak{S} \times_{\mathfrak{X}} \mathfrak{X}' \to \mathfrak{X}'$ is a closed immersion (respectively finite and radicial), then so is $\mathfrak{S} \to \mathfrak{X}$. $\square$

Proposition 2.15. *Let $\mathfrak{X}$ be a stack satisfying (†) and consider the map*

$$u : \mathrm{Grad}(\mathfrak{X}) \to \mathfrak{X}$$

induced by restriction along $\mathrm{pt} \to \mathrm{pt}/\mathbb{G}_m$. Any Θ-stratification $\{\mathfrak{X}_{\leq c}\}_{c \in \Gamma}$ of $\mathfrak{X}$ induces a Θ-stratification of $\mathrm{Grad}(\mathfrak{X})$.

Proof. Corollary 1.7.1 implies that $\mathrm{Grad}(\mathfrak{X}_{\leq c}) = u^{-1}(\mathfrak{X}_{\leq c})$ for all $c \in \Gamma$, so it suffices to focus on a single Θ-stratum $\mathfrak{S}_c \subset \mathrm{Filt}(\mathfrak{X}_{\leq c}) \to \mathfrak{X}_{\leq c}$, and we will drop the subscript c from the notation. We first apply the functor $\mathrm{Grad}(-)$ to the sequence $\mathfrak{S} \subset \mathrm{Filt}(\mathfrak{X}) \xrightarrow{\mathrm{ev}_1} \mathfrak{X}$ to obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Grad}(\mathfrak{S}) \arrow[r] \arrow[d, "u"] & \mathrm{Grad}(\mathrm{Filt}(\mathfrak{X})) \arrow[r, "\mathrm{Grad}(\mathrm{ev}_1)"] \arrow[d, "u_{\mathrm{Filt}(\mathfrak{X})}"] & \mathrm{Grad}(\mathfrak{X}) \arrow[d, "u"] \\
\mathfrak{S} \arrow[r] & \mathrm{Filt}(\mathfrak{X}) \arrow[r] & \mathfrak{X}
\end{tikzcd}.$$

Because $\mathfrak{S} \subset \mathrm{Filt}(\mathfrak{X})$ is a union of connected components and $\mathfrak{S} \to \mathfrak{X}$ is a locally closed immersion, Corollary 1.7.1 implies that both the outer square and the left square in this diagram are cartesian.

As an immediate consquence of their description as mapping stacks we can identify

$$\mathrm{Filt}(\mathrm{Grad}(\mathfrak{X})) \simeq \underline{\mathrm{Map}}(\Theta \times (\mathrm{pt}/\mathbb{G}_m), \mathfrak{X}) \simeq \mathrm{Grad}(\mathrm{Filt}(\mathfrak{X})), \tag{11}$$

and under this isomorphism we have a canonical identification between the map $u_{\mathrm{Filt}(\mathfrak{X})}$ in the commutative diagram above and the application of the functor $\mathrm{Filt}(-)$ to the map $u : \mathrm{Grad}(\mathfrak{X}) \to \mathfrak{X}$, in other words $u_{\mathrm{Filt}(\mathfrak{X})} \simeq \mathrm{Filt}(u_{\mathfrak{X}})$. Therefore, (11) identifies the union of connected components $\mathrm{Filt}(u)^{-1}(\mathfrak{S})$ with $\mathrm{Grad}(\mathfrak{S})$, and the map $\mathrm{ev}_1 : \mathrm{Filt}(u)^{-1}(\mathfrak{S}) \to \mathrm{Grad}(\mathfrak{X})$ is the base change of $\mathrm{ev}_1 : \mathfrak{S} \to \mathfrak{X}$ along $u : \mathrm{Grad}(\mathfrak{X}) \to \mathfrak{X}$, which establishes that the Θ-stratum $\mathfrak{S} \to \mathfrak{X}$ induces a Θ-stratum in $\mathrm{Grad}(\mathfrak{X})$. $\square$

Remark 2.16. One concrete consequence is that if $g \in \mathrm{Grad}(\mathfrak{X})$ is a graded object whose underlying point $u(g) \in \mathfrak{X}$ is unstable, then the HN filtration of $u(g)$ lifts uniquely to an HN filtration of g in $\mathrm{Grad}(\mathfrak{X})$. This can be shown directly for any graded object for which $u(g)$ admits a unique HN filtration, even when μ does not define a Θ-stratum. The general statement underlying this fact is that if $x \in \mathfrak{X}(\bar{k})$ is a geometric point and $\mathfrak{S} \subset \mathrm{Filt}(\mathfrak{X})$ a connected component such that $\mathrm{ev}_1^{-1}(x) \cap \mathfrak{S}$ has a single $\bar{k}$ point f, then any homomorphism from a reduced k-group $G \to \mathrm{Aut}_{\mathfrak{X}}(x)$ lifts uniquely to a homomorphism $G \to \mathrm{Aut}_{\mathrm{Filt}(\mathfrak{X})}(f)$.

Let $\{\mathfrak{X}_{\leq c}\}_{c \in \Gamma}$ be a Θ-stratification of $\mathfrak{X}$, and consider the center $\mathfrak{Z}_c^{\mathrm{ss}}$ of the stratum $\mathfrak{S}_c \hookrightarrow \mathfrak{X}_{\leq c}$ (Definition 2.4). Assume for simplicity that the stratification is induced by a locally constant μ as in Simplification 2.8, which will always be the case for the stratifications constructed in the remainder of this paper. For every $c \in \Gamma$, let $\mathfrak{Z}_c \subset \mathrm{Grad}(\mathfrak{X})$ be a the union of all connected components meeting the open substack $\mathfrak{Z}_c^{\mathrm{ss}} \subset \mathrm{Grad}(\mathfrak{X}_{\leq c}) \subset \mathrm{Grad}(\mathfrak{X})$. The main consequence of Proposition 2.15 is

Corollary 2.16.1. *The center $\mathfrak{Z}_c^{\mathrm{ss}}$ is the semistable locus of a Θ-stratification $\{(\mathfrak{Z}_c)_{\leq c'}\}_{c' \geq c}$ of $\mathfrak{Z}_c$ induced by the forgetful map $u : \mathfrak{Z}_c \to \mathfrak{X}$, where $c' = c$ is the minimal index. In particular $(\mathfrak{Z}_c)_{\leq c'}$ is the preimage of $\mathfrak{X}_{\leq c'}$ under u, and $(\mathfrak{Z}_c)_{\leq c'} = \emptyset$ for $c' < c$.*

Proof. The claim that $u : \mathfrak{Z}_c \to \mathfrak{X}$ induces a Θ-stratification on $\mathfrak{Z}_c$ follows from Proposition 2.15 applied to $\mathrm{Grad}(\mathfrak{X}) \to \mathfrak{X}$ and Lemma 2.13 applied to $\mathfrak{Z}_c \to \mathrm{Grad}(\mathfrak{X})$.

Let $\bar{\mathfrak{S}}_c \subset \mathrm{Filt}(\mathfrak{X})$ be the union of all connected components which meet the open substack $\mathfrak{S}_c \subset \mathrm{Filt}(\mathfrak{X}_{\leq c}) \subset \mathrm{Filt}(\mathfrak{X})$. Because we have assumed μ is locally constant, μ is defined and takes the value c on all irreducible components of $\bar{\mathfrak{S}}_c$. It follows that $M^\mu(p) \geq c$ for every point in $\mathrm{ev}_1(|\bar{\mathfrak{S}}_c|) \subset \mathfrak{X}$ and that $M^\mu(p) = c$ if and only if $p \in \mathrm{ev}_1(|\mathfrak{S}_c|)$. In particular $\mathfrak{S}_c$ is the preimage of $\mathfrak{X}_{\leq c} \subset \mathfrak{X}$ under the restriction of ev_1 to $\bar{\mathfrak{S}}_c \subset \mathrm{Filt}(\mathfrak{X})$.

Now making use of Lemma 1.24, we see that $\mathfrak{Z}_c = \sigma^{-1}(\bar{\mathfrak{S}}_c) \subset \mathrm{Grad}(\mathfrak{X})$, and combining this with the fact that $\mathrm{ev}_1 \circ \sigma \simeq u$ shows that $\mathfrak{Z}_c^{\mathrm{ss}} \subset \mathfrak{Z}_c$ is the preimage of $\mathfrak{X}_{\leq c} \subset \mathfrak{X}$ under u. In order to identify $\mathfrak{Z}_c^{\mathrm{ss}}$ with the "semistable" locus for the induced Θ-stratification of $\mathfrak{Z}_c$ in the sense of Definition 2.2, we must verify the preimage of $X_{\leq c'}$ is empty for any $c' < c$. This follows from the fact that $M^\mu(u(g)) = M^\mu(\mathrm{ev}_1(\sigma(g))) \geq c$ for any $g \in |\mathfrak{Z}_c|$ ☐

3. Combinatorial structures in moduli theory

In this section, we introduce objects, the degeneration fan and degeneration space, whose structure encodes the various possible filtrations of a point in an algebraic stack. Our main technical result in Theorem 3.60, which identifies small perturbations of a filtration f with small perturbations of the canonical filtration of the associated graded object $\mathrm{ev}_0(f)$. In Section 4 we will use the concepts introduced here to reformulate the Harder-Narasimhan problem in general moduli problems.

3.1. **Formal fans and their geometric realizations.** The basic combinatorial objects which we study, formal fans, are analogous to semisimplicial sets.

Definition 3.1. We define a category of *integral simplicial cones* $\mathfrak{Cone}$ to have

- objects: positive integers $[n]$ with $n > 0$,
- morphisms: a morphism $\phi : [k] \to [n]$ is an injective group homomorphism $\mathbb{Z}^k \to \mathbb{Z}^n$ which maps the standard basis of $\mathbb{Z}^k$ to the cone spanned by the standard basis of $\mathbb{Z}^n$.

We define the category of *formal fans*

$$\mathrm{Fan} := \mathrm{Fun}(\mathfrak{Cone}^{op}, \mathrm{Set})$$

For $F \in \mathrm{Fan}$ we use the abbreviated notation $F_n = F([n])$ and refer to this as the set of *cones*. We refer to the elements of F_1 as *rays*.

Given a formal fan $F_\bullet$, in analogy with the theory of semisimplicial sets we define the *face maps* $d_i : F_n \to F_{n-1}$ for $i = 1, \ldots, n$ to be the maps induced by the homomorphism $\mathbb{Z}^{n-1} \to \mathbb{Z}^n$ mapping $(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_n)$. We also introduce the *vertex maps* $v_i : F_n \to F_1$ for $i = 1, \ldots, n$ induced by the group homomorphism $\mathbb{Z} \to \mathbb{Z}^n$ which is the inclusion of the i^{th} factor.

For any $F \in \mathrm{Fan}$, we can define two notions of geometric realization. First form the comma category $(\mathfrak{Cone}|F)$ whose objects are elements $\xi \in F_n$ and morphisms $\xi_1 \to \xi_2$ are given by morphisms $\phi : [n_1] \to [n_2]$ with $\phi^* \xi_2 = \xi_1$. There is a canonical functor $(\mathfrak{Cone}|F) \to \mathrm{Top}$ assigning $\xi \in F_n$ to the cone $(\mathbb{R}_{\geq 0})^n$ spanned by the standard basis of $\mathbb{R}^n$. Using this we can define the *geometric realization* of F

$$|F| := \operatorname*{colim}_{(\mathfrak{Cone}|F)} (\mathbb{R}_{\geq 0})^n = \operatorname*{colim}_{[n] \in \mathfrak{Cone}} F_n \times (\mathbb{R}_{\geq 0})^n$$

This is entirely analogous to the geometric realization functor for semisimplicial sets.

The map $\mathbb{R}^k_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ corresponding to a map $[k] \to [n]$ in $\mathfrak{Cone}$ takes integral points $\mathbb{Z}^k_{\geq 0} \subset \mathbb{R}^k_{\geq 0}$ to $\mathbb{Z}^n_{\geq 0}$. For any fan $F_\bullet$, we define the set of *integral points* of $|F_\bullet|$ to be the image of the defining maps $\mathbb{Z}^n_{\geq 0} \to |F_\bullet|$. Alternatively, the set of non-zero integral points of $|F_\bullet|$ is in bijection with the set of rays, where $\sigma \in F_1$ is identified with the image of $1 \in \mathbb{R}_{\geq 0}$ under the map $\mathbb{R}_{\geq 0} \to |F_\bullet|$ corresponding to σ.

Given a map $[k] \to [n]$ in $\mathfrak{Cone}$, the corresponding linear map $\phi : \mathbb{R}^k \to \mathbb{R}^n$ is injective and equivariant with respect to scaling by $\mathbb{R}^\times_{>0}$. ϕ descends to a an injective map $\Delta^{k-1} \to \Delta^{n-1}$, where $\Delta^{n-1} = ((\mathbb{R}_{\geq 0})^n - \{0\}) / (\mathbb{R}_{>0})^\times$ is the standard $(n-1)$-simplex realized as the space of rays in $(\mathbb{R}_{\geq 0})^n$. Thus for any $F \in \mathrm{Fan}$ we have a functor $(\mathfrak{Cone}|F) \to \mathrm{Top}$ assigning $\xi \in F_n$ to Δ_{n-1}. We define the *projective realization* of F to be

$$\mathbb{P}(F) := \operatorname*{colim}_{\xi \in (\mathfrak{Cone}|F)} \Delta_\xi = \operatorname*{colim}_{[n] \in \mathfrak{Cone}} F_n \times \Delta^{n-1}$$

where the notation Δ_ξ denotes a copy of the standard $n-1$ simplex formally indexed by $\xi \in F_n$.[5] We refer to the map $\xi : \Delta_\xi \to \mathbb{P}(F_\bullet)$ as a *rational simplex* of $\mathbb{P}(F_\bullet)$ and when $n = 1$ we call this a rational point of $\mathbb{P}(F_\bullet)$.

[5] When we wish to emphasize the dimension of the simplex, we will denote it Δ^{n-1}_ξ.

Example 3.2. For a representable fan, $h_{[n]}(\bullet) = \mathrm{Hom}_{\mathfrak{Cone}}(\bullet, [n])$, the category $(\mathfrak{Cone}|h_{[n]})$ has a single terminal object, the identity map on $[n]$. Hence $|h_{[n]}| \simeq (\mathbb{R}_{\geq 0})^n$ and $\mathbb{P}(h_{[n]}) \simeq \Delta^{n-1}$. For any other fan $F_\bullet$ and $\xi \in F_n$, the corresponding map on geometric realizations $\mathbb{P}(h_{[n]}) \to \mathbb{P}(F_\bullet)$ is the rational simplex $\xi : \Delta_\xi \to \mathbb{P}(F_\bullet)$.

Example 3.3. One can define a fan $F_\bullet$ whose cones are the same as the cones of $h_{[2]}$, except that F_1 is the quotient of $h_{[2]}([1])$ by the equivalence relation which identifies the two boundary rays (corresponding the homomorphisms $\phi(e_0) = e_1$ and $\phi(e_0) = e_0$). Then $|F_\bullet|$ has the shape of a rolled paper cone, $\mathbb{P}(F_\bullet)$ is a circle, and the rational simplex $\mathbb{P}(h_{[2]}) \to \mathbb{P}(F_\bullet)$ is the non-injective map from the closed unit interval to the circle obtained by identifying its endpoints.

Remark 3.4. As an alternative description of $\mathbb{P}(F_\bullet)$, we observe that injective maps $(\mathbb{R}_{\geq 0})^k \to (\mathbb{R}_{\geq 0})^n$ arising in the construction of $|F|$ are equivariant with respect to the action of $\mathbb{R}^\times_{>0}$ on $(\mathbb{R}_{\geq 0})^n$ by scalar multiplication, and thus $|F|$ has a canonical continuous $\mathbb{R}^\times_{>0}$-action. Commuting colimits shows that

$$\mathbb{P}(F) \simeq (|F| - \{*\})/\mathbb{R}^\times_{>0}$$

as topological spaces, where $* \in |F|$ is the cone point corresponding to the origin in each copy of $(\mathbb{R}_{\geq 0})^n$.

The terminology of formal fans and cones is motivated by the following construction, which establishes a relationship between formal fans and the classical notion of a fan in a vector space.

Construction 3.5. A subset $K \subset \mathbb{R}^N$ which is invariant under multiplication by $\mathbb{R}^\times_{\geq 0}$ is called a *cone in* $\mathbb{R}^N$. Given a set of cones $K_\alpha \subset \mathbb{R}^N$, we define

$$R_n(\{K_\alpha\}) := \left\{ \begin{array}{c} \text{injective homomorphisms } \phi : \mathbb{Z}^n \to \mathbb{Z}^N \text{ s.t.} \\ \exists \alpha \text{ s.t. } \phi(e_i) \subset K_\alpha, \forall i \end{array} \right\} \qquad (12)$$

These sets naturally form an object $R_\bullet(\{K_\alpha\}) \in \mathrm{Fan}$.

Remark 3.6. We use the phrase *classical fan* to denote a collection of rational polyhedral cones in $\mathbb{R}^N$ such that a face of any cone is also in the collection, and the intersection of two cones is a face of each. If $K_\alpha \subset \mathbb{R}^N$ are the cones of a classical fan Σ, it is possible to reconstruct the partially ordered set of cones in Σ partially ordered by inclusion from the data of $R_\bullet(\{K_\alpha\})$. Likewise one can recover the original $K_\alpha \subset \mathbb{R}^n$ from the data of the inclusion $R_\bullet(\{K_\alpha\}) \subset R_\bullet(\{\mathbb{R}^n\})$.

Lemma 3.7. *Let $K_\alpha \subset \mathbb{R}^N$ be a finite collection of cones, each of which is a finite union of rational polyhedral cones. Then the canonical map $|R_\bullet(\{K_\alpha\})| \to \bigcup K_\alpha$ is a homeomorphism. Furthermore, $\mathbb{P}(R_\bullet(\{K_\alpha\})) \simeq S^{N-1} \cap \bigcup_\alpha K_\alpha$ via the evident quotient map $\mathbb{R}^N - \{0\} \to S^{N-1}$.*

Proof. By considering all intersections of rational polyhedral cones and their faces appearing in the description of some K_α, we may assume there is a classical fan $\Sigma = \{\sigma_i\}$ in $\mathbb{R}^N$ such that each σ_i is contained in some K_α and each K_α is the union of some collection of σ_i. By further refinement we may assume that Σ is simplicial, and that the ray generators of each σ_i form a basis for the lattice generated by $\sigma_i \cap \mathbb{Z}^N$. Consider the formal fans $F_\bullet = R_\bullet(\{K_\alpha\})$ and $F'_\bullet = R_\bullet(\{\sigma_i\})$. By hypothesis $F'_\bullet$ is a subfunctor of $F_\bullet$. Hence we have a functor of comma categories $(\mathfrak{Cone}|F') \to (\mathfrak{Cone}|F)$ and thus a map of topological spaces $|F'| \to |F|$ which commutes with the map to $\mathbb{R}^N$.

We can reduce to the case when $\{K_\alpha\} = \{\sigma_i\}$. Indeed the map $|F'| \to |F|$ is surjective because any point in K_α lies in some σ_i. If the composition $|F'| \to |F| \to \bigcup_\alpha K_\alpha = \bigcup \sigma_i$ were a homeomorphism it would follow that $|F'| \to |F|$ was injective as well, and one could use the inverse of $|F'| \to \bigcup_\alpha K_\alpha$ to construct and inverse for $|F| \to \bigcup_\alpha K_\alpha$.

For a single simplicial cone $\sigma \subset \mathbb{R}^N$ of dimension n whose ray generators form a basis for the lattice generated by $\sigma \cap \mathbb{Z}^N$, we have $R_\bullet(\sigma) \subset R_\bullet(\mathbb{R}^N)$ is isomorphic to $h_{[n]}(\bullet)$. The terminal object of $(\mathfrak{Cone}|h_{[n]})$ corresponds to the linear map $\mathbb{R}^n \to \mathbb{R}^N$ mapping the standard basis vectors to the ray generators of σ, and thus $|R_\bullet(\sigma)| \to \sigma$ is a homeomorphism.

Let $\sigma_1^{\max}, \ldots, \sigma_r^{\max} \in \Sigma$ be the cones which are maximal with respect to inclusion and let n_i be the dimension of each. Let $\sigma'_{ij} := \sigma_i^{\max} \cap \sigma_j^{\max} \in \Sigma$, and let n_{ij} denote its dimension. Then by construction

$$F := R_\bullet(\{\sigma_i\}) \simeq \operatorname{coeq}\left(\bigsqcup_{i,j} h_{n_{ij}} \rightrightarrows \bigsqcup_i h_{n_i} \right)$$

as functors $\mathfrak{Cone}^{op} \to \mathrm{Set}$. Our geometric realization functor commutes with colimits, so it follows that

$$|F| = \operatorname{coeq}\left(\bigsqcup_{i,j} \sigma'_{ij} \rightrightarrows \bigsqcup_i \sigma_{ij}^{\max} \right)$$

Which is homeomorphic to $\bigcup \sigma_i$ under the natural map $|F| \to \mathbb{R}^N$. The final claim follows from the fact that $\mathbb{P}(F) \simeq (|F| - \{0\})/\mathbb{R}^\times_{>0}$. $\square$

Example 3.8. Objects of Fan describe a wider variety of structures than classical fans. For instance if K_1 and K_2 are two simplicial cones which intersect but do not meet along a common face, then $R_\bullet(K_1, K_2)$ will not be equivalent to $R_\bullet(\{\sigma_i\})$ for any classical fan $\Sigma = \{\sigma_i\}$.

Example 3.9. The category Fan is broad enough to include some pathological examples. For instance, if $K \subset \mathbb{R}^3$ is the cone over a circle which is not contained in a linear subspace, then $|R_\bullet(K)|$ consists of the rational rays of K equipped with the discrete topology and is not homeomorphic to K.

Another example: if $K \subset \mathbb{R}^2$ is a convex cone generated by two irrational rays, then $|R_\bullet(K)|$ is the interior of that cone along with the origin. There are also examples of fans whose geometric realizations are not Hausdorff, such as multiple copies of the standard cone in $\mathbb{R}^2$ glued to each other along the set of rational rays.

3.1.1. *Some useful lemmas.*

Lemma 3.10. *Let F be a fan and let $x \in \mathbb{P}(F)$ be a point lying in the image of two rational simplices $\Delta_{\sigma_i} \to \mathbb{P}(F)$. Then there is a $\xi \in (\mathfrak{Cone}|F)$ with maps $\xi \to \sigma_i$ such that x lies in the image of $\Delta_\xi \to \mathbb{P}(F)$.*

Proof. By the definition of $\mathbb{P}(F)$ as a colimit, it is the quotient of the set $\bigsqcup_{\sigma \in (\mathfrak{C}|F)} \Delta_\sigma$ by the equivalence relation generated by $x \sim \phi(x)$ for the maps $\phi : \Delta_{\sigma_1} \to \Delta_{\sigma_2}$ induced by morphisms in $(\mathfrak{Cone}|F)$. The lemma follows from an alternative description of this equivalence relation: points $x_1 \in \Delta_{\sigma_1}$ and $x_2 \in \Delta_{\sigma_2}$ are equivalent if there exists a diagram in $(\mathfrak{Cone}|F)$ of the form $\sigma_1 \leftarrow \xi \to \sigma_2$ and a point $y \in \Delta_\xi$ mapping to x_1 and x_2. This relation is clearly symmetric and reflexive, so we must show transitivity.

Consider a diagram of the form $\sigma_1 \leftarrow \xi_1 \to \sigma_2 \leftarrow \xi_2 \to \sigma_3$ and points $y_i \in \Delta_{\xi_i}$ and $x_i \in \Delta_{\sigma_i}$ which are identified by these maps. The corresponding maps $\Delta_{\xi_1} \hookrightarrow \Delta_{\sigma_2}$ and $\Delta_{\xi_2} \hookrightarrow \Delta_{\sigma_2}$ are embeddings of simplices with rational vertices, and hence their intersection in Δ_{σ_2} is a polyhedron with rational vertices, which contains the point $y_1 = y_2 = x_2$. It follows that there is some simplex with rational vertices in Δ_{σ_2} containing x_2, which defines a $\xi' \in (\mathfrak{Cone}|F)$ mapping to ξ_1 and ξ_2 and a point $y' \in \Delta_{\xi'}$ mapping to y_1 and y_2. Thus we have a diagram of the form $\sigma_1 \leftarrow \xi' \to \sigma_3$ exhibiting the equivalence $x_1 \sim x_3$. □

We can formulate the previous lemma a bit more generally.

Lemma 3.11. *For any maps of fans $F, F' \to G$, the canonical map $\mathbb{P}(F \times_G F') \to \mathbb{P}(F) \times_{\mathbb{P}(G)} \mathbb{P}(F')$ is surjective.*

Proof. Let $S = \mathbb{P}(F) \times_{\mathbb{P}(G)} \mathbb{P}(F')$. Any point in S can be represented by a pair of rational simplices $\xi \in F_n$ and $\xi' \in F'_{n'}$ and two points $x \in \Delta_\xi$ and $x' \in \Delta_{\xi'}$ which map to the same point in $\mathbb{P}(G)$. Lemma 3.10 implies that we can find subcones of ξ and ξ' which map to the same cone in G and whose corresponding rational simplex contains a point which maps to both x and x'. This defines a point in $\mathbb{P}(F \times_G F')$ mapping to our original point in S. □

Lemma 3.12. *If $F \to G$ is a map of fans such that $\mathbb{P}(F) \to \mathbb{P}(G)$ is surjective, then for any fan F' and map $F' \to G$, the map $\mathbb{P}(F' \times_G F) \to \mathbb{P}(F')$ is surjective.*

Proof. This is an immediate corollary of the previous lemma and the fact that surjective maps of topological spaces are stable under base change. □

Lemma 3.13. *If $F_\bullet \to G_\bullet$ is an injective (respectively surjective) map of fans, then so is $\mathbb{P}(F_\bullet) \to \mathbb{P}(G_\bullet)$.*

Proof. Lemma 3.10 implies that if two points in $x_0, x_1 \in \mathbb{P}(F_\bullet)$, represented by cones σ_1, σ_2 in $F_\bullet$ with a choice of real ray in each, map to the same point in $\mathbb{P}(G_\bullet)$, then there is a cone and real ray in $G_\bullet$ which is a subcone of the image of both σ_1 and σ_2. Because the map $F_\bullet \to G_\bullet$ is injective and its image is closed under taking subcones, this new cone must lie in $F_\bullet$ as well. The corresponding claim for surjectivity is immediate from the definition of $\mathbb{P}(G_\bullet)$. □

Corollary 3.13.1. *Let $f : F_\bullet \to G_\bullet$ be a map of fans and let $\mathbb{P}(f) : \mathbb{P}(F_\bullet) \to \mathbb{P}(G_\bullet)$ be the induced map. Then*

(1) $\mathrm{im}(\mathbb{P}(f)) = \mathbb{P}(\mathrm{im}(f))$ *where* im *denotes the image of a map of fans or topological spaces, and*
(2) if $G'_\bullet \to G_\bullet$ is a map and $f' : F_\bullet \times_{G_\bullet} G'_\bullet \to G'_\bullet$ is the base change, then $\mathrm{im}(\mathbb{P}(f'))$ *is the preimage of* $\mathrm{im}(\mathbb{P}(f)) \subset \mathbb{P}(G_\bullet)$ *in* $\mathbb{P}(G'_\bullet)$.

Proof. The first claim is an immediate consequence of Lemma 3.13 and the definition of the image in both categories as the unique factorization of a map as a surjection followed by an injection. The second then follows from this and Lemma 3.12. □

Note that as a consequence of part (2) of this corollary, for any two subfans $F_\bullet, F'_\bullet \subset G_\bullet$, we have $\mathbb{P}(F_\bullet) \cap \mathbb{P}(F'_\bullet) = \mathbb{P}(F_\bullet \cap F'_\bullet)$ as subsets of $\mathbb{P}(G_\bullet)$.

3.1.2. *Boundedness of fans.* We shall sometimes restrict to a class of fans whose projective realizations exhibit fewer pathologies.

Definition 3.14. We say that a fan is *bounded* if $\mathbb{P}(F)$ is covered by finitely many rational simplices, and a map $F \to G$ is *bounded* if for any cone $h_{[n]} \to G$, the fiber product $F \times_G h_{[n]}$ is bounded. We say that a fan F is *quasi-separated* if for any pair of cones $h_{[n_0]}, h_{[n_1]} \to F$, the fan $h_{[n_0]} \times_F h_{[n_1]}$ is bounded.

Remark 3.15. Note that if $K \subset \mathbb{R}^n$ is a rational polyhedral cone which is not simplicial, then $R_\bullet(\{K\})$ does not admit a surjection from a finite union of representable fans $h_{[n]}$, but $R_\bullet(\{K\})$ is still bounded in the above sense. On the other hand, a surjective map of fans $F_\bullet \to G_\bullet$ induces a surjection $\mathbb{P}(F_\bullet) \to \mathbb{P}(G_\bullet)$. Therefore $F_\bullet$ is bounded if it admits a surjection $\bigsqcup_{i=1}^N h_{[n_i]} \to F_\bullet$.

Lemma 3.16. *A fan G is quasi-separated if and only if for any bounded fan F, the map $F \to G$ is bounded.*

Proof. The "if" direction is immediate from the definitions, taking $F = h_{[n]}$ in the statement of the lemma. For the "only if" direction, let $h_{[m]} \to G$ be a cone, and let $\bigsqcup_i h_{[n_i]} \to F$ be a map which is surjective after applying $\mathbb{P}(-)$. Then $\bigsqcup_i \mathbb{P}(h_{[n_i]} \times_G h_{[m]}) \to \mathbb{P}(F \times_G h_{[m]})$ is surjective by Lemma 3.12, so the latter is bounded if the former is. □

Remark 3.17. The notion of boundedness is analogous to quasi-compactness in algebraic geometry, and the notion of a quasi-separated fan is analogous to that of a quasi-separated stack, which is a stack $\mathfrak{X}$ such that for any pair of maps $\mathrm{Spec}(A_i) \to \mathfrak{X}$ the fiber produce $\mathrm{Spec}(A_1) \times_{\mathfrak{X}} \mathrm{Spec}(A_2)$ is quasi-compact.

An algebraic stack is quasi-separated if and only if the diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is quasi-compact. For fans, however, the condition that $F \to F \times F$ is bounded is weaker than F being quasi-separated. The equivalence fails because $h_{[n]} \times h_{[m]}$ is not bounded for $m \neq n$, which is related to the failure of $\mathbb{P}(-)$ to commute with products. Presumably this deficiency can be addressed by reformulating the theory using a category $\mathfrak{Cone}$ which allows degenerate maps between cones, just as the theory of simplicial sets addresses similar deficiencies in the theory of semisimplicial sets. We hope to return to this point elsewhere.

Lemma 3.18. *Let $\psi : F \to G$ be a bounded map of fans. Then $\psi : \mathbb{P}(F) \to \mathbb{P}(G)$ is a closed map whose fibers are finite sets. The image of this map restricted to any rational simplex of $\mathbb{P}(G)$ is a finite union of closed rational sub-simplices.*

Proof. Given a rational simplex $\xi : \Delta^n_\xi \to \mathbb{P}(G)$, one can find a surjection from a finite union of rational simplices

$$\bigsqcup_i \Delta_{\sigma_i} \twoheadrightarrow \mathbb{P}(F \times_G h_{[n+1]}) \twoheadrightarrow \mathbb{P}(F) \times_{\mathbb{P}(G)} \Delta^n_\xi,$$

by Lemma 3.11. Let p_1 and p_2 denote the projection from $\bigsqcup_i \Delta_{\sigma_i}$ to $\mathbb{P}(F)$ and Δ^n_ξ respectively. Then for any closed set $S \subset \mathbb{P}(F)$,

$$\xi^{-1}(\psi(S)) = p_2(p_1^{-1}(S)).$$

All three claims follow from the fact that p_2 is a closed map with finite fibers. □

Lemma 3.19. *If $\psi : F_\bullet \to G_\bullet$ is a bounded map of fans, then $\mathbb{P}(\psi) : \mathbb{P}(F_\bullet) \to \mathbb{P}(G_\bullet)$ is injective (resp. surjective) if and only if it is injective (resp. surjective) on rational points.*

Proof. Assume that $\mathbb{P}(\psi)$ is surjective on rational points. Lemma 3.18 implies that the image of $\mathbb{P}(\psi)$ restricted to each rational simplex $\Delta_\xi \to \mathbb{P}(G_\bullet)$ is a finite union of rational subsimplices. If this finite union contains all rational points of Δ_ξ, then it must cover Δ_ξ, so $\mathbb{P}(\psi)$ is surjective. □

Corollary 3.19.1. *If G is a quasi-separated fan and $F \subset G$ is a bounded sub-fan, then $\mathbb{P}(F) \to \mathbb{P}(G)$ is a closed embedding, and its preimage under any rational simplex $\Delta_\xi \to \mathbb{P}(G)$ is a finite union of rational sub-simplices.*

Proof. Combine Lemma 3.18 and Lemma 3.13. □

Proposition 3.20. *Let F be a quasi-separated fan. Then any rational simplex $\Delta_\xi \to \mathbb{P}(F)$ is a closed map, and $\mathbb{P}(F)$ is a Hausdorff space whose topology is compactly generated by the images of rational simplices.*

Proof. The fact that an rational simplex is a closed map is Lemma 3.18. Once we know this, we see from the definition of the colimit topology that a subset of $\mathbb{P}(F)$ is closed iff its intersection with the compact subset $\xi(\Delta_\xi)$ is closed for all rational simplices, so the topology on $\mathbb{P}(F)$ is compactly generated.

We show that $\mathbb{P}(F)$ is Hausdorff by showing that the diagonal $\mathbb{P}(F) \hookrightarrow \mathbb{P}(F) \times \mathbb{P}(F)$ is a closed subset. Writing $F = \bigcup_\alpha F_\alpha$ as a union of bounded subfans, we have that $\mathbb{P}(F) = \bigcup \mathbb{P}(F_\alpha)$, because $\mathbb{P}(-)$ commutes with colimits. This colimit is filtered, so $\mathbb{P}(F) \times \mathbb{P}(F) = \bigcup_{\alpha,\beta} \mathbb{P}(F_\alpha) \times \mathbb{P}(F_\beta)$ with the colimit topology. It therefore suffices to show that for each α and β the preimage of the diagonal under the map

$$\mathbb{P}(F_\alpha) \times \mathbb{P}(F_\beta) \to \mathbb{P}(F) \times \mathbb{P}(F)$$

is a closed subset.

The fans F_α are bounded by hypotheses and quasi-separated because they are sub-fans of a quasi-separated fan, so $\mathbb{P}(F_\alpha)$ admit closed surjections from a finite disjoint union of rational simplices. This allows one to reduce the claim to showing that for any two rational simplices $\xi_i : \Delta_{\xi_i} \to \mathbb{P}(F)$, the preimage of the diagonal under the map $\Delta_{\xi_1} \times \Delta_{\xi_2} \to \mathbb{P}(F) \times \mathbb{P}(F)$ is a closed subset. We can identify this preimage with $\Delta_{\xi_1} \times_{\mathbb{P}(F)} \Delta_{\xi_2}$, which by Lemma 3.11 is covered by finitely many rational simplices because $h_{[n_1]} \times_F h_{[n_2]}$ is bounded. Hence the preimage of the diagonal in $\Delta_{\xi_1} \times \Delta_{\xi_2}$ is closed. □

In fact we can be even more precise:

Lemma 3.21. *Let G be a quasi-separated fan. Then for any finite collection of cones $\xi_i \in G_{n_i}$, the image of $\bigsqcup \Delta_{\xi_i} \to \mathbb{P}(G)$ is the quotient of $\bigsqcup \Delta_{\xi_i}$ by a closed equivalence relation*

$$R \subset (\bigsqcup_i \Delta_{\xi_i}) \times (\bigsqcup_i \Delta_{\xi_i})$$

which is a union of finitely many closed subsets $\Delta^k \hookrightarrow (\bigsqcup_i \Delta_{\xi_i}) \times (\bigsqcup_i \Delta_{\xi_i})$ obtained from a pair of rational sub-simplices $\Delta^k \hookrightarrow \Delta_{\xi_i}$ and $\Delta^k \hookrightarrow \Delta_{\xi_j}$.

Proof. Writing $\bigsqcup_i \Delta_{\xi_i} = \mathbb{P}(\bigsqcup h_{[n_i]})$, $R = \mathbb{P}(\bigsqcup h_{[n_i]}) \times_{\mathbb{P}(G)} \mathbb{P}(\bigsqcup h_{[n_i]})$ is covered by the image of

$$\bigsqcup_{i,j} \mathbb{P}(h_{[n_i]} \times_F h_{[n_j]}) \to \mathbb{P}(\bigsqcup h_{[n_i]}) \times \mathbb{P}(\bigsqcup h_{[n_i]}),$$

by Lemma 3.11, which in turn is covered by finitely many rational simplices by hypotheses. For each rational simplex $\Delta^k_\sigma \to \mathbb{P}(h_{[n_i]} \times_F h_{[n_j]})$, the projection onto each factor $\mathbb{P}(\bigsqcup h_{[n_i]})$ is a closed embedding, and hence so is the map from Δ^k_σ to the product. □

Corollary 3.21.1. *Let $\psi : F_\bullet \to G_\bullet$ be a map of fans, with $F_\bullet$ quasi-separated. Then $\mathbb{P}(\psi) : \mathbb{P}(F) \to \mathbb{P}(G)$ is injective if and only if it is injective on rational points.*

Proof. Assume that $\mathbb{P}(\psi)$ is injective on rational points. Consider two points $x, x' \in \mathbb{P}(F)$ which map to the same point in $\mathbb{P}(G)$. Using Lemma 3.10 one can find two cones $\xi, \xi' \in F_n$ which map to the same cone in G_n along with a point $y \in \Delta^{n-1}$ such that x and x' are the image of y under the two rational simplices $\xi, \xi' : \Delta^{n-1} \to \mathbb{P}(F)$. By Lemma 3.21 the image of the map $\Delta^{n-1}_\xi \sqcup \Delta^{n-1}_{\xi'} \to \mathbb{P}(F)$ is the quotient of these two simplices by a closed equivalence relation $R \subset (\Delta^{n-1}_\xi \sqcup \Delta^{n-1}_{\xi'}) \times (\Delta^{n-1}_\xi \sqcup \Delta^{n-1}_{\xi'})$. Because $\mathbb{P}(\psi)$ is injective on rational points, the relation R identifies corresponding rational points on Δ^{n-1}_ξ and $\Delta^{n-1}_{\xi'}$, but because it is closed it must identify non-rational points as well. Hence $x = x' \in \mathbb{P}(F)$. □

3.1.3. *Realizable sets.*

Definition 3.22. Let $F_\bullet$ be a fan. We say that a subset of $\mathbb{P}(F_\bullet)$ is *realizable* if it is of the form $\mathbb{P}(G_\bullet) \subset \mathbb{P}(F_\bullet)$ for some sub-fan $G_\bullet \subset F_\bullet$.

By part (1) of Corollary 3.13.1, a subset of $\mathbb{P}(F_\bullet)$ is realizable if and only if it is a union of the images of some collection of rational simplices $\Delta_\xi \to \mathbb{P}(F)$. In addition, a subset $U \subset \mathbb{P}(F_\bullet)$ is realizable if and only if its preimage under any rational simplex $\Delta_\xi \to \mathbb{P}(F_\bullet)$ is realizable.

Example 3.23. Any open subset of $\mathbb{P}(F_\bullet)$ is realizable, because any open subset of a simplex Δ_ξ is a union of sub-simplices.

Note that for a realizable set $U \subset \mathbb{P}(F_\bullet)$, there is a *canonical* subfan $G_\bullet \subset F_\bullet$ for which $\mathbb{P}(G_\bullet) = U$ – $G_\bullet$ consists of all cones ξ for which $\Delta_\xi \to \mathbb{P}(F_\bullet)$ factors through U. The canonical fan realizing U is also the largest fan realizing U.

Lemma 3.24. *The set of realizable subsets of* $\mathbb{P}(F_\bullet)$ *is closed under finite intersections and arbitrary unions.*

Proof. This follows from the fact that $\mathbb{P}(-)$ commutes with arbitrary colimits and finite intersections of subfans (see Corollary 3.13.1). □

3.1.4. *Convexity and fans.* Let $U \subset \mathbb{R}^n_{\geq 0}$ be a convex subset. For any map $[m] \to [n]$, the preimage in $\mathbb{R}^m_{\geq 0}$ of U under the corresponding linear map $\mathbb{R}^m_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ is also convex. For any $F_\bullet \in \mathrm{Fan}$, the space $|F_\bullet|$ consists of copies of $\mathbb{R}^n_{\geq 0}$ glued along linear maps.

Definition 3.25. For $F_\bullet \in \mathrm{Fan}$, a subset $U \subset |F_\bullet|$ is said to be *locally convex* if for any cone $\sigma \in F_n$, the preimage of U under the corresponding map $\mathbb{R}^n_{\geq 0} \to |F_\bullet|$ is convex. Likewise a subset $U \subset \mathbb{P}(F_\bullet)$ is said to be *locally convex* if its preimage under the projection $|F_\bullet| - \{*\} \to \mathbb{P}(F_\bullet)$ is convex. We say that $U \subset \mathbb{P}(F_\bullet)$ is *convex* if it is locally convex, realizable, and any two rational points are connected by a rational 1-simplex. [6]

[6]The conventional notion of a convex subset of the standard simplex Δ^n does not include the realizablity condition. Realizability is necessary to avoid certain pathologies

Many of the formal properties of convex sets are local and thus generalize to locally convex subsets of $|F_\bullet|$ and $\mathbb{P}(F_\bullet)$. For instance, the whole space is a locally convex subset, and the intersection of any collection of locally convex subsets is locally convex, so for every $S \subset \mathbb{P}(F_\bullet)$ or $S \subset |F_\bullet|$ there is a unique smallest locally convex subset containing S, which we call the *locally convex hull* of S.

Lemma 3.26. *A subset $U \subset \mathbb{P}(F_\bullet)$ is locally convex if and only if for any rational simplex $\Delta^n \to \mathbb{P}(F_\bullet)$ and points $x_1, x_2 \in \Delta \to \mathbb{P}(F_\bullet)$ mapping to U, every point in the line segment joining x_1 and x_2 in Δ^n also maps to U.*[7]

Proof. From the definition and the fact that colimits commute with quotients reduces the lemma to the following claim: if $\pi : \mathbb{R}^{n+1}_{\geq 0} - \{0\} \to \Delta^n$ is the quotient projection and $\iota : \Delta^n \hookrightarrow \mathbb{R}^{n+1}_{\geq 0}$ is the standard embedding, then for any subset $U \subset \Delta^n$ the subset $\iota(U) \subset \mathbb{R}^{n+1}$ is convex if and only if $\pi^{-1}(U) \subset \mathbb{R}^{n+1}$ is convex. This follows from the fact that the image of a line segment under π is a line segment in Δ^n. $\square$

Remark 3.27. If $\{K_\alpha\}$ is a collection of cones in $\mathbb{R}^n$ and $U \subset |R_\bullet(\{K_\alpha\})| \simeq \bigcup_\alpha K_\alpha$ is convex in the usual sense for subsets of $\mathbb{R}^n$, then it is locally convex in the sense of Definition 3.25. The converse is not true, though. For example if $\bigcup K_\alpha = \mathbb{R}^n$, then the open subset $\mathbb{R}^n - \{0\} \subset \mathbb{R}^n$ is locally convex.

Recall that a function Φ defined on a convex subset $U \subset \mathbb{R}^n_{\geq 0}$ is concave if $\Phi(tx + (1-t)y) \geq t\Phi(x) + (1-t)\Phi(y)$ for distinct points $x, y \in U$ and $t \in (0,1)$, and it is strictly concave if strict inequality holds. This notion is preserved by restriction along a linear map $\mathbb{R}^m_{\geq 0} \to \mathbb{R}^n_{\geq 0}$, so we may define a function $\Phi(x)$ on a locally convex subset $U \subset |F_\bullet|$ to be *(strictly) concave* if for every cone $\sigma \in F_n$ the restriction of $\Phi(x)$ along the canonical map $\mathbb{R}^n_{\geq 0} \to |F_\bullet|$ is a (strictly) concave function on the preimage of U.

Warning 3.28. The line segment joining two points in Δ^n does not have a canonical parameterization. Consider two vectors $v_0, v_1 \in \mathbb{R}^{n+1}_{\geq 0} - \{0\}$ with distinct image under the quotient map $\pi : \mathbb{R}^{n+1}_{\geq 0} - \{0\} \to \Delta^n$. Define $v_t = tv_1 + (1-t)v_0$, so that $\{\pi(v_t) | t \in [0,1]\}$ is the line segment joining the two points in Δ^n. If we instead let $v_i' = \lambda_i v_i$ for $i = 0, 1$ and define $v_t' = tv_1' + (1-t)v_0'$, then

$$\pi(v_t') = \pi\left(\frac{t\alpha_1}{t\alpha_1 + (1-t)\alpha_0} v_1 + \frac{(1-t)\alpha_0}{t\alpha_1 + (1-t)\alpha_0} v_0\right) = \pi(v_\tau),$$

that are introduced by only considering only line segments with rational endpoints. For instance, consider two cones $\sigma_1, \sigma_2 \subset \mathbb{R}^2$ which are disjoint away from the origin, and let $F_\bullet = R_\bullet\{\sigma_1, \sigma_2\}$. Let $U \subset \mathbb{P}(F_\bullet)$ be the union of a non-rational point in Δ_{σ_1} and a non-rational point in Δ_{σ_2}. This U satisfies the definition of convexity without the realizability condition, but it does not have the usual properties of a convex set.

[7]For the purposes of defining the line segment joining two points, we regard Δ^n as embedded in $\mathbb{R}^{n+1}_{\geq 0}$ as the points $(p_0, \ldots, p_n)$ where $p_0 + \cdots + p_n = 1$. This notion of line segment does not depend on the choice of affine linear embedding $\Delta^n \hookrightarrow \mathbb{R}^m$.

where $\tau = t\alpha_1/(t\alpha_1 + (1-t)\alpha_0)$. Both $\pi(v'_t)$ and $\pi(v_\tau)$ are parameterizations of the same line segment in Δ^n, but the change of coordinate from τ to t is not affine-linear and does not preserve concave functions, so *there is no notion of a concave function on* $\mathbb{P}(F_\bullet)$. Informally, while $|F_\bullet|$ has a canonical affine structure, the space $\mathbb{P}(F_\bullet)$ only has a canonical convex structure.

Definition 3.29. Let $U \subset \mathbb{P}(F_\bullet)$ be a locally convex subset. We say that a function $\Phi : U \to \mathbb{R}$ is *quasi-concave* if for any rational simplex $\Delta^n \to \mathbb{P}(F_\bullet)$ and any three distinct points $x_0, x_1, x_2 \in \Delta^n$ mapping to U such that x_1 lies in the interior of the line segment joining x_0 and x_2, we have

$$\Phi(x_1) \geq \min(\Phi(x_0), \Phi(x_2)).$$

We say Φ is *strictly quasi-concave* if this inequality holds strictly for all such points.

Note that by Lemma 3.26, Φ is quasi-concave if and only if $\Phi^{-1}((c,\infty)) \subset U$ is locally convex for any $c \in \mathbb{R}$. Another consequence of Lemma 3.26 is that Φ is quasi-concave if and only if its restriction along the composition $\mathbb{R}^n_{\geq 0} - \{0\} \to \Delta^{n-1}_\xi \to \mathbb{P}(F_\bullet)$ is quasi-concave on its domain of definition for any rational simplex ξ. One nice property of quasi-concave functions is that if Φ is quasi-concave and $m : \mathbb{R} \to \mathbb{R}$ is monotone increasing, the $m \circ \Phi$ is quasi-concave.

Lemma 3.30. *Let Φ be a continuous quasi-concave function on a convex subspace $U \subset \mathbb{P}(F_\bullet)$. Then any rational point which is a local maximum for Φ is a global maximum. If Φ is strictly quasi-concave, then at most one rational point is a local maximum for Φ.*

Proof. Let $x \in U$ be a rational point which is a local maximum. If $y \in U$ is a rational point, then we can find a rational 1-simplex in U which connects x and y. The restriction of Φ to this 1-simplex is a quasi-concave function with a local maximum at the endpoint corresponding to x, and it follows that x is a global maximum along this 1-simplex, so $\Phi(x) \geq \Phi(y)$. Now if $y \in U$ is any point, the fact that U is realizable implies that we can find a sequence of rational points converging to y, and then the fact that Φ is continuous implies that $\Phi(x) \geq \Phi(y)$ as well. Hence x is a global maximum.

The definition of strictly quasi-concave function implies that there is at most one maximum obtained along any rational 1-simplex. This implies the uniqueness of a rational point maximizing Φ, because any two rational points can be joined by a rational 1-simplex. □

This will suffice for our purposes, but we note that without further hypotheses, Φ can achieve a maximum at more than one non-rational point. For instance, let $F_\bullet$ be the quotient of $h_{[2]} \sqcup h_{[2]}$ by the equivalence relation identifying corresponding 1-cones in each summand. Then $\mathbb{P}(F_\bullet)$ is two copies of Δ^1 glued along corresponding pairs of rational points, and $U = \mathbb{P}(F_\bullet)$ is convex. Choose a strictly quasi-concave function on Δ^1 which achieves its maximum at a non-rational point. Then this function defines a strictly quasi-concave function $\Phi : U \to \mathbb{R}$ which has two global maxima. For completeness we note the following

Lemma 3.31. *Let $F_\bullet$ be a quasi-separated fan, and let $U \subset \mathbb{P}(F_\bullet)$ be a compact convex subspace. If $\Phi : U \to \mathbb{R}$ is a continuous strictly quasi-concave function then Φ obtains a unique a local maximum, which is also a global maximum.*

Proof. For any $c \in \mathbb{R}$ which lies below the maximum value of Φ, let S_c denote the closure in U of the subset $\Phi^{-1}((c,\infty))$. Any such S_c is connected, because the open set $\Phi^{-1}((c,\infty))$ is convex. Let $x \in U$ be a local maximum for Φ, and define $S' = \Phi^{-1}([\Phi(x),\infty))$ then we have

$$S' = \bigcap_{c<\Phi(x)} \Phi^{-1}((c,\infty)) = \bigcap_{c<\Phi(x)} S_c$$

So it follows from the fact that U is Haudorff by Proposition 3.20 and compact by hypothesis that S' is connected [E, Theorem 6.1.18].

For any rational simplex $\Delta_\xi \to U$ and point $\tilde{x}$ mapping to x, the strictly quasi-concave function $\Phi|_{\Delta_\xi}$ obtains a local maximum at $\tilde{x}$ and hence $\tilde{x}$ is the unique global maximum of $\Phi|_{\Delta_\xi}$. It follows that if $\xi^{-1}(S') \subset \Delta_\xi$ contains a point in the preimage of x, it cannot contain any other point. This implies that the singleton $\{x\} \subset S'$ is a closed subset whose complement is also closed. Because S' is connected, this implies $S' = \{x\}$. Hence any local maximum is the unique global maximum. The existence of such a maximum follows from compactness of U. □

3.2. **The degeneration space $\mathscr{D}eg(\mathfrak{X}, p)$.** For any field k, we introduce the 2-category of pointed k-stacks, St_k as follows: objects are k-stacks $\mathfrak{X}$ whose inertia group is representable and separated over $\mathfrak{X}$ along with a fixed k-point $p : \operatorname{Spec} k \to \mathfrak{X}$ over $\operatorname{Spec}(k)$. A 1-morphism in this category is a 1-morphism of k-stacks $\psi : \mathfrak{X} \to \mathfrak{X}'$ along with an isomorphism $p' \simeq \psi \circ p$, subject to the constraint that $\psi_* : \operatorname{Aut}(q) \to \operatorname{Aut}(\psi(q))$ has finite kernel for all $q \in \mathfrak{X}(k)$.[8] A 2-isomorphism is a 2-isomorphism $\psi \to \tilde{\psi}$ which is compatible with the identification of marked points.

We let the point 1^n denote the canonical k-point in Θ_k^n coming from $(1,\ldots,1) \in \mathbb{A}_k^n$, and regard Θ_k^n as an object of St_k.

Lemma 3.32. *The assignment $[n] \mapsto (\Theta_k^n, 1^n)$ extends to a functor $\mathfrak{Cone} \to \mathrm{St}_k$.*

[8]When $\mathfrak{X}$ and $\mathfrak{Y}$ are algebraic stacks, we interpret this to mean the kernel of the homomorphism of group schemes $\operatorname{Aut}(q) \to \operatorname{Aut}(\psi(q))$ is a finite k-group scheme. More generally for any map $\psi : \mathfrak{X} \to \mathfrak{Y}$ whose inertia $I_\psi := \mathfrak{X} \times_{\mathfrak{X}\times_{\mathfrak{Y}}\mathfrak{X}} \mathfrak{X}$ is representable over $\mathfrak{X}$, one should require that the fiber of I_ψ over $q \in \mathfrak{X}(k)$ is a finite k-group scheme. If one would like to consider this definition for non-algebraic stacks $\mathfrak{X}$ and $\mathfrak{Y}$, then one should instead require $\ker(\operatorname{Aut}(q) \to \operatorname{Aut}(f(q)))$ to be finite after base change to an arbitrary extension k'/k.

Proof. A morphism $\phi : [k] \to [n]$ in $\mathfrak{Cone}$ is represented by a matrix of nonnegative integers ϕ_{ij} for $i = 1, \ldots, n$ and $j = 1, \ldots, k$. One has a map of stacks $\phi_* : \Theta^k \to \Theta^n$ defined by the map $\mathbb{A}^k \to \mathbb{A}^n$

$$(z_1, \ldots, z_k) \mapsto (z_1^{\phi_{11}} \cdots z_k^{\phi_{1k}}, \ldots, z_1^{\phi_{n1}} \cdots z_k^{\phi_{nk}})$$

which is intertwined by the group homomorphism $\mathbb{G}_m^k \to \mathbb{G}_m^n$ defined by the same formula. It is clear that these 1-morphisms are compatible with composition in $\mathfrak{Cone}$, and that this construction gives canonical identifications $\phi_*(1^k) \simeq 1^n \in \Theta^n(k)$, hence these are morphisms in St_k. $\square$

We call a $\mathbb{Z}^n$-weighted filtration of a k-point $f : \Theta_k^n \to \mathfrak{X}$ *non-degenerate* if the resulting homomorphism $(\mathbb{G}_m)_k^n \to \mathrm{Aut}(f(0))$ of group sheaves over $\mathrm{Spec}(k)$ has finite kernel. A non-degenerate filtration defines a map $(\Theta_k^n, 1^n) \to (\mathfrak{X}, f(1^n))$ in St_k. Morphisms between pointed k-stacks form a groupoid in general, but Remark 1.11 shows that $\mathrm{Map}_{\mathrm{St}_k}((\Theta_k^n, 1^n), (\mathfrak{X}, p))$ is equivalent to a set whenever the inertia of $\mathfrak{X}$ is representable by separated algebraic spaces, and we regard it as such.

Definition 3.33. Let $\mathfrak{X}$ be a stack with representable separated inertia over a fixed base stack B. Any k-point $p : \mathrm{Spec}(k) \to \mathfrak{X}$ induces a k-point of B and a canonical pointed k-stack $\mathfrak{X}_p := \mathrm{Spec}(k) \times_B \mathfrak{X}$. We define the *degeneration fan*

$$\mathbf{DF}(\mathfrak{X}, p)_n := \mathrm{Map}_{\mathrm{St}_k}((\Theta_k^n, 1^n), (\mathfrak{X}_p, p)), \tag{13}$$

which defines a functor $\mathfrak{Cone}^{op} \to \mathrm{Set}$ via pre-composition with the morphisms of Lemma 3.32: a morphism $\phi : [k] \to [n]$ of $\mathfrak{Cone}$ defines a map $\mathbf{DF}(\mathfrak{X}, p)_n \to \mathbf{DF}(\mathfrak{X}, p)_k$ by $f \mapsto f \circ \phi_*$. We define the *degeneration space* $\mathscr{D}eg(\mathfrak{X}, p) := \mathbb{P}(\mathbf{DF}(\mathfrak{X}, p)_\bullet)$. We will sometimes abbreviate the notation to $\mathbf{DF}(p)_\bullet$ and $\mathscr{D}eg(p)$ respectively, as the stack $\mathfrak{X}$ is implicitly specified by the map $p : \mathrm{Spec}(k) \to \mathfrak{X}$.

Concretely, elements of the set $\mathbf{DF}(\mathfrak{X}, p)_n$ consist of non-degenerate filtrations $f : \Theta_k^n \to \mathfrak{X}$ along with an isomorphism $f(1^n) \simeq p$. We will give a complete description of the degeneration space of a point in a quotient stack in Section 3.3.

Remark 3.34. When $\mathfrak{X}$ is an algebraic stack satisfying (†), Proposition 1.2 implies that (13) is the set of k-points of the algebraic space $\mathrm{Flag}^n(p)$ whose underlying filtration is non-degenerate (See also (5)), so this would have been an equivalent definition of $\mathbf{DF}(\mathfrak{X}, p)_n$. As an alternative, one could consider the fan whose n-cones consist of *all* finite type points of $|\,\mathrm{Flag}^n(p)|$ and use this as a basis for the theory of stability. The resulting fan, however, would be equivalent to $\mathbf{DF}(\mathfrak{X}, \bar{p})$ where $\bar{p} : \mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(k) \to \mathfrak{X}$ is a geometric point associated to p, so Definition 3.33 is more general.

Remark 3.35. When $\mathfrak{X}$ is not an algebraic stack, we can still discuss the groupoid of k-points of

$$\underline{\mathrm{Map}}(\Theta^n, \mathfrak{X})_p := \underline{\mathrm{Map}}(\Theta^n, \mathfrak{X}) \times_{\mathrm{ev}_1, \mathfrak{X}, p} \mathrm{Spec}(k),$$

but this groupoid need not be equivalent to a set. For instance, if $\mathfrak{X}$ is the stack which associates every k-scheme T to the groupoid of coherent sheaves on T and isomorphisms between them, the homotopy fiber of $\mathrm{ev}_1 : \underline{\mathrm{Map}}(\Theta, \mathfrak{X}) \to \mathfrak{X}$ has non-trivial automorphism groups. These automorphisms arise as automorphisms of coherent sheaves on Θ which are supported on the origin in $\mathbb{A}^1$.

3.2.1. *The degeneration fan and toric geometry.* The simplest degeneration space to compute is the following:

Example 3.36. If $G = T$ is a torus, then $\mathbf{DF}(\mathrm{pt}/T, \mathrm{pt})_n$ is the set of all injective homomorphisms $\mathbb{Z}^n \to \mathbb{Z}^r$ where $r = \operatorname{rank} T$. This fan is equivalent to $R_\bullet(\mathbb{R}^r)$ where $\mathbb{R}^r \subset \mathbb{R}^r$ is thought of as a single cone. Because this cone admits a simplicial subdivision, Lemma 3.7 implies that $|\mathbf{DF}(\mathrm{pt}/T, \mathrm{pt})| \simeq \mathbb{R}^r$ and $\mathbb{P}(\mathbf{DF}(\mathrm{pt}/T, \mathrm{pt})) \simeq S^{r-1}$.

Now let us compute the degeneration space for the action of a split torus T on a finite type k-scheme X. Let $p \in X(k)$, and define $T' = T/\operatorname{Aut}(p)$. Define $Y \subset X$ to be the closure of $T \cdot p$ and $\tilde{Y} \to Y$ its normalization. The subgroup $\operatorname{Aut}(p)$ acts trivially on Y and $\tilde{Y}$. $\tilde{Y}$ is a toric variety for the torus T' and thus defines and is defined by a classical fan consisting of rational polyhedral cones $\sigma_i \subset N'_{\mathbb{R}}$, where N' is the cocharacter lattice of T'.

Lemma 3.37. *Let $\pi : N_{\mathbb{R}} \to N'_{\mathbb{R}}$ be the linear map induced by the surjection of cocharacter lattices $N \to N'$ corresponding to the quotient homomorphism $T \to T'$. Then the cones $\pi^{-1}\sigma_i \subset N_{\mathbb{R}}$ define a classical fan, and the canonical map*
$$\mathbf{DF}(X/T, p)_\bullet \to \mathbf{DF}(\mathrm{pt}/T, \mathrm{pt})_\bullet \simeq R_\bullet(N_{\mathbb{R}})$$
identifies $\mathbf{DF}(X/T, p)_\bullet$ with the sub fan $R_\bullet\{\pi^{-1}(\sigma_i)\} \subset R_\bullet(N_{\mathbb{R}})$.

Proof. Corollary 1.20.1 implies that because $\tilde{Y} \to X$ is finite, the canonical composition map induces an isomorphism
$$\mathbf{DF}(\tilde{Y}/T, p)_\bullet \xrightarrow{\simeq} \mathbf{DF}(X/T, p)_\bullet,$$
where on the left p denotes the unique lift of p to the normalization $\tilde{Y}$. So it suffices to prove the claim when $X = \tilde{Y}$ is normal and p has a dense orbit.

Theorem 1.37, along with the fact that for any $\psi \in \operatorname{Hom}(\mathbb{G}_m^n, T)$ the map $Y_\psi \to X$ is a monomorphism and an immersion locally on Y_ψ implies that p has at most one preimage in each Y_ψ/T. In particular $\mathbf{DF}(X/T, p)_n$ is precisely the set of non-degenerate $\psi \in \operatorname{Hom}(\mathbb{G}_m^n, T)$ for which there is an equivariant map $\mathbb{A}^n \to X$ mapping $1^n \mapsto p$. Such an equivariant map exists if and only if the same is true when we regard $\tilde{Y}$ as a T' scheme and consider the composition $\psi' : (\mathbb{G}_m^n)_k \to T \to T'$.

Equivariant maps between toric varieties preserving a marked point in the open orbit are determined by maps of lattices such that the image of any cone in the first lattice is contained in some cone of the second [F1]. Applying this to the toric variety $\mathbb{A}^n$ under the torus $\mathbb{G}_m^n$ and to $\tilde{Y}$ under the torus T, pointed non-degenerate equivariant maps from $\mathbb{A}^n$ to $\tilde{Y}$ correspond exactly to non-degenerate homomorphisms $\phi : \mathbb{Z}^n \to N$ such that the composition $\mathbb{R}^n \to N'_{\mathbb{R}} \to N'_{\mathbb{R}}$ maps $\mathbb{R}^n_{\geq 0}$ to some cone $\sigma_i \subset N'_{\mathbb{R}}$. This is exactly the sub-fan $R_\bullet\{\pi^{-1}(\sigma_i)\} \subset R_\bullet(N_{\mathbb{R}})$. $\square$

Example 3.38. Let X be an affine toric variety defined by a rational polyhedral cone $\sigma \subset \mathbb{R}^n$ and let $p \in X$ be generic. Then $\mathbf{DF}(X/T, p)_\bullet \simeq R_\bullet(\sigma)$ as defined in (12). For instance, $\mathbf{DF}(\mathbb{A}^n/\mathbb{G}_m^n, 1^n)_\bullet \simeq h_{[n]}$ is represented by the object $[n] \in \mathfrak{Cone}$.

The previous lemma and Lemma 3.7 implies that $|\,\mathbf{DF}(X/T, p)_\bullet| \simeq \bigcup_i \sigma_i \subset N_{\mathbb{R}}$ is the support of the fan of the toric variety $\tilde{Y}$, and $\mathscr{D}eg(X/T, p)$ can be identified with the intersection of this set and the unit sphere in $N_{\mathbb{R}}$, after choosing some metric.

3.2.2. *Properties of the degeneration space.* The construction of $\mathbf{DF}(-)_\bullet$ is functorial in three ways. First, for any map $\psi : \mathfrak{X} \to \mathfrak{Y}$ with quasi-finite inertia and any $p \in \mathfrak{X}(k)$, composition of a filtration $f : \Theta^n_k \to \mathfrak{X}$ with ψ defines a functorial map of fans

$$\psi_* : \mathbf{DF}(\mathfrak{X}, p)_\bullet \to \mathbf{DF}(\mathfrak{Y}, \psi(p))_\bullet.$$

Second, for any pair $p, p' \in \mathfrak{X}(k)$ and an isomorphism of k-points $p \xrightarrow{\sim} p'$, one has a functorial map

$$\mathbf{DF}(\mathfrak{X}, p)_\bullet \to \mathbf{DF}(\mathfrak{X}, p')_\bullet$$

which maps a filtration $f : \Theta^n_k \to \mathfrak{X}$ along with an isomorphism $f(1) \simeq p$ to the same filtration with the composed isomorphism $f(1) \simeq p \simeq p'$. In particular, the group $\mathrm{Aut}(p)$ acts on $\mathbf{DF}(\mathfrak{X}, p)_\bullet$.

Finally, one also has canonical base change maps. Given a field extension k'/k and $p \in \mathfrak{X}(k)$, extending the map $\Theta^n_k \to \mathfrak{X}_p$ to k' defines a map of fans $\mathbf{DF}(\mathfrak{X}, p)_\bullet \to \mathbf{DF}(\mathfrak{X}, p')_\bullet$, where $p' \in \mathfrak{X}(k')$ is the image of $p \in \mathfrak{X}(k)$. Faithfully flat descent implies

Lemma 3.39. $\mathbf{DF}(\mathfrak{X}, p)_\bullet \to \mathbf{DF}(\mathfrak{X}, p')_\bullet$ *is injective. If k'/k is Galois, then* $\mathbf{DF}(\mathfrak{X}, p)_\bullet$ *is the sub-fan of cones in* $\mathbf{DF}(\mathfrak{X}, p')_\bullet$ *which are fixed points for the natural* $\mathrm{Gal}(k'/k)$*-action on* $\mathbf{DF}(\mathfrak{X}, p')_\bullet$.

Consider a pair of maps of stacks $\mathfrak{X} \to \mathfrak{Y}$ and $\mathfrak{X}' \to \mathfrak{Y}$, and let $\mathfrak{Z} := \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}'$. Fix a point $z \in \mathfrak{Z}(k)$ corresponding to points $p \in \mathfrak{X}(k), p' \in \mathfrak{X}'(k)$, and $q \in \mathfrak{Y}(k)$ along with isomorphisms $p \simeq q \simeq p'$ in $\mathfrak{Y}(k)$. Assume that the resulting maps of pointed k-stacks $\mathfrak{X}_p \to \mathfrak{Y}_q$ and $\mathfrak{X}'_{p'} \to \mathfrak{Y}_q$ lie in St_k. Then we have

$$\mathbf{DF}(\mathfrak{Z}, z)_\bullet = \mathbf{DF}(\mathfrak{X}, p)_\bullet \times_{\mathbf{DF}(\mathfrak{Y}, q)_\bullet} \mathbf{DF}(\mathfrak{X}', p')_\bullet \tag{14}$$

from the definition of $\mathbf{DF}(\mathfrak{Z}, z)_\bullet$. The first consequence of this observation is the following:

Lemma 3.40. *Let $\mathfrak{X}$ be an algebraic stack whose diagonal is represented by separated finite type schemes, and let $p \in \mathfrak{X}(k)$. Then $\mathbf{DF}(\mathfrak{X}, p)_\bullet$ is quasi-separated.*

Proof. Consider the maps $f_i : \Theta^{n_i+1} \to \mathfrak{X}$ with isomorphisms $f_i(1) \simeq p$ corresponding to two cones of $\mathbf{DF}(\mathfrak{X}, p)_\bullet$. Form the fiber product and consider following the diagram, where every square is Cartesian

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y_0 \arrow[r] \arrow[d] & \mathbb{A}^{n_0+1} \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\ Y_1 \arrow[r] \arrow[d] & \mathfrak{Y} \arrow[r] \arrow[d] & \Theta^{n_0+1} \arrow[r] \arrow[d] & \mathrm{pt}/\mathbb{G}_m^{n_0+1} \\ \mathbb{A}^{n_1+1} \arrow[r] & \Theta^{n_1+1} \arrow[r] & \mathfrak{X} & \end{tikzcd}$$

In other words, there is $\mathbb{G}_m^{n_0+n_1+2}$-torsor over $\mathfrak{Y}$ whose total space is Y, which is a separated finite type k-scheme by the hypotheses on the diagonal of $\mathfrak{X}$. Hence $\mathfrak{Y}$ is a global quotient of a finite type k-scheme by a torus. The lemma follows from the explicit computation of the degeneration space of a quotient of a scheme by a torus action in Lemma 3.37, which shows that $\mathbb{P}(h_{[n_1]} \times_{\mathbf{DF}(\mathfrak{X},p)_\bullet} h_{[n_2]}) = \mathbb{P}(\mathfrak{Y}, q)$ is covered by finitely many simplices. $\square$

The second consequence of the observation above is the following:

Proposition 3.41. *Let $\pi : \mathfrak{X} \to \mathfrak{Y}$ be a finite type morphism of pointed stacks which is representable by quasi-separated algebraic spaces. Let $p \in \mathfrak{X}(k)$ and let $q = \pi(p)$. Then the induced map of fans $\pi_* : \mathbf{DF}(\mathfrak{X}, p)_\bullet \to \mathbf{DF}(\mathfrak{Y}, q)_\bullet$ is bounded. Furthermore,*

(1) if π is finite, then π_* is an isomorphism;
(2) if π is separated, then π_* is injective, hence $\mathscr{D}eg(\mathfrak{X}, p) \to \mathscr{D}eg(\mathfrak{Y}, q)$ is a closed embedding whose restriction along each rational simplex of $\mathscr{D}eg(\mathfrak{Y}, q)$ is a finite union of rational sub-simplices;
(3) if π is proper, then $\mathscr{D}eg(\mathfrak{X}, p) \to \mathscr{D}eg(\mathfrak{Y}, q)$ is a homeomorphism; and
(4) if π is affine, then the closed subspace $\mathscr{D}eg(\mathfrak{X}, p) \to \mathscr{D}eg(\mathfrak{Y}, q)$ restricted to any rational simplex of $\mathscr{D}eg(\mathfrak{Y}, q)$ is a convex rational polyhedron.

Lemma 3.42. *Let Z be a qc.qs. algebraic space of finite type over a field k with an action of $T = (\mathbb{G}_m^n)_k$, and let $z \in Z(k)$ be a point whose orbit closure contains every point of Z. Then $\mathbf{DF}(Z/T, z)_\bullet$ is bounded.*

Proof. First, we claim that Z admits a $\mathbb{G}_m^n$ equivariant Nisnevich cover from an affine scheme $W \to Z$. Indeed in each $(\mathbb{G}_m^n)_k$ orbit of Z we may choose a closed point $z' : \mathrm{Spec}(k') \hookrightarrow Z$. The stabilizer $\mathrm{Stab}(z') \subset (\mathbb{G}_m^n)_{k'}$ is a group of multiplicative type, hence reductive. Using Theorem B.1 we may extend the resulting locally closed immersion $\mathrm{Spec}(k')/\mathrm{Stab}(z') \hookrightarrow Z/(\mathbb{G}_m^n)_k$ to a representable étale map $U/\operatorname{Stab}(z') \to Z/(\mathbb{G}_m^n)_k$ with U affine. Because $(\mathbb{G}_m^n)_{k'}/\operatorname{Stab}(z')$ is affine, we may present this as a $\mathbb{G}_m^n$-equivariant map $U' = U \times_{\operatorname{Stab}(z')} (\mathbb{G}_m^n)_{k'} \to Z$, where U' is affine. Repeating this construction for each of the (finitely many) $\mathbb{G}_m^n$-orbits and taking a disjoint union defines our equivariant Nisnevich cover $W \to Z$.

The map $W \to Z$ must induce an isomorphism on the reduced connected component of every stabilizer group because it is étale. It follows that for any extension k'/k and any map $\operatorname{Spec}(k')/\mathbb{G}_m^n \to Z/T$ admits a lift to W/T. If follows from Lemma 3.54 below and the remark immediately following it[9] that because $W \to Z$ is étale, any map $\Theta_k^n \to Z/T$ lifts uniquely to W/T once a lift of the map $\operatorname{Spec}(k)/\mathbb{G}_m^n \to Z/T$ is chosen. Let $w_1, \ldots, w_r \in W(k)$ be the points in the fiber over $z \in Z(k)$. Then our previous discussion shows that

$$\bigsqcup_{i=1}^{r} \mathbf{DF}(W/T, w_i)_\bullet \to \mathbf{DF}(Z/T, z)_\bullet$$

is surjective. The left hand side is bounded by Lemma 3.37, which implies that $\mathbf{DF}(Z/T, z)_\bullet$ is bounded. □

Proof of Proposition 3.41. The fact that $\pi_* : \mathbf{DF}(\mathfrak{X}, p)_\bullet \to \mathbf{DF}(\mathfrak{Y}, q)_\bullet$ is an isomorphism if π is finite follows immediately from Corollary 1.20.1, because cones in the degeneration fan correspond to k-points of the flag space, and a filtration $f : \Theta_k^n \to \mathfrak{X}$ is non-degenerate if and only if $\pi \circ f$ is non-degenerate.

Consider a rational simplex $\xi : \Delta_\xi \to \mathscr{D}eg(\mathfrak{Y}, q)$, corresponding to a non-degenerate pointed map $f : \Theta_k^n \to \mathfrak{Y}$. Let $(\mathfrak{Z}, z)$ denote the fiber product $\Theta^n \times_{\mathfrak{Y}} \mathfrak{X}$ along with its canonical k-point. From (14) we have an equivalence

$$\mathbf{DF}(\mathfrak{Z}, z)_\bullet \simeq h_{[n]} \times_{\mathbf{DF}(\mathfrak{Y},q)_\bullet} \mathbf{DF}(\mathfrak{X}, p)_\bullet.$$

Furthermore the map $\mathfrak{Z} \to \Theta_k^n$ is representable by finite type quasi-separated algebraic spaces, so $\mathfrak{Z} \simeq Z/(\mathbb{G}_m^n)_k$ for a finite type quasi-separated algebraic space Z. The canonical point in $\mathfrak{Z}(k)$ can be lifted to a point $z \in Z(k)$, and we can re-define Z to be the normalization of the $(\mathbb{G}_m^n)_k$ orbit closure of z without changing $\mathbf{DF}(\mathfrak{Z}, z)_\bullet$ because the map from the normalization is finite and a point with an open orbit lifts uniquely to the normalization. Lemma 3.42 implies that this fan is bounded, hence the map $\mathbf{DF}(\mathfrak{X}, p)_\bullet \to \mathbf{DF}(\mathfrak{Y}, q)_\bullet$ is bounded. Lemma 3.18 implies that the map $\mathscr{D}eg(\mathfrak{X}, x) \to \mathscr{D}eg(\mathfrak{Y}, y)$ is closed and its image restricted to any rational simplex of $\mathscr{D}eg(\mathfrak{Y}, y)$ is a finite union of rational sub-simplices.

The set of cones in $\mathbf{DF}(\mathfrak{X}, x)_n$ mapping to the given cone of $\mathbf{DF}(\mathfrak{Y}, y)_n$ is in bijection with the set of equivariant sections of the equivariant map $Z \to \mathbb{A}_k^n$. If π is separated, then Z is separated, so a section is uniquely determined by its restriction to the open orbit. This implies that $\mathbf{DF}(\mathfrak{X}, x)_\bullet \to \mathbf{DF}(\mathfrak{Y}, y)_\bullet$ is injective and therefore a closed embedding which is locally a union of rational simplices by Corollary 3.19.1. Furthermore, if π is proper and we consider

[9] Lemma 3.54 appears below for purely expository reasons. Its proof is independent of any other results proved in this paper.

the case of $n = 1$, the map $Z \to \mathbb{A}^1_k$ is proper, so the point $z \in Z(k)$ over $1 \in \mathbb{A}^1_k$ extends uniquely to an equivariant section over $\mathbb{A}^1_k$ by the valuative criterion. This implies that $\mathscr{D}eg(\mathfrak{X}, x) \to \mathscr{D}eg(\mathfrak{Y}, y)$ is also surjective on rational points, hence a homeomorphism by Lemma 3.19.

Finally, if π is affine, then Z is an affine normal toric variety with a toric map to $\mathbb{A}^n_k$. It follows from our computation in Lemma 3.37 that the sub-fan $\mathbf{DF}(\mathfrak{Z}, z)_\bullet \hookrightarrow \mathbf{DF}(\Theta^n_k, 1^n)_\bullet$ has the form $R_\bullet(\sigma) \subset R_\bullet(\mathbb{R}^n_{\geq 0})$ for some rational polyhedral cone $\sigma \subset \mathbb{R}^n_{\geq 0}$ corresponding to Z. Thus $\mathscr{D}eg(\mathfrak{Z}, z) \subset \Delta^{n-1}$ is a rational polytope. □

Remark 3.43. In the proof of Proposition 3.41, we explicitly described the preimage of the closed subspace $\mathscr{D}eg(\mathfrak{X}, x) \hookrightarrow \mathscr{D}eg(\mathfrak{Y}, y)$ under an rational simplex $\Delta_\xi \to \mathscr{D}eg(\mathfrak{Y}, y)$ when $\mathfrak{X} \to \mathfrak{Y}$ is representable by separated finite type schemes. If the rational simplex is represented by a map $\Theta^n_k \to \mathfrak{Y}$, then we consider the fiber product $Z := \mathbb{A}^n_k \times_{\mathfrak{Y}} \mathfrak{X}$, a separated finite type scheme with a $(\mathbb{G}^n_m)_k$-action. There is a canonical orbit in Z lying above $1^n \in \mathbb{A}^n_k$, and we can let Z' be the normalization of the closure of this orbit. Then the restriction of $\mathscr{D}eg(\mathfrak{X}, x)$ to Δ_ξ is the projectivization of the support of the fan of the toric variety Z' inside $\mathbb{R}^n_{\geq 0}$.

3.2.3. *Separated flag spaces.* When an algebraic stack has separated flag spaces, which is always the case when $\mathfrak{X}$ has affine diagonal by Proposition 1.41, we can say more about the degeneration space.

Lemma 3.44. *Let $\mathfrak{X}$ be a stack satisfying (†) such that $\mathrm{ev}_1 : \mathrm{Filt}(\mathfrak{X}) \to \mathfrak{X}$ is separated, and let $p \in \mathfrak{X}(k)$. The map $(v_0, \ldots, v_n) : \mathrm{Flag}^{n+1}(p)(k) \to \prod_{i=0}^n \mathrm{Flag}(p)(k)$ is injective, so any cone in $\mathbf{DF}(\mathfrak{X}, p)_\bullet$ is uniquely determined by its vertices.*

Proof. This follows by induction from the claim that the map

$$(v_0, d_0) : \mathrm{Flag}^{n+1}(p)(k) \to \mathrm{Flag}(p)(k) \times \mathrm{Flag}^n(p)(k)$$

is injective. Using the equivalence $\mathrm{Filt}^{n+1}(\mathfrak{X}) \simeq \mathrm{Filt}^n(\mathrm{Filt}(\mathfrak{X}))$, we can identify $\mathrm{Flag}^{n+1}(p)$ with the set of points $f \in \mathrm{Filt}(\mathfrak{X})(k)$ with an isomorphism $f(1) \simeq p$ and a pointed map $\Theta^n_k \to \mathrm{Filt}(\mathfrak{X})$, where the latter is regarded as pointed stack with point f. The latter data is equivalent to a pointed map $\sigma : \Theta^n_k \to \mathfrak{X}$ along with a section of the projection

$$\Theta^n_k \times_{\mathfrak{X}} \mathrm{Filt}(\mathfrak{X}) \to \Theta^n_k$$

taking the point $1^n \in \Theta^n_k$ to the point $(1^n, f) \in \Theta^n_k \times_{\mathfrak{X}} \mathrm{Filt}(\mathfrak{X})$. The point 1^n is open and dense in Θ^n_k, so if the map ev_1 is separated, any section is uniquely determined by its restriction to a dense open subset. It follows that the existence of a section is a condition, and not data, so the pointed map $\sigma : \Theta^{n+1}_k \to \mathfrak{X}$ is uniquely determined by $f = v_0(\sigma)$ and $d_0(\sigma) \in \mathrm{Flag}^n(p)(k)$. □

Lemma 3.45. *Let $\mathfrak{X}$ be a stack satisfying (†) such that $\mathrm{ev}_1 : \mathrm{Filt}(\mathfrak{X}) \to \mathfrak{X}$ is separated. Then for any point $p \in \mathfrak{X}(k)$ any rational simplex $\Delta_\xi \to \mathscr{D}eg(\mathfrak{X}, p)$ is a closed embedding.*

Proof. It suffices by Corollary 3.21.1 and Lemma 3.18 to show that $\Delta^n_\xi \to \mathscr{D}eg(\mathfrak{X},p)$ is injective on rational points. Assume that two points $x_0, x_1 \in \Delta_\xi$ map to the same rational point in $\mathscr{D}eg(\mathfrak{X},p)$. Let $f_0, f_1 \in \mathbb{Z}^{n+1}_{>0}$ represent the points x_0, x_1, and correspond to non-degenerate pointed maps $\Theta_k \to \Theta_k^{n+1}$. f_0 and f_1 together define a unique non-degenerate pointed map $\Theta_k^2 \to \Theta_k^{n+1}$. Composing this with the map $\xi : \Theta_k^{n+1} \to \mathfrak{X}$ gives a (possibly degenerate) point $\gamma \in \mathrm{Flag}^2(p)(k)$ such that $v_0(\gamma) = f_0$ and $v_1(\gamma) = f_1$. By hypothesis after replacing f_i with a positive multiple we can arrange that the two compositions

$$\Theta_k \xrightarrow{f_i} \Theta_k^{n+1} \xrightarrow{\xi} \mathfrak{X}$$

are the *same* pointed map. It follows from the uniqueness of fillings, Lemma 3.44, that γ must correspond to a 2-dimensional filtration which factors through the projection $\Theta_k^2 \to \Theta_k$ corresponding to the homomorphism $\phi : \mathbb{Z}^2 \to \mathbb{Z}$ defined by $\phi(a,b) = a+b$. As a result, every 1-dimensional sub-cone of γ corresponds to the same rational point in $\mathscr{D}eg(\mathfrak{X},p)$ as well. In other words the map $\Delta_\xi \to \mathscr{D}eg(\mathfrak{X},p)$ collapses all rational points in the 1-simplex connecting $x_0, x_1 \in \Delta_\xi$ to a single point in $\mathscr{D}eg(\mathfrak{X},p)$. This contradicts the finiteness of the fibers of the map $\Delta_\xi \to \mathscr{D}eg(\mathfrak{X},p)$, established in Lemma 3.18. □

3.3. **The $\mathscr{D}eg(X/G,p)$ is a generalized spherical building.** We have already seen in Lemma 3.37 that for a toric variety X, the degeneration fan $\mathbf{DF}(X/T,\mathbf{1})_\bullet$ is essentially equivalent to a more classical construction, the toric fan associated to X. In this section we provide a more classical description of $\mathbf{DF}(X/G,p)_\bullet$ when G is not abelian.

First we address the case where $\mathfrak{X} = \mathrm{pt}/G$ for a semisimple group G. Recall that the spherical building $\Delta(G)$ is a simplicial complex whose vertices are the maximal parabolic subgroups of G, where a set of vertices spans a simplex if and only if the corresponding maximal parabolics contain a common parabolic subgroup of G.

Proposition 3.46. *Let $\mathfrak{X} = \mathrm{pt}/G$ where G is a split semisimple group, and let $p \in \mathfrak{X}(k)$ correspond to the trivial G-bundle. Then $\mathscr{D}eg(\mathfrak{X},p)$ is homeomorphic to the spherical building $\Delta(G)$.*

Proof. Theorem 1.37 implies that $\mathrm{Flag}^n(p) \simeq \bigsqcup_\psi G/P_\psi$ where ψ ranges over $\mathrm{Hom}((\mathbb{G}_m^n)_k, T)/W$, and $\mathbf{DF}(\mathfrak{X},p)_n$ is the set of non-degenerate k-points of this flag space. Thus to every cone $\xi \in \mathbf{DF}(\mathfrak{X},p)_n$ we associate a $\psi \in \mathrm{Hom}((\mathbb{G}_m^n)_k, T)$ up to conjugation by W and a subgroup conjugate to $P_\psi \subset G$. P_ψ need not be parabolic if $n > 1$. We denote $F = \mathbf{DF}(\mathfrak{X},p)_\bullet$, and $F^{par} \subset F$ the sub-fan consisting of all cones such that the associated P_ψ is parabolic.

Fix a split maximal torus $T \subset G$ with cocharacter lattice N, and let $\mathcal{W}$ denote the classical fan in $N_\mathbb{R}$ defined by the Weyl chambers. We also choose a dominant Weyl chamber and an ordering on the set of minimal generators

$\{v_1, \ldots, v_r\} \in N$ for the rays of the dominant Weyl chamber. Every subset of $\{v_1, \ldots, v_r\}$ of cardinality n defines a homomorphism $\psi : \mathbb{G}_m^n \to G$ such that P_ψ is parabolic. Let $\mathcal{S} \subset (\mathfrak{Cone}|F^{par})$ be the full subcategory of cones $\xi \in \mathbf{DF}(\mathfrak{X}, p)_n$ whose associated conjugacy class of homomorphism $\psi : \mathbb{G}_m^n \to G$ is one of these. The inclusions $\mathcal{S} \subset (\mathfrak{Cone}|F^{par}) \subset (\mathfrak{Cone}|F)$ induce canonical maps

$$\operatorname*{colim}_{\xi \in \mathcal{S}} \Delta_\xi \xrightarrow{(a)} \operatorname*{colim}_{\xi \in (\mathfrak{Cone}|F^{par})} \Delta_\xi \xrightarrow{(b)} \operatorname*{colim}_{\xi \in (\mathfrak{Cone}|F)} \Delta_\xi .$$

The statement of the proposition follows from the claim that (a) and (b) are homeomorphisms, because we may identify the first colimit with the spherical building $\Delta(G)$. Indeed, k-cones in $\mathcal{S}$ correspond bijectively to proper parabolic subgroups of G which are contained in k distinct maximal parabolics, i.e. $k-1$-simplices in $\Delta(G)$, and morphisms in $\mathcal{S}$ correspond to containment of parabolics, which corresponds to the containment of faces in $\Delta(G)$.

Step 1: $F^{par} \subset F$ is a bounded inclusion of fans which is surjective on 1-cones, so the map (b) is a homeomorphism by Lemma 3.19 and Corollary 3.19.1.

Surjectivity on 1-cones follows from the fact that for any $\lambda : \mathbb{G}_m \to G$, the subgroup P_λ is parabolic. To show that the map $F^{par} \to F$ is bounded, consider a $\xi : h_{[n]} \to F$, corresponding under Theorem 1.37 to a $\psi \in \mathrm{Hom}((\mathbb{G}_m^n)_k, T)$ and a k-point of G/P_ψ. For any sub-cone $h_{[m]} \subset h_{[n]}$, corresponding to a homomorphism $\mathbb{Z}^m \hookrightarrow \mathbb{Z}^n$ with nonnegative matrix coefficients, the composition $h_{[m]} \to h_{[n]} \to F$ lies in F^{par} if and only if for the resulting homomorphism

$$\psi' : (\mathbb{G}_m^m)_k \to (\mathbb{G}_m^n)_k \xrightarrow{\psi} T$$

the subgroup P_ψ is parabolic. This happens if and only if the image of $\mathbb{R}^n_{\geq 0}$ under the homomorphism on cocharacter lattices $\mathbb{Z}^m \to N$ is contained in some cone of the fan $\mathcal{W}$. In other words

$$h_{[n]} \times_F F^{par} = R_\bullet \mathcal{W}' \subset R_\bullet(\mathbb{R}^n_{\geq 0}) = h_{[n]},$$

where $\mathcal{W}'$ is the classical fan in $\mathbb{R}^n$ obtained by taking the preimage of $\mathcal{W}$ under the homomorphism $\mathbb{R}^n \to N_\mathbb{R}$ induced by ψ and intersecting with the nonnegative cone $\mathbb{R}^n_{\geq 0}$.

Step 2: $\mathcal{S} \subset (\mathfrak{Cone}|F^{par})$ is cofinal, so the map (a) is a homeomorphism.

It suffices to show that the inclusion $\mathcal{S} \subset (\mathfrak{Cone}|F^{par})$ admits a left adjoint. Let $\xi \in F_n^{par}$, and let $\psi : \mathbb{G}_m^n \to T$ represent the conjugacy class of homomorphism associated to ξ. Because P_ψ is parabolic, the cone in $N_\mathbb{R}$ associated to ψ must be in $R_\bullet \mathcal{W}$. Up to conjugation by an element of the Weyl group we may assume that ψ maps $(\mathbb{R}_{\geq 0})^n$ to the dominant Weyl chamber of $\mathcal{W}$.

Let $\sigma \in \mathcal{W}$ be the smallest cone containing the image of $(\mathbb{R}_{\geq 0})^n$. $P_\sigma = P_\psi$, and because cones in the dominant Weyl chamber classify conjugacy classes of parabolics, σ is thus uniquely determined by the conjugacy class of ψ. As the ray generators of σ form a basis for the subgroup generated by $\sigma \cap N$, it follows that there is a unique n-cone $\xi' \in \mathcal{S}$ and a morphism $\phi : [n] \to [n]$ such that $\xi = \phi^* \xi'$. A similar argument shows that any morphism in $(\mathfrak{Cone}|F_\bullet^{par})$ from ξ to an element of $\mathcal{S}$ factors uniquely through this ξ', and thus that the assignment $\xi \mapsto \xi'$ is a left adjoint for the inclusion $\mathcal{S} \subset (\mathfrak{Cone}|F^{par})$. $\square$

Proposition 3.46 justifies the following

Definition 3.47. For an algebraic group G over a field k, the *generalized spherical building* of G is the degeneration space $\Delta(G) := \mathscr{D}eg(\mathrm{pt}/G, \mathrm{pt})$.

For non-semisimple G, this space does not canonically have the structure of a simplicial complex. For instance, when G is a torus we have seen in Example 3.36 that $\Delta(G)$ is a sphere. Nevertheless for a split reductive group G of rank r, $\Delta(G)$ has a canonical cover by rational simplices after fixing a maximal torus $T \subset G$. For any Borel subgroup $B \subset G$ and rational basis $\{v_1, \ldots, v_p\}$ of $N_{\mathbb{R}}^W$, where N is the cocharacter lattice of T and W is the Weyl group, we shall construct a rational simplex $\Delta^{r-1}_{B,v_1,\ldots,v_p} \subset \Delta(G)$:

The Borel subgroup B selects a dominant Weyl chamber in $N'_{\mathbb{R}} := N_{\mathbb{R}}/N_{\mathbb{R}}^W$. There is a unique and hence canonical W-equivariant splitting $N_{\mathbb{R}} = N_{\mathbb{R}}^W \oplus N'_{\mathbb{R}}$, so we can using generators for the extremal rays of the dominant Weyl chamber in $N'_{\mathbb{R}}$ we can extend the vectors above to a rational basis $\{v_1, \ldots, v_r\}$ of $N_{\mathbb{R}}$. Clearing denominators so that v_i are integral, they define a homomorphism $\psi : \mathbb{G}_m^r \to G$ for which P_ψ is conjugate to B. Hence B defines a k point of G/P_ψ and thus an r-cone $\xi \in \mathbf{DF}(\mathrm{pt}/G, \mathrm{pt})_r$. The basis $v_1, \ldots, v_r$ is uniquely defined from the original data $(B, v_1, \ldots, v_p)$ up to positive rescaling of the v_i, which implies that the resulting rational simplex

$$\Delta_{B,v_1,\ldots,v_p} \hookrightarrow \mathscr{D}eg(\mathrm{pt}/G, \mathrm{pt})$$

is determined uniquely by this data. These simplices are top dimensional, and they cover $\Delta(G)$. When G is semisimple, $p = 0$ and these are the simplices arising in the usual description of the spherical building.

Remark 3.48. One can deduce from the above discussion that if $G' := G/Z(G)$, then $\Delta(G)$ is a sphere bundle over $\Delta(G')$. It seems likely that in general if $\pi : \mathfrak{X} \to \mathfrak{Y}$ is a gerbe for the group $\mathbb{G}_m^n$, then one can canonically identify $\mathscr{D}eg(\mathfrak{X}, p)$ with a fiber bundle over $\mathscr{D}eg(\mathfrak{Y}, \pi(p))$ with fiber S^n. Note that the map π is not quasi-finite, and therefore does not induce a map of fans $\mathbf{DF}(\mathfrak{X}, p)_\bullet \to \mathbf{DF}(\mathfrak{Y}, \pi(p))_\bullet$.

We can now complete the description of $\mathscr{D}eg(X/G, p)$ for $p \in X(k)$, by regarding it as a closed subspace $\mathscr{D}eg(X/G, p) \subset \mathscr{D}eg(\mathrm{pt}/G, \mathrm{pt})$ using Proposition 3.41. It suffices to describe the intersection with any rational simplex. Theorem 1.37 shows that a pointed map $\xi : \Theta_k^n \to \mathrm{pt}/G$ corresponds to a choice of $\psi : \mathrm{Hom}(\mathbb{G}_m^n, T)/W$ and a point $g \in G/P_\psi$. One can identify the fiber product

$$\begin{tikzcd} (\mathbb{A}^n \times X)/\mathbb{G}_m^n \arrow[r] \arrow[d] & X/G \arrow[d] \\ \Theta^n \arrow[r, "\xi"] & \mathrm{pt}/G \end{tikzcd},$$

where the action of $\mathbb{G}_m^n$ on $\mathbb{A}^n \times X$ is the diagonal action induced by the standard action of $\mathbb{G}_m^n$ on $\mathbb{A}^n$ and the action on X via $g\psi g^{-1}$. The point $p \in X(k)$ defines a point $(1^n, p) \in \mathbb{A}^n \times X$. Let $\tilde{X}$ be the normalization of the closure of $\mathbb{G}_m^n \cdot (1^n, p)$ in $\mathbb{A}^n \times X$. Then Proposition 3.41 and the discussion leading up to it imply that the intersection of $\mathscr{D}eg(X/G, p) \subset \Delta(G)$ with the simplex $\Delta_\xi^{n-1} \subset \Delta(G)$ is the degeneration space $\mathscr{D}eg(\tilde{X}/\mathbb{G}_m^n, (1^n, p))$. By Lemma 3.37 this is the support of the fan of the toric variety $\tilde{X}$ modulo the scaling action of $\mathbb{R}_{>0}^\times$, where the toric map $\tilde{X} \to \mathbb{A}^n$ is used to identify the support of this fan with a union of subcones of $(\mathbb{R}_{\geq 0})^n$. In particular $\mathscr{D}eg(\tilde{X}/\mathbb{G}_m^n, (1^n, p))$ is a sub-polyhedron of Δ_ξ^{n-1}.

Example 3.49. Let V be an affine scheme and G be a split reductive k-group. For any k-point in V, the morphism $V/G \to BG$ identifies $\mathscr{D}eg(V/G, p)$ with a closed subspace of the spherical building $\Delta(G)$ whose intersection with any rational simplex $\Delta_\xi \subset \Delta(G)$ is a rational polytope $P \subset \Delta_\xi$. We call such a subset of $\Delta(G)$, whose intersection with each Δ_ξ is a rational polytope, a rational polytope in $\Delta(G)$. This suggests a non-abelian analog of toric geometry where one encodes a normal affine G-variety with dense open orbit by a rational polytope in $\Delta(G)$, analogous to the rational polyhedral cone which encodes a normal affine toric variety.

3.4. **A result on extension of filtrations.** In this section we study the relationship between $n+1$-dimensional filtrations σ of a point $p \in \mathfrak{X}(k)$ with $v_0(\sigma) = f$, and n-dimensional filtrations of $\mathrm{ev}_0(f)$ as a point in $\mathrm{Grad}(\mathfrak{X})$. Our main result, Proposition 3.52, is essentially the first half of the main theorem on perturbation of filtrations in the next section Theorem 3.60.

Given a graded object $g \in \mathrm{Grad}(\mathfrak{X})(k)$ and a filtration $\nu \in \mathbf{DF}(\mathrm{Grad}(\mathfrak{X}), g)_1$, we consider the corresponding map

$$\nu : (\mathrm{pt}/\mathbb{G}_m)_k \times \Theta_k \simeq \mathbb{A}^1_k/(\mathbb{G}_m^2)_k \to \mathfrak{X}.$$

We regard the coordinate t on $\mathbb{A}^1_k$ as having weight $(0, -1)$ with respect to the torus $(\mathbb{G}_m^2)_k$, and consider the weight decomposition of the representation of $(\mathbb{G}_m^2)_k$

$$H^0((\mathrm{pt}/\mathbb{G}_m^2)_k, \nu^* \mathbb{L}_{\mathfrak{X}}|_{\{0\}}) \oplus H^1((\mathrm{pt}/\mathbb{G}_m^2)_k, \nu^* \mathbb{L}_{\mathfrak{X}}|_{\{0\}}) = \bigoplus_{a_0, a_1 \in \mathbb{Z}} W_{a_0, a_1}. \quad (15)$$

We say that (a_0, a_1) is a *cotangent weight* for $\nu \in \mathbf{DF}(\mathrm{Grad}(\mathfrak{X}), g)_1$ if $W_{a_0, a_1} \neq 0$ in the decomposition above.

Definition 3.50. Let $\mathfrak{X}$ be an algebraic stack, and let $g \in \mathrm{Grad}(\mathfrak{X})(k)$. Then we define the sub-fan

$$\mathbf{DF}(\mathrm{Grad}(\mathfrak{X}), g)^{\mathfrak{c}}_{\bullet} \subset \mathbf{DF}(\mathrm{Grad}(\mathfrak{X}), g)_{\bullet}$$

to consist of cones σ such that if $\nu \in \mathbf{DF}(\mathrm{Grad}(\mathfrak{X}), g)_1$ is a sub-cone of σ, then any cotangent weight (a_0, a_1) of ν with $a_0 < 0$ has $a_1 \leq 0$. We denote the projective realization $\mathscr{D}eg(\mathrm{Grad}(\mathfrak{X}), g)^{\mathfrak{c}}$.

For a general map of stacks $\phi : \mathfrak{X} \to \mathfrak{X}'$ and point $p \in \mathfrak{X}(k)$, the composition map $\mathrm{Flag}^n(p) \to \mathrm{Flag}^n(\phi(p))$ does not preserve non-degenerate filtrations.[10] Nevertheless we have

Lemma 3.51. *For any stack $\mathfrak{X}$ with separated inertia, the map of stacks $\mathrm{ev}_0 : \mathrm{Filt}(\mathfrak{X}) \to \mathrm{Grad}(\mathfrak{X})$ preserves non-degenerate n-dimensional filtrations and thus induces a map of fans*

$$\mathrm{ev}_0 : \mathbf{DF}(\mathrm{Filt}(\mathfrak{X}), f)_{\bullet} \to \mathbf{DF}(\mathrm{Grad}(\mathfrak{X}), \mathrm{ev}_0(f))_{\bullet}$$

for any $f \in \mathrm{Filt}(\mathfrak{X})(k)$.

Proof. This is just a matter of unraveling definitions. An n-dimensional filtration of f is a map $\sigma : \Theta_k^{n+1} = \mathbb{A}_k^{n+1}/(\mathbb{G}_m^{n+1})_k \to \mathfrak{X}$ with an isomorphism $v_0(\sigma) \simeq f$. Denote the coordinates on $\mathbb{A}_k^{n+1}$ by $(t_0, \ldots, t_n)$. Then $\mathrm{ev}_0(\sigma)$ is the restriction of σ to the closed substack $\{t_0 = 0\}/(\mathbb{G}_m^{n+1})_k \simeq (\mathrm{pt}/\mathbb{G}_m)_k \times \Theta_k^n \subset \Theta_k^{n+1}$. Non-degeneracy of either filtration σ or $\mathrm{ev}_0(\sigma)$ amounts to the same condition: that the homomorphism $(\mathbb{G}_m^{n+1})_k \to \mathrm{Aut}(\sigma(0^{n+1}))$ induced by σ at $0^{n+1} \in \Theta_k^{n+1}$ has finite kernel when restricted to the subtorus $\{1\} \times (\mathbb{G}_m^n)_k$. $\square$

We can now state our main extension result:

Proposition 3.52. *Let $\mathfrak{X}$ be a stack satisfying (†), and let $f \in \mathrm{Filt}(\mathfrak{X})(k)$ be a non-degenerate filtration. Every cone in $\mathbf{DF}(\mathrm{Grad}(\mathfrak{X}), \mathrm{ev}_0(f))_{\bullet}$ lifts uniquely to $\mathbf{DF}(\mathrm{Filt}(\mathfrak{X}), f)_{\bullet}$, so there is a unique map of fans* ext *making the following diagram commute*

$$\begin{tikzcd} & \mathbf{DF}(\mathrm{Filt}(\mathfrak{X}), f)_{\bullet} \arrow[d, "\mathrm{ev}_0"] \\ \mathbf{DF}(\mathrm{Grad}(\mathfrak{X}), \mathrm{ev}_0(f))^{\mathfrak{c}}_{\bullet} \arrow[r, hook] \arrow[ur, dashed, "\mathrm{ext}"] & \mathbf{DF}(\mathrm{Grad}(\mathfrak{X}), \mathrm{ev}_0(f))_{\bullet} \end{tikzcd} .$$

The proof of this proposition will appear at the end of this section, and is a straightforward application of a slightly more technical result, which we now formulate. Say we are given the following data:

(1) a filtration $f : \Theta \to \mathfrak{X}$,
(2) an $\mathbb{Z}^n$-weighted filtration $f_0 : \Theta^n \to \mathrm{Grad}(\mathfrak{X})$, and
(3) an isomorphism $f_0(1) \simeq \mathrm{ev}_0(f) \in \mathrm{Grad}(\mathfrak{X})$.

[10] We have shown that this is the case for maps ϕ with quasi-finite inertia.

We can reinterpret this data in terms of certain substacks of Θ^{n+1}. Consider the following subschemes of $\mathbb{A}^{n+1}$, with coordinates $t_0, \ldots, t_n$:

$$Y_0 := \{t_0 = 0\} \quad \text{and} \quad U := \{t_1 \cdots t_n \neq 0\}.$$

Note that $U/\mathbb{G}_m^{n+1} \simeq \Theta$ with coordinate t_0, and that a map

$$f_0 : Y_0/\mathbb{G}_m^{n+1} \simeq (\mathrm{pt}/\mathbb{G}_m)_{t_0} \times \Theta^n \to \mathfrak{X}$$

classifies an n-dimensional filtration $\Theta^n \to \mathrm{Grad}(\mathfrak{X})$ of the graded object classified by the restriction of f_0 to a map $\{(0,1,\ldots,1)\}/(\mathbb{G}_m)_{t_0} \to \mathfrak{X}$. Note also that the inclusion of this point induces an equivalence of stacks $\{(0,1,\ldots,1)\}/(\mathbb{G}_m)_{t_0} \simeq U \cap Y_0/\mathbb{G}_m^{n+1}$. Thus the data above is equivalent to specifying:

(1') a map $f : U/\mathbb{G}_m^{n+1} \to \mathfrak{X}$,
(2') a map $f_0 : Y_0/\mathbb{G}_m^{n+1} \to \mathfrak{X}$,
(3') an isomorphism between f and f_0 after restricting to $U \cap Y_0/\mathbb{G}_m^{n+1} \simeq \{(0,1,\ldots,1)\}/(\mathbb{G}_m)_{t_0}$.

We can now formulate the unusual gluing question of when the data (1')-(3') results from restricting a $\mathbb{Z}^{n+1}$-weighted filtration $f' : \Theta^{n+1} \to \mathfrak{X}$ to $Y_0/\mathbb{G}_m^{n+1}$ and $U/\mathbb{G}_m^{n+1}$. In other words, we want to know then there is a dotted arrow which makes the following diagram commute:

$$\begin{tikzcd}
 & Y_0/\mathbb{G}_m^{n+1} \arrow[dr] \arrow[drr, bend left, "f_0"] & & \\
Y_0 \cap U/\mathbb{G}_m^{n+1} \arrow[ur] \arrow[dr] & & \mathbb{A}^{n+1}/\mathbb{G}_m^{n+1} \arrow[r, dashed, "f'"] & \mathfrak{X} \\
 & U/\mathbb{G}_m^{n+1} \arrow[ur] \arrow[urr, bend right, "f"'] & &
\end{tikzcd} \tag{16}$$

If such an extension f' exists, then we can restrict f' to $\Theta^n \simeq \{t_0 \neq 0\}/\mathbb{G}_m^{n+1} \subset \Theta^{n+1}$ to obtain a new $\mathbb{Z}^n$-weighted filtration of $f(0)$. This is the perturbation of the filtration f discussed above in the case of coherent sheaves.

Proposition 3.53. *Let $\mathfrak{X}$ be a stack satisfying (†), and assume we are given data (1')-(3'). Consider the space $H^i((f_0^*\mathbb{L}_\mathfrak{X})|_{\{(0,\ldots,0)\}})$ as a representation of $\mathbb{G}_m^{n+1}$, and let $H^i((f_0^*\mathbb{L}_\mathfrak{X})|_{\{(0,\ldots,0)\}})_{a_0,\ldots,a_n}$ denote the summand of weight $(a_0,\ldots,a_n)$ with respect to $\mathbb{G}_m^{n+1}$. If*

$$H^i((f_0^*\mathbb{L}_\mathfrak{X})|_{\{(0,\ldots,0)\}})_{a_0,\ldots,a_n} = 0$$

whenever $i = 0, 1$, $a_0 < 0$, and $a_j > 0$ for some j, then there exists an extension f' making the diagram (16) commute, and it is unique up to unique isomorphism.

We shall prove the claim after collecting some initial lemmas. The key observation is that maps $\Theta \to \mathfrak{X}$ are the same as maps from the formal completion of Θ at $\{0\}$. We shall denote the r^{th} infinitesimal neighborhood of $\{0\}$ in Θ by

$$\mathcal{Q}_r := \mathrm{Spec}(k[t]/(t^{r+1}))/\mathbb{G}_m$$

where t has weight -1 with respect to $\mathbb{G}_m$. Note the canonical closed immersions $\mathrm{pt}/\mathbb{G}_m = \mathcal{Q}_0 \hookrightarrow \mathcal{Q}_1 \hookrightarrow \mathcal{Q}_2 \hookrightarrow \cdots \hookrightarrow \Theta$.

Lemma 3.54. *For any stack satisfying (†), the restriction map*

$$\mathrm{Map}(\Theta, \mathfrak{X}) \to \operatorname*{holim}_r \mathrm{Map}(\mathcal{Q}_r, \mathfrak{X})$$

is an equivalence of stacks.

Proof. We will show that for any finite type affine B-scheme $\mathrm{Spec}(R)$, the restriction functor induces an equivalence of symmetric monoidal ∞-categories

$$\iota^* : \mathrm{APerf}(\Theta_R) \to \varprojlim \mathrm{APerf}((\mathcal{Q}_n)_R). \tag{17}$$

The only open substack of Θ_R containing $(\mathcal{Q}_0)_R$ is Θ_R itself, so Nakayama's lemma implies that an object in $\mathrm{APerf}(\Theta_R)$ is connective if and only if its restriction to $(\mathcal{Q}_0)_R$ is connective, and hence ι^* induces an equivalence $\mathrm{APerf}(\Theta_R)^{cn} \to \varprojlim_r \mathrm{APerf}((\mathcal{Q}_r)_R)^{cn}$ as well. It follows from [BHL, Theorem 5.1, Lemma 3.13] that for any prestack S over $\mathrm{Spec}(R)$,

$$\mathrm{Map}_R(S, \mathfrak{X}_R) \simeq \mathrm{Fun}^c_{\mathrm{Perf}(R)^{\otimes}}(\mathrm{APerf}(\mathfrak{X}_R)^{cn}, \mathrm{APerf}(S)^{cn}), \tag{18}$$

where the latter denotes colimit preserving R-linear symmetric monoidal functors. The statement of the lemma follows formally from .

Showing ι^* is fully faithful:

We need to show that for any $F, G \in \mathrm{APerf}(\Theta_R)$, the natural map

$$R\operatorname{Hom}_{\Theta_R}(F, G) \to \varprojlim_r R\operatorname{Hom}_{(\mathcal{Q}_r)_R}(F|_{(\mathcal{Q}_r)_R}, G|_{(\mathcal{Q}_r)_R}) \tag{19}$$

is an equivalence. First consider the case where $F = \mathcal{O}_{\Theta_R}$ and $G = \mathcal{O}_{\Theta_R}\langle m\rangle$, which denotes the twist of the structure sheaf by a character of $\mathbb{G}_m$ such that the fiber weight at $\{0\}$ is $-m$. Then the limit on the right hand side of (19) is the limit of the weight m subspaces of $R[t]/(t^r)$ as $r \to \infty$, which clearly stabilizes to the weight m subspace of $R[t]$. It follows formally that (19) is an equivalence when $F = \mathcal{O}_{\Theta_R}$ and G is a finite complex whose terms are direct sums of copies of $\mathcal{O}_{\Theta_R}\langle m\rangle$.

The objects $\mathcal{O}_{\Theta_R}\langle m\rangle \in \mathrm{QCoh}(\Theta_R)$ are projective, and every quasi-coherent sheaf admits a surjection from a direct sum of invertible sheaves of this form. It follows that every $G \in D_{qc}(\Theta_R)$ which is homologically bounded below admits a presentation as a right-bounded complex of quasi-coherent sheaves of the form $\bigoplus_{i\in I} \mathcal{O}_{\Theta_R}\langle m_i\rangle$. Furthermore, because R is Noetherian and $G \in \mathrm{APerf}(\Theta_R)$ by hypothesis, one can even find a presentation of this form where each I is finite. So for any $n \in \mathbb{Z}$, one can find a finite complex G' whose terms are finite direct sums of $\mathcal{O}_{\Theta_R}\langle m\rangle$ and a map $G' \to G$ such that $\tau_{\leq n}G' \to \tau_{\leq n}G$ is an equivalence. From the analysis of the previous paragraph, it follows that (19) is an equivalence for G', so because $F = \mathcal{O}_{\Theta_R}$ is projective the map (19) is also an equivalence for G in homological degree $\leq n$. Letting $n \to \infty$ implies the result for $F = \mathcal{O}_{\Theta_R}$ and arbitrary $G \in \mathrm{APerf}(\Theta_R)$.

Next if $F \in \mathrm{Perf}(\Theta_R) \subset D_{qc}(\Theta_R)$, then

$$R\operatorname{Hom}_{\Theta_R}(F, G) \simeq R\operatorname{Hom}_{\Theta_R}(\mathcal{O}_{\Theta_R}, G \otimes F^\vee)$$

and likewise after restriction to $(\mathcal{Q}_r)_R$. So we have fully faithfulness for $F \in \mathrm{Perf}(\Theta_R)$ and $G \in \mathrm{APerf}(\Theta_R)$. Finally, any $F \in D_{qc}(\Theta_R)$ can be written as a filtered homotopy colimit $F = \operatorname{colim}_\alpha F_\alpha$ with $F_\alpha \in \mathrm{Perf}(\Theta_R)$, which by formally commuting homotopy limits implies that (19) is an equivalence whenever $F \in D_{qc}(\Theta_R)$ and $G \in \mathrm{APerf}(\Theta_R)$.

Showing ι^* is essentially surjective:

First consider a pro-system $\{F_r\} \in \varprojlim \mathrm{APerf}((\mathcal{Q}_r)_R)$ such that each F_r is a locally free sheaf whose restriction to $(\mathcal{Q}_0)_R = \operatorname{Spec}(R) \times \mathrm{pt}/\mathbb{G}_m$ is concentrated in weight w. Then F_r is canonically isomorphic to $\mathcal{O}_{(\mathcal{Q}_r)_R} \otimes_R F_0$, where F_0 is regarded as a graded R-module concentrated in weight w. Therefore we have a canonical map

$$\mathcal{O}_{\Theta_R} \otimes_R F_0 \to \varprojlim i_*(F_r)$$

is an equivalence, where we abuse notation slightly to denote the inclusion $i : (\mathcal{Q}_r)_R \hookrightarrow \Theta_R$ for each r so that we have an inverse system $\cdots \to i_*(F_2) \to i_*(F_1) \to i_*(F_0)$. Indeed, a map in $D_{qc}(\Theta_R)$ is an equivalence if and only if it is an equivalence on weight w pieces of the corresponding graded complex of $R[t]$-modules. The weight w piece is extracted by the functor $R\Gamma(\Theta_R, (-) \otimes \mathcal{O}_{\Theta_R}\langle w \rangle)$, which commutes with limits. The map $\mathcal{O}_{\Theta_R} \otimes F_0 \to \varprojlim i_*(F_r)$ can therefore be seen to be an equivalence because the limit $\varprojlim_r R\Gamma(\Theta, F_r\langle w \rangle)$ stabilizes for r sufficiently large.

This shows that any pro-system $\{F_r\} \in \varprojlim_r \mathrm{APerf}((\mathcal{Q}_r)_R)$ of the form above has the property that

$$\varprojlim_r i_*(F_r) \in \mathrm{APerf}(\Theta) \quad \text{and } \iota^*(\varprojlim i_*(F_r)) \to \{F_r\} \text{ is an equivalence} \tag{20}$$

Note that the full subcategory of pro-systems with this property is closed under extensions. Any pro-system where each F_r is a locally free sheaf can be canonically filtered such that $\operatorname{gr}_w F_r$ is a locally free sheaf whose restriction to $(\mathcal{Q}_0)_R$ has weight w, and so the property (20) holds for pro-systems of locally free sheaves.

Next we use [S2, Tag 09AV], with appropriate modifications for the category of graded $R[t]$ modules rather than modules over a ring: for any pro-system $\{F_r\} \in \varprojlim_r \mathrm{APerf}((\mathcal{Q}_r)_R)$, we can find bounded below complexes $(P_r)_\bullet$ of locally free graded modules representing F_r, such that for any $r \geq 1$

$$(P_r)_\bullet \otimes_{k[t]/(t^{r+1})} k[t]/(t^r) \to (P_{r-1})_\bullet$$

is an isomorphism of complexes representing the given quasi-isomorphisms $F_r \otimes_{k[t]/(t^{r+1})} k[t]/(t^r) \to F_{r-1}$. Thus we can consider the naive truncation of this pro-system of complexes $\{(P_r)_{\bullet \leq N}\} \to \{(P_r)_\bullet\} \simeq \{F_r\}$. The truncated pro-system can be built as a sequence of extensions of pro-systems of locally free sheaves, hence (20) holds, and $\varprojlim_r i_*((P_r)_{\bullet\leq N})$ is equivalent to $\varprojlim_r i_*(F_r)$ in low homological degree, so (20) holds for $\{F_r\}$ as well. This shows that ι^* is essentially surjective. $\square$

Using the canonical isomorphism $\underline{\mathrm{Map}}(\Theta^n, \underline{\mathrm{Map}}(\Theta, \mathfrak{X})) \simeq \underline{\mathrm{Map}}(\Theta^{n+1}, \mathfrak{X})$, we conclude that the restriction map

$$\mathrm{Map}(\Theta^{n+1}_k, \mathfrak{X}) \simeq \mathrm{holim}_m \mathrm{Map}(\mathfrak{Q}_m \times \Theta^n_k, \mathfrak{X})$$

is an equivalence of groupoids as well.

Lemma 3.55. *Consider the open subscheme*

$$V = \{(t_0, \ldots, t_n) | t_n \neq 0\} \subset \mathbb{A}^{n+1}.$$

Fix maps $f : V/\mathbb{G}_m^{n+1} \to \mathfrak{X}$, $f_0 : Y_0/\mathbb{G}_m^{n+1} \to \mathfrak{X}$, and an isomorphism between f and f_0 after restricting to $V \cap Y_0/\mathbb{G}_m^{n+1}$. Assume that for all $m < 0, l > 0$ and for $i = 0, 1$ we have

$$\left[H^i((f_0^* \mathbb{L}_{\mathfrak{X}})|_{\{(0,\ldots,0)\}})\right]_{\substack{t_0\text{-weight } m, \\ t_n\text{-weight } l \\ \text{summand}}} = 0.$$

Then f, f_0, and the given isomorphism are induced by restriction from a map $f' : \mathbb{A}^{n+1}/\mathbb{G}_m^{n+1} \to \mathfrak{X}$, which is unique up to unique isomorphism.

Remark 3.56. The extension problem in this lemma is identical to that of (16), but with V instead of U. The gluing data in this case is more complicated, though, because $Y_0 \cap V/\mathbb{G}_m^{n+1} \simeq (\mathrm{pt}/\mathbb{G}_m)_{t_0} \times (\Theta^{n-1})_{t_1,\ldots,t_{n-1}}$.

Proof. Let $Y_m = \mathrm{Spec}(k[t_0, \ldots, t_n]/(t_0^{m+1}))$ be the m^{th} infinitesimal neighborhood of $Y_0 = \{0\} \times \mathbb{A}^n \hookrightarrow \mathbb{A}^{n+1}$. Then $Y_m/\mathbb{G}_m^{n+1} \simeq \mathfrak{Q}_m \times \Theta^n$, so by the previous lemma

$$\mathrm{Map}(\Theta^{n+1}, \mathfrak{X}) \to \mathrm{holim}_m \mathrm{Map}(Y_m/\mathbb{G}_m^{n+1}, \mathfrak{X})$$

is an equivalence of groupoids. We likewise define $V_m = V \cap Y_m$. The same reasoning as above imples that the canonical map

$$\mathrm{Map}(V/\mathbb{G}_m^{n+1}, \mathfrak{X}) \to \mathrm{holim}_m \mathrm{Map}(V_m/\mathbb{G}_m^{n+1}, \mathfrak{X})$$

is an equivalence of groupoids.

We can now rephrase the goal: we are given a map $f_0 : Y_0/\mathbb{G}_m^{n+1} \to \mathfrak{X}$ and an extension of the restriction of f_0 to $V_0/\mathbb{G}_m^{n+1} \subset Y_0/\mathbb{G}_m^{n+1}$ to a pro-system of maps $V_m/\mathbb{G}_m^{n+1} \to \mathfrak{X}$, and we wish to extend this pro-system by filling in the dotted arrows in the diagram

$$\begin{tikzcd}
V_0 \arrow[r, hook] \arrow[d, phantom, "\cap"] & V_1 \arrow[r, hook] \arrow[d, phantom, "\cap"] & V_2 \arrow[r, hook] \arrow[d, phantom, "\cap"] & V_3 \arrow[r, hook] \arrow[d, phantom, "\cap"] & \cdots \\
Y_0 \arrow[r, hook, "i_0"] \arrow[d, "f_0"'] & Y_1 \arrow[r, hook, "i_1"] \arrow[dl, dashed, "f_1"'] & Y_2 \arrow[r, hook, "i_2"] \arrow[dll, dashed, "f_2"'] & Y_3 \arrow[r, hook, "i_3"] \arrow[dlll, dashed, "f_3"] & \cdots \\
\mathfrak{X} & & & &
\end{tikzcd}$$

The maps f_m can be constructed inductively, so it suffices to show that the dotted arrow has a unique filling making the following diagram commute:

$$\begin{tikzcd} V_{m-1}/\mathbb{G}_m^{n+1} \arrow[r, hook] \arrow[d, phantom, "\cap"] & V_m/\mathbb{G}_m^{n+1} \arrow[d, phantom, "\cap"] \arrow[ddr, bend left] & \\ Y_{m-1}/\mathbb{G}_m^{n+1} \arrow[r, hook, "i_{m-1}"] \arrow[drr, bend right, "f_{m-1}"'] & Y_m/\mathbb{G}_m^{n+1} \arrow[dr, dashed, "f_m"] & \\ & & \mathfrak{X} \end{tikzcd} \tag{21}$$

This now amounts to a deformation theory problem. Each inclusion $i_{m-1} : Y_{m-1}/\mathbb{G}_m^{n+1} \to Y_m/\mathbb{G}_m^{n+1}$ is a square-zero extension of algebraic stacks by the coherent sheaf $i_*\mathcal{O}_{Y_0}\langle m, 0, \ldots, 0\rangle$, where we abuse notation slightly to denote the composed inclusion

$$i : Y_0/\mathbb{G}_m^{n+1} \to Y_{m-1}/\mathbb{G}_m^{n+1}.$$

Given an extension of f_0 to a map $f_{m-1} : Y_{m-1}/\mathbb{G}_m^{n+1} \to \mathfrak{X}$, the map can be extended to $Y_m/\mathbb{G}_m^{n+1}$ if and only if the composition

$$f_{m-1}^*\mathbb{L}_{\mathfrak{X}} \to \mathbb{L}_{Y_{m-1}/\mathbb{G}_m^{n+1}} \to i_*(\mathcal{O}_{Y_0}\langle m, 0, \ldots, 0\rangle[1])$$

vanishes as an element of $H^0(R\operatorname{Hom}(f_{m-1}^*\mathbb{L}_{\mathfrak{X}}, i_*(\mathcal{O}_{Y_0}\langle m, 0, \ldots, 0\rangle[1])))$. Furthermore, if this map vanishes, then the set of extensions to $Y_m/\mathbb{G}_m^{n+1}$ up to isomorphism is a torsor for the group

$$H^0(R\operatorname{Hom}(f_{m-1}^*\mathbb{L}_{\mathfrak{X}}, i_*(\mathcal{O}_{Y_0}\langle m, 0, \ldots, 0\rangle))).$$

The same analysis applies to the square zero extension $V_{m-1}/\mathbb{G}_m^{n+1} \hookrightarrow V_m/\mathbb{G}_m^{n+1}$, so the extension f_m in (21) exists and is unique as long as the restriction map

$$R\operatorname{Hom}_{Y_{m-1}/\mathbb{G}_m^{n+1}}(f_{m-1}^*\mathbb{L}_{\mathfrak{X}}, i_*(\mathcal{O}_{Y_0}\langle m, 0, \ldots, 0\rangle)) \to \\ R\operatorname{Hom}_{V_{m-1}/\mathbb{G}_m^{n+1}}(f_{m-1}^*\mathbb{L}_{\mathfrak{X}}|_{V_{m-1}}, i_*(\mathcal{O}_{V_0}\langle m, 0, \ldots, 0\rangle))$$

is injective in cohomological degree 1 and an equivalence in degree 0, for every $m \geq 1$. Using the adjunction and the isomorphism $f_0 \simeq f_{m-1} \circ i$, we can rewrite the $R\operatorname{Hom}$ complexes above only in terms of f_0 and $Y_0/\mathbb{G}_m^{n+1}$, identifying the map above with the restriction map

$$R\operatorname{Hom}_{Y_0/\mathbb{G}_m^{n+1}}(f_0^*\mathbb{L}_{\mathfrak{X}}, \mathcal{O}_{Y_0}\langle m, 0, \ldots, 0\rangle) \to \\ R\operatorname{Hom}_{V_0/\mathbb{G}_m^{n+1}}(f_0^*\mathbb{L}_{\mathfrak{X}}|_{V_0}, \mathcal{O}_{V_0}\langle m, 0, \ldots, 0\rangle).$$

We apply the long exact sequence in local cohomology for the closed substack $\mathcal{S}_0 := \{t_n = 0\}/\mathbb{G}_m^{n+1} \subset Y_0/\mathbb{G}_m^{n+1}$, whose complement it $V_0/\mathbb{G}_m^{n+1}$. This sequence implies that for the map above to be injective in degree 1 and bijective in degree 0, it suffices to show that for all $m > 0$

$$H^i R\operatorname{Hom}_{Y_0/\mathbb{G}_m^{n+1}}(f_0^*\mathbb{L}_{\mathfrak{X}}, R\Gamma_{\mathcal{S}_0}\mathcal{O}_{Y_0} \otimes \mathcal{O}_{Y_0}\langle m, 0, \ldots, 0\rangle)) = 0 \text{ for } i = 0, 1, \tag{22}$$

where $R\Gamma_{\mathcal{S}_0}\mathcal{O}_{Y_0} \simeq k[t_1,\ldots,t_{n-1},t_n^{\pm}]/k[t_1,\ldots,t_n][-1]$ is the derived subsheaf with supports on $\{t_n = 0\}$. Note that this is supported in cohomological degree 1 and $f_0^*\mathbb{L}_{\mathfrak{X}}$ is supported in cohomological degree ≤ 1, so the $R\operatorname{Hom}$ complex in (22) is supported in cohomological degree ≥ 0.

Now observe that $R\Gamma_{\mathcal{S}_0}\mathcal{O}_{Y_0}$ has a filtration whose associated graded is isomorphic to $\bigoplus_{l>0}\mathcal{O}_{\mathcal{S}_0}\langle 0,\ldots,0,-l\rangle[-1]$. A spectral sequence argument shows that for the vanishing of (22) to hold for all $m>0$, it suffices to show that

$$H^i\left(\bigoplus_{l,m>0} R\operatorname{Hom}_{\mathcal{S}_0}((f_0^*\mathbb{L}_{\mathfrak{X}})|_{\mathcal{S}_0}[1], \mathcal{O}_{\mathcal{S}_0}\langle m,0,\ldots,0,-l\rangle)\right) = 0 \text{ for } i = 0,1.$$

Now observe that $\mathcal{S}_0 = \operatorname{Spec}(k[t_1,\ldots,t_{n-1}])/\mathbb{G}_m^{n+1}$ where the first and last copies of $\mathbb{G}_m$ act trivially. This implies that the complex $f_0^*\mathbb{L}_{\mathfrak{X}}|_{\mathfrak{S}_0}$ decomposes into a direct sum of weight complexes under the action of $(\mathbb{G}_m)_{t_0} \times (\mathbb{G}_m)_{t_n}$. For the homological vanishing condition above, it is sufficient to check that the t_0-weight $-m$ and t_n-weight l summand of $f_0^*\mathbb{L}_{\mathfrak{X}}|_{\mathfrak{S}_0}$ is supported in cohomological degree <0. By Nakayama's lemma, and using the fact that every closed substack of $\mathcal{S}_0$ meets the origin, it suffices to check that the t_0-weight $-m$ and t_n-weight l summand of the homology space $H^i(f_0^*\mathbb{L}_{\mathfrak{X}}|_{\{0,\ldots,0\}})$ is zero for $i = 0,1$ and all $l,m>0$. $\square$

Proof of Proposition 3.53. We use Lemma 3.55 to argue by induction on n. The base case is $n=1$, where the statement of Proposition 3.53 is the same as Lemma 3.55. Now let $n>1$ and $V = \{t_n \neq 0\}$ and $U := \{t_1\cdots t_n \neq 0\}$ as defined above. We expand the diagram (16) to the following commutative diagram

$$\begin{tikzcd}
Y_0 \cap U/\mathbb{G}_m^{n+1} \arrow[r, phantom, "\subset"] \arrow[d, hook] \arrow[dr, phantom, "(A)"] & Y_0 \cap V/\mathbb{G}_m^{n+1} \arrow[r, phantom, "\subset"] \arrow[d, hook] \arrow[dr, phantom, "(B)"] & Y_0/\mathbb{G}_m^{n+1} \arrow[d, hook] \arrow[dd, bend left, "f_0"] \\
U/\mathbb{G}_m^{n+1} \arrow[r, phantom, "\subset"] \arrow[drr, bend right, "f"'] & V/\mathbb{G}_m^{n+1} \arrow[r, phantom, "\subset"] \arrow[dr, dashed, "f'"] & \mathbb{A}^{n+1}/\mathbb{G}_m^{n+1} \arrow[d, dashed, "f''"] \\
 & & \mathfrak{X}
\end{tikzcd}$$

Note that $(\mathbb{G}_m)_{t_n}$ acts freely on V, so we have a natural equivalence

$$\mathbb{A}^n/\mathbb{G}_m^n \simeq \{t_n = 1\}/\mathbb{G}_m^n \simeq V/\mathbb{G}_m^{n+1}.$$

This equivalence identifies the square (A) above with the square in (16). Semi-continuity of fiber homology of coherent complexes implies that

$$\bigoplus_{a_n\in\mathbb{Z}} H^i((f_0^*\mathbb{L}_{\mathfrak{X}})|_{\{(0,\ldots,0)\}})_{a_0,\ldots,a_n} = 0 \Rightarrow H^i((f_0^*\mathbb{L}_{\mathfrak{X}})|_{\{(0,\ldots,0,1)\}})_{a_0,\ldots,a_{n-1}} = 0.$$

We use this to verify the weight conditions needed to apply the inductive hypothesis to the square (A), which implies the existence and uniqueness of the extension f'. Once we have f', we are in the situation of Lemma 3.55, and the appropriate weight hypotheses are satisfied, so we can conclude the existence and uniqueness of f''. □

3.4.1. *Proof of Proposition 3.52.* We have seen in the proof of Lemma 3.51 that for $\sigma \in \mathrm{Flag}^n(f)(k)$, the non-degeneracy of σ is equivalent to the non-degeneracy of $\mathrm{ev}_0(\sigma)$. So the proof of Proposition 3.52 amounts to showing that for any $f_0 \in \mathbf{DF}(\mathrm{ev}_0(f))^{\mathfrak{c}}_\bullet$, there is a unique filtration $\mathrm{ext}(f_0) \in \mathrm{Flag}^{n+1}(p)(k)$ such that $v_0(\mathrm{ext}(f_0)) = f$ and $\mathrm{ev}_0(\mathrm{ext}(f_0)) = f_0$. This follows immediately from Proposition 3.53 and the following:

Lemma 3.57. *For any $f_0 \in \mathbf{DF}(\mathrm{Grad}(\mathfrak{X}), \mathrm{ev}_0(f))^{\mathfrak{c}}_n$, the resulting map $f_0 : (\mathrm{pt}/\mathbb{G}_m) \times \Theta_k^n = \{t_0 = 0\}/(\mathbb{G}_m^{n+1})_k \to \mathfrak{X}$ satisfies the weight conditions of Proposition 3.53.*

Proof. We shall prove the contrapositive. Consider the weight decomposition of the representation of $(\mathbb{G}_m^{n+1})_k$

$$H^0((\mathrm{pt}/\mathbb{G}_m^{n+1})_k, f_0^*\mathbb{L}_{\mathfrak{X}}|_{\{(0,\ldots,0)\}}) \oplus H^1((\mathrm{pt}/\mathbb{G}_m^{n+1})_k, f_0^*\mathbb{L}_{\mathfrak{X}}|_{\{(0,\ldots,0)\}}) = \bigoplus_{a_0,\ldots,a_n \in \mathbb{Z}} W_{a_0,\ldots,a_n}. \quad (23)$$

The condition in Proposition 3.53 is that if $(a_0, \ldots, a_n)$ is a weight for which $W_{a_0,\ldots,a_n} \neq 0$ and $a_0 < 0$, then $a_i \leq 0$ for all $i = 1, \ldots, n$.

Let us compute the cotangent weights (15) corresponding to a rational ray in the *interior* of the cone f_0. Such a ray is determined by a vector with non-zero entries $(r_1, \ldots, r_n) \in \mathbb{Z}^n_{>0}$, corresponding to a pointed map $\Theta_k \to \Theta_k^n$ which maps $0 \mapsto 0^n$. If we let $\tilde{W}_{a_0,a_1}$ denote the weight spaces in the decomposition (15) corresponding to the composition $\Theta_k \to \Theta_k^n \to \mathrm{Grad}(\mathfrak{X})$, then

$$\tilde{W}_{b_0,b_1} = \bigoplus_{r_1 a_1 + \cdots r_n a_n = b_1} W_{b_0,a_1,\ldots,a_n}.$$

If $W_{a_0,\ldots,a_n} \neq 0$ with $a_0 < 0$ and $a_i > 0$ for some i, then we can choose $r_i \gg r_j$ for $j \neq i$, and we will have $\tilde{W}_{a_0,b_1} \neq 0$ with $a_0 < 0$ and $b_1 = r_1 a_1 + \cdots + r_n a_n > 0$. So if f_0 does not satisfy the weight condition of Proposition 3.53, we can find a 1-dimensional sub-cone of f_0 which does not lie in $\mathbf{DF}(\mathrm{ev}_0(f))^{\mathfrak{c}}_1$. □

3.5. **Local structure of $\mathscr{D}eg(\mathfrak{X}, p)$.** Let $0 \subset \cdots \subset \mathcal{E}_{i+1} \subset \mathcal{E}_i \subset \cdots \subset \mathcal{E}$ be a $\mathbb{Z}$-weighted filtered coherent sheaf, and let $\mathcal{G} := \bigoplus_w \mathrm{gr}_w(\mathcal{E}_\bullet)$ denote its associated graded coherent sheaf. Let $F_\bullet \mathcal{G}$ be a filtration of $\mathcal{G}$ in the category of graded coherent sheaves, which simply means a filtration of each weight summand $\mathcal{G}_w := \mathrm{gr}_w(\mathcal{E}_\bullet) \subset \mathcal{G}$. We make the following assumption on F: if $F_i \mathcal{G}_w \neq 0$, then $F_i \mathcal{G}_v = \mathcal{G}_v$ for all $v > w$. Then we may induce a new filtration on $\mathcal{E}$ by defining:

$$F'_i \mathcal{E} := \sum_{w \text{ s.t. } F_i \mathcal{G}_w \neq 0} \{\text{preimage of } F_i \mathcal{G}_w \subset \mathcal{E}_w/\mathcal{E}_{w+1} \text{ in } \mathcal{E}_w\} \subset \mathcal{E}.$$

This construction admits a slight generalization in which a $\mathbb{Z}^n$-weighted filtration of $\mathcal{G}$ induces a $\mathbb{Z}^n$-weighted filtration of $\mathcal{E}$. We regard the filtration $0 \cdots \subset F'_{i+1}\mathcal{E} \subset F'_i\mathcal{E} \subset \cdots \subset \mathcal{E}$ as a "perturbation" of the original filtration of $\mathcal{E}$ by the filtration F of the graded coherent sheaf $\mathrm{gr}_\bullet(\mathcal{E}_\bullet)$. We will justify this terminology in the next section. In this section we describe this construction intrinsically and extend it to filtrations in an arbitrary algebraic stack $\mathfrak{X}$.

Observe that the associated graded coherent sheaf $\mathcal{G}$ above acquires a canonical filtration, whose weight-i subsheaf is

$$F_i\mathcal{G} := \bigoplus_{j \geq i} \mathrm{gr}_j(\mathcal{E}_\bullet) \subset \mathcal{G}.$$

The condition placed on the filtration $F_\bullet\mathcal{G}$ above can be phrased as saying that it is "close" to the canonical filtration in the degeneration space of $\mathcal{G}$. In fact, we will identify a neighborhood of the filtration of the original filtration in the degeneration space of $\mathcal{E}$ with a neighborhood of the canonical filtration in the degeneration space of $\mathcal{G}$ as a graded coherent sheaf.

To generalize the canonical filtration of a graded coherent sheaf above, consider a stack $\mathfrak{X}$ and a point $g \in \mathrm{Grad}(\mathfrak{X})(k)$. Then g has a canonical filtration classified by the composition

$$\Theta_k \times (\mathrm{pt}/\mathbb{G}_m)_k \longrightarrow (\mathrm{pt}/\mathbb{G}_m)_k \xrightarrow{g} \mathfrak{X}\ , \tag{24}$$

where the first map is determined by the group homomorphism $(\mathbb{G}_m)_t \times (\mathbb{G}_m)_z \to (\mathbb{G}_m)_z$ given by $(t,z) \mapsto tz$ and the equivariant projection $\mathbb{A}^1_t \to \mathrm{pt}$. The restriction of this map to $\{1\} \times (\mathrm{pt}/\mathbb{G}_m)_k$ is canonically isomorphic to g. We denote the resulting point $\mathfrak{c} \in \mathrm{Flag}(g)(k)$. If g corresponds to a non-degenerate map $(\mathrm{pt}/\mathbb{G}_m)_k \to \mathfrak{X}$, then $\mathfrak{c}$ will be a non-degenerate filtration of g in $\mathrm{Grad}(\mathfrak{X})$, so we may regard it as an element of $\mathbf{DF}(\mathrm{Grad}(\mathfrak{X}), g)_1$. By a slight abuse of notation use the same notation for the corresponding rational point $\mathfrak{c} \in \mathscr{D}eg(\mathrm{Grad}(\mathfrak{X}), g)$.

Remark 3.58. We will refer to the fan $\mathbf{DF}(\mathrm{Grad}(\mathfrak{X}), g)^{\mathfrak{c}}_\bullet$ as the fan of simplices which are *near* $\mathfrak{c}$, justifying the notation of Definition 3.50. Note that all of the cotangent weights (a_0, a_1) of $\mathfrak{c} \in \mathbf{DF}(\mathrm{Grad}(\mathfrak{X}), g)_1$ have $a_1 = a_0$, so $\mathfrak{c} \in \mathbf{DF}(\mathrm{Grad}(\mathfrak{X}), g)^{\mathfrak{c}}_1$. Note also that if $f \in \mathbf{DF}(\mathrm{Grad}(\mathfrak{X}), g)_1$ is antipodal to $\mathfrak{c}$, then all of its cotangent weights (a_0, a_1) have $a_0 = -a_1$, so $\mathscr{D}eg(\mathrm{Grad}(\mathfrak{X}), g)^{\mathfrak{c}}$ contains no points which are antipodal to $\mathfrak{c}$.

Informally stated, for a non-degenerate filtration f, our main results identify small perturbations of the canonical filtration of $\mathrm{ev}_0(f)$ with small perturbations of f itself. The comparison uses the map of fans defined as composition $\mathbb{T} := (\mathrm{ev}_1)_* \circ \mathrm{ext}$,

$$\begin{tikzcd} \mathbf{DF}(\mathrm{Grad}(\mathfrak{X}), \mathrm{ev}_0(f))^{\mathfrak{c}}_\bullet \arrow[r, "\mathrm{ext}", "\substack{\textit{see}\\ \textit{Proposition } 3.52}"'] \arrow[rr, bend left, "\mathbb{T}"] & \mathbf{DF}(\mathrm{Filt}(\mathfrak{X}), f)_\bullet \arrow[r, "(\mathrm{ev}_1)_*"] & \mathbf{DF}(\mathfrak{X}, f(1))_\bullet \end{tikzcd}\ . \tag{25}$$

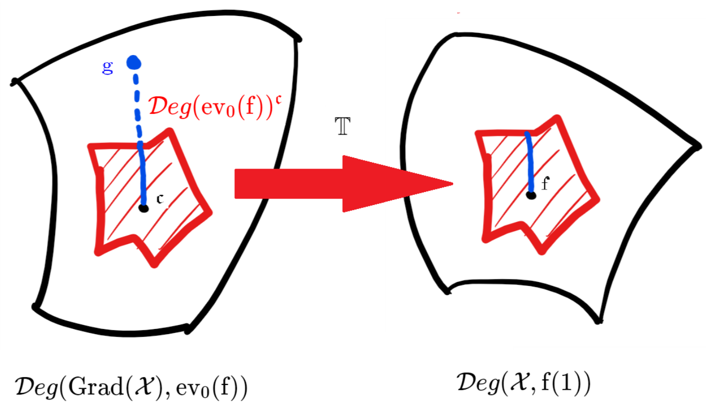


FIGURE 1. Visual summary of Theorem 3.60, Proposition 3.59, and Theorem 3.61: For any non-degenerate filtration in $\mathfrak{X}$, the map $\mathbb{T}$ identifies a neighborhood of the canonical filtration of $\mathrm{ev}_0(f)$ with a neighborhood of the filtration f in $\mathscr{D}eg(\mathfrak{X}, f(1))$. Furthermore, for any point $g \in \mathscr{D}eg(\mathrm{Grad}(\mathfrak{X}, \mathrm{ev}_0(f)))$ which is not antipodal to $\mathfrak{c}$, there is a canonical rational 1-simplex connecting g with $\mathfrak{c}$ which passes through this neighborhood.

Concretely, given an $\sigma \in \mathbf{DF}(\mathrm{ev}_0(f))^{\mathfrak{c}}_n$ corresponding to a map $\sigma : (\mathrm{pt}/\mathbb{G}_m)_k \times \Theta^n_k \to \mathfrak{X}$, the cone $\mathbb{T}(\sigma)$ is constructed by filling in the vertical arrows in the following commutative diagram from left to right:

$$\begin{tikzcd}
(\mathrm{pt}/\mathbb{G}_m)_k \times \Theta^n_k \arrow[r, hook, "t_0=0"] \arrow[dr, "\sigma"'] & \Theta^{n+1}_k \arrow[d, dashed, "\mathrm{ext}(\sigma)"'] & \Theta^n_k \arrow[l, hook', "t_0=1"'] \arrow[dl, dashed, "\mathbb{T}(\sigma)"] \\
& \mathfrak{X} &
\end{tikzcd}$$

Using this one can see that $\mathbb{T}(\mathfrak{c}) = f \in \mathbf{DF}(\mathfrak{X}, f(1))_1$. For simplicity we first state our results under the hypothesis that $\mathfrak{X}$ has separated flag spaces, which holds when $\mathfrak{X}$ has affine diagonal (Proposition 1.41).

Proposition 3.59. *Let $\mathfrak{X}$ be a stack satisfying (†) whose flag spaces are separated, and let $g \in \mathrm{Grad}(\mathfrak{X})(k)$. Then*

(1) the inclusion $\mathbf{DF}(\mathrm{ev}_0(f))^{\mathfrak{c}}_\bullet \subset \mathbf{DF}(\mathrm{ev}_0(f))_\bullet$ is bounded,
(2) $\mathfrak{c}$ lies in the interior of the closed subset $\mathscr{D}eg(\mathrm{ev}_0(f))^{\mathfrak{c}} \subset \mathscr{D}eg(\mathrm{ev}_0(f))$, and
(3) for any rational point $x \in \mathscr{D}eg(\mathrm{ev}_0(f))$ which is not antipodal to $\mathfrak{c}$, there is a unique rational 1-simplex

$$\Delta^1_\sigma \to \mathscr{D}eg(\mathrm{ev}_0(f))$$

with $v_0(\sigma) = \mathfrak{c}$ and $v_1(\sigma) = x$. The set of points in Δ^1_σ mapping to $\mathscr{D}eg(\mathrm{ev}_0(f))^{\mathfrak{c}}$ is a closed subinterval with non-empty interior containing the vertex v_0.

Theorem 3.60 (Perturbation of filtrations)**.** *Let $\mathfrak{X}$ be a stack satisfying (†) whose flag spaces are separated, and let $f \in \mathrm{Filt}(\mathfrak{X})(k)$ be a non-degenerate filtration. The map of fans $\mathbb{T}$ of (25) is injective and maps $\mathfrak{c}$ to f. The subset $\mathbb{T}(\mathscr{D}eg(\mathrm{ev}_0(f))^{\mathfrak{c}}) \subset \mathscr{D}eg(f(1))$ contains an open neighborhood of $f \in \mathscr{D}eg(f(1))$. Furthermore*

(1) $\mathbb{T}$ is "bounded near f" in the sense that or any rational simplex $\Delta_\xi \to \mathscr{D}eg(f(1))$ mapping $x \in \Delta_\xi$ to f, the preimage of $\mathbb{T}(\mathscr{D}eg(\mathrm{ev}_0(f))^{\mathfrak{c}})$ in Δ_ξ is a finite union of rational simplices; and

(2) If $f \in \mathrm{Filt}(\mathfrak{X})(k)$ is contained in a closed substack $\mathfrak{Y} \subset \mathrm{Filt}(\mathfrak{X})$ such that $\mathrm{ev}_1 : \mathfrak{Y} \to \mathfrak{X}$ is quasi-compact, then $\mathbb{T}$ is bounded inclusion, hence $\mathbb{T}(\mathscr{D}eg(\mathrm{ev}_0(f))^{\mathfrak{c}}) \subset \mathscr{D}eg(f(1))$ is closed.

Note that the stronger hypothesis on $\mathfrak{X}$ in the second part of Theorem 3.60 is satisfied when $\mathfrak{X}$ has quasi-compact flag spaces (See Definition 3.88 below). Proposition 3.59 and Theorem 3.60 are synopses of the lemmas which we prove in the remainder of this section. The lemmas also lead to a more lightweight version of the statement which does not require ev_1 to be separated. We state it here for reference, as we will use it below:

Theorem 3.61. *Let $\mathfrak{X}$ be a stack satisfying (†), and let $f \in \mathrm{Filt}(\mathfrak{X})(k)$ be a non-degenerate filtration. Then for any rational point in $\mathscr{D}eg(\mathrm{ev}_0(f))$ which is not antipodal to $\mathfrak{c}$, there is a canonical rational 1-simplex $\Delta^1_\sigma \to \mathscr{D}eg(\mathrm{ev}_0(f))$ with $v_0(\sigma) = \mathfrak{c}$ and $v_1(\sigma) = x$, and the preimage of $\mathscr{D}eg(\mathrm{ev}_0(f))^{\mathfrak{c}} \subset \mathscr{D}eg(\mathrm{ev}_0(f))$ contains an open neighborhood of v_0. Conversely, for any rational 1-simplex $\Delta^1_\sigma \to \mathscr{D}eg(f(1))$ with $v_0(\sigma) = f$, there is a rational 1-simplex $\Delta^1_{\sigma'} \to \mathscr{D}eg(\mathrm{ev}_0(f))^{\mathfrak{c}}$ such that $v_0(\sigma') = \mathfrak{c}$ and $\mathbb{T}(\sigma')$ is a subcone of σ containing v_0.*

Remark 3.62. Note that Theorem 3.60 essentially identifies a closed neighborhood of $f \in \mathbf{DF}(\mathfrak{X}, p)$ with non-empty interior such that every point in this neighborhood is connected to f by a canonical 1-simplex. If $\mathfrak{X}$ admits a positive definite class in $H^4(\mathfrak{X}; \mathbb{R})$, one can canonically parameterize this 1-simplex by arc length using the spherical metric on $\mathbf{DF}(\mathfrak{X}, p)$. It should be possible to use this to show that this neighborhood of f is contractible, hence $\mathbf{DF}(\mathfrak{X}, p)$ is a locally contractible space.

3.5.1. *Proof of Proposition 3.59: Lemmas on the degeneration fan of a graded object.*

Lemma 3.63. *Let $\mathfrak{X}$ be an algebraic stack and let $g \in \mathrm{Grad}(\mathfrak{X})(k)$. The inclusion $\mathbf{DF}(g)^{\mathfrak{c}}_\bullet \subset \mathbf{DF}(g)_\bullet$ is bounded.*

Proof. Given a cone $\xi : h_{[n]} \to \mathbf{DF}(\mathrm{ev}_0(f))_n$, we must show that the preimage of $\mathbf{DF}(g)^{\mathfrak{c}}_\bullet \subset \mathbf{DF}(g)_\bullet$ is a bounded sub-fan of $h_{[n]}$. Because cones in $\mathbf{DF}(g)^{\mathfrak{c}}_\bullet$ are characterized by 1-dimensional sub-cones, it suffices to show that the preimage of $\mathscr{D}eg(g)^{\mathfrak{c}} \to \mathscr{D}eg(g)$ under a rational simplex $\Delta_\xi \to \mathscr{D}eg(g)$ is a finite union of rational simplices.

Say that $\Delta_\xi \to \mathscr{D}eg(g)$ corresponds to a map $f : (\mathrm{pt}/\mathbb{G}_m)_k \times \Theta_k^n \to \mathfrak{X}$, we consider the representation $\bigoplus_{a_0,\ldots,a_n} W_{a_0,\ldots,a_n}$ of $(\mathbb{G}_m^{n+1})_k$ described in (23). Points in Δ_ξ correspond to $(r_1,\ldots,r_n) \in \mathbb{R}^n_{\geq 0} - \{0\}$ up to positive scale. One can check that the set of points in the interior of Δ_ξ which map to $\mathscr{D}eg(g)^{\mathfrak{c}}$ are the points $(r_1,\ldots,r_n)$ with all $r_i > 0$ for which

$$r_1 a_1 + \cdots + r_n a_n \leq 0 \quad \forall (a_1,\ldots,a_n) \text{ s.t. } a_0 < 0 \text{ for some } W_{a_0,\ldots,a_n} \neq 0. \tag{26}$$

Note that these constraints are a finite list of linear inequalities. Rational points in the boundary of Δ_ξ correspond to maps $\Theta_k \to \Theta_k^n$ which do not map 0 to 0^n. However, semicontinuity for the fiber cohomology of the complex $f^*\mathbb{L}_{\mathfrak{X}}$ implies that any point in the boundary of Δ_ξ satisfying the above inequalities still maps to $\mathscr{D}eg(g)^{\mathfrak{c}}$.

We have produced a rational polytope in Δ_ξ mapping to $\mathscr{D}eg(g)^{\mathfrak{c}}$ and containing every point in the interior of Δ_ξ which maps to $\mathscr{D}eg(g)^{\mathfrak{c}}$. Applying this argument to the boundary of Δ_ξ inductively shows that the set of points in Δ_ξ mapping to $\mathscr{D}eg(g)^{\mathfrak{c}}$ is a finite union of rational polytopes, and can thus be covered by finitely many simplices. □

Lemma 3.64. *Let $F_\bullet$ be a quasi-separated fan, and let $\{x_i\}_{i=0}^\infty$ be a sequence of points in $\mathbb{P}(F_\bullet)$ converging to $x \in \mathbb{P}(F_\bullet)$. Then there is a rational simplex $\Delta_\sigma \to \mathbb{P}(F_\bullet)$ and a subsequence $\{x_{i_j}\}_{j=1}^\infty$ which lifts to a sequence $\{\tilde{x}_{i_j}\} \subset \Delta_\sigma$ which converges to a lift $\tilde{x} \in \Delta_\sigma$ of x.*

Proof. There must be a rational simplex $\Delta_\sigma \to \mathbb{P}(F_\bullet)$ whose image contains infinitely many of the x_i. If $x = x_i$ for infinitely many i, then we chose any rational simplex containing x. If not, then passing to a subsequence we may assume that $x \neq x_i$ for any i. Now there must be a rational simplex whose image contains infintitely many of the x_i, or else the subset $\{x_i\} \subset \mathbb{P}(F_\bullet)$ would be closed, implying that any limit point appears in the sequence.

Fix such a rational simplex $\Delta_\sigma \to \mathbb{P}(F_\bullet)$. Select the subsequence of points which lift to Δ_σ, and choose lifts. Because Δ_ξ is compact we can pass to a smaller subsequence which converges to some point $\tilde{x} \in \Delta_\xi$. Because $\mathbb{P}(F_\bullet)$ is Hausdorff by Proposition 3.20, the image of $\tilde{x}$ under the map $\Delta_\xi \to \mathbb{P}(F_\bullet)$ must be x. □

Lemma 3.65. *If $\mathfrak{X}$ is a stack satisfying (†), then $\mathfrak{c}$ lies in the interior of the closed subset $\mathscr{D}eg(g)^{\mathfrak{c}} \subset \mathscr{D}eg(g)$.*

Proof. To show that $\mathfrak{c}$ lies in the interior of $\mathscr{D}eg(g)^{\mathfrak{c}}$, it suffices to show that any sequence in $\mathscr{D}eg(g)$ converging to $\mathfrak{c}$ must contain points in $\mathscr{D}eg(g)^{\mathfrak{c}}$. By Lemma 3.64 we may assume that this sequence lifts to a sequence $\{x_i\}_{i=1}^\infty \subset \Delta_\sigma$ for some rational simplex $\Delta_\xi \to \mathscr{D}eg(g)$ converging to a point which maps to $\mathfrak{c}$. By subdividing Δ_ξ we may assume that $v_0(\xi) = \lim_{i\to\infty} x_i$ maps to $\mathfrak{c}$. By definition of the canonical filtration $\mathfrak{c}$, the fact that $v_0(\xi) = \mathfrak{c} \in \mathscr{D}eg(g)$ implies that in the linear inequalities (26) defining the preimage of $\mathscr{D}eg(g)^{\mathfrak{c}} \subset \mathscr{D}eg(g)$ in Δ_ξ, the integer a_1 is always some positive multiple of a_0. It follows that the constraints are satisfied in some open neighborhood of the point corresponding to $(r_1, \ldots, r_n) = (1, 0, \ldots, 0)$. Thus the sequence $\{x_i\}$, which converges to $(1, 0, \ldots, 0)$, must map to $\mathscr{D}eg(g)^{\mathfrak{c}}$ for i sufficiently large. □

Finally, we show that both the degeneration space $\mathscr{D}eg(g)$ and $\mathscr{D}eg(g)^{\mathfrak{c}}$ are "star shaped" with respect to the canonical point $\mathfrak{c}$.

Lemma 3.66. *If $\mathfrak{X}$ is a stack satisfying (†), then for any rational point $x \in \mathscr{D}eg(g)$ which is not antipodal to $\mathfrak{c}$, there is a canonical rational 1-simplex $\Delta^1_\sigma \to \mathscr{D}eg(g)$ with $v_0(\sigma) = \mathfrak{c}$ and $v_1(\sigma) = x$. The set of points in Δ^1_σ mapping to $\mathscr{D}eg(g)^{\mathfrak{c}}$ is a closed subinterval with non-empty interior containing the vertex v_0. If $\mathfrak{X}$ has separated flag spaces then σ is unique.*

Proof. The point x is represented by some map $\nu : (\mathrm{pt}/\mathbb{G}_m)_k \times \Theta_k \to \mathfrak{X}$. We can compose this with the map $(\mathrm{pt}/\mathbb{G}_m)_k \times \Theta_k^2 = (\Theta_k \times (\mathrm{pt}/\mathbb{G}_m)_k) \times \Theta_k \to (\mathrm{pt}/\mathbb{G}_m)_k \times \Theta_k$ which is the identity on the first factor and the canonical map (24) on the second factor. It follows[11] from Lemma 4.23 that if the resulting map $\Theta_k^2 \to \mathrm{Grad}(\mathfrak{X})$ is non-degenerate, then either $v_0(\sigma), v_1(\sigma) \in \mathbf{DF}(\mathrm{Grad}(\mathfrak{X}), g)_1$ are positive multiples of one another, or they are antipodal, so under the hypothesis of the lemma σ is non-degenerate. If $\mathfrak{X}$ has separated flag spaces, then σ is uniquely determined by v_0 and v_1 by Lemma 3.44.

Finally, we know from the proof of Lemma 3.63, that the condition for a point in the interior of Δ_σ to map to $\mathscr{D}eg(g)^{\mathfrak{c}}$ is given by a finite list of linear inequalities which hold strictly at v_0 and thus hold in a neighborhood of v_0. These inequalities define a closed interval with non-empty interior containing v_0. To complete the proof we must argue that if $x \in \mathscr{D}eg(g)^{\mathfrak{c}}$, then all of σ must map to $\mathscr{D}eg(g)^{\mathfrak{c}}$ as well. Say x is represented by ν as above, and consider a point in the interior of Δ^1_σ, represented by a nonvanishing pair $(r_0, r_1) \in \mathbb{Z}^2_{>0}$ which defines a map

$$\nu' : \Theta_k \times (\mathrm{pt}/\mathbb{G}_m)_k \xrightarrow{(r_0,r_1)} \Theta_k^2 \times (\mathrm{pt}/\mathbb{G}_m)_k \xrightarrow{\sigma} \mathfrak{X}.$$

We can directly compare the two representations $\nu^*\mathbb{L}_{\mathfrak{X}}|_{(0,0)}$ and $(\nu')^*\mathbb{L}_{\mathfrak{X}}|_{(0,0)}$ of $(\mathbb{G}_m)^2_k$: the latter is the pullback of the former along the homomorphism $\mathbb{G}_m^2 \to \mathbb{G}_m^2$ given by

$$(z_0, z_1) \mapsto (z_0 z_1^{r_0}, z_1^{r_1}).$$

It follows that (a_0, a_1) is a cotangent weight of ν if and only if $(a_0, a_0 r_0 + a_1 r_1)$ is a cotangent weight of ν'. In particular if $\nu \in \mathbf{DF}(g)^{\mathfrak{c}}_1$, then any cotangent weight of ν' with $a_0 < 0$ is of the form $(a_0, a_0 r_0 + a_1 r_1)$ with $a_1 \leq 0$, so $\nu' \in \mathscr{D}eg(g)^{\mathfrak{c}}_1$ as well. □

[11] Although it appears below, Lemma 4.23 is logically independent from the other results of this paper and thus does not introduce cyclic reasoning here.

Remark 3.67. In fact, with more work one could probably show that for any rational simplex $\sigma : \Delta^n_\sigma \to \mathscr{D}eg(\mathrm{Grad}(\mathfrak{X}), g)$ which does not contain a point which is antipodal to $\mathfrak{c}$, one can express σ as a finite union of rational sub-simplices σ_i such that each of these smaller simplices is contained in an $n+1$-simplex $\tilde{\sigma}_i$ such that $v_0(\tilde{\sigma}_a) = \mathfrak{c}$.

3.5.2. *Proof of Theorem 3.60.*

Lemma 3.68. *If $\mathfrak{X}$ is a stack satisfying (†), then the map* $\mathrm{ext} : \mathbf{DF}(\mathrm{ev}_0(f))^{\mathfrak{c}}_\bullet \to \mathbf{DF}(f)_\bullet$ *is bounded, and* $\mathrm{ext}(\mathfrak{c})$ *lies in the interior of the closed subset* $\mathscr{D}eg(\mathrm{ev}_0(f))^{\mathfrak{c}} \subset \mathscr{D}eg(f)$.

Proof. Consider a cone $\xi : h_{[n]} \to \mathbf{DF}(f)_\bullet$, then Proposition 3.52 implies that a subcone $\xi' : h_{[k]} \to h_{[n]} \to \mathbf{DF}(f)_\bullet$ lies in the image of ext if and only if $\mathrm{ev}_0(\xi') \in \mathbf{DF}(\mathrm{ev}_0(f))^{\mathfrak{c}}_n$. Hence the preimage of $\mathbf{DF}(\mathrm{ev}_0(f))^{\mathfrak{c}}_\bullet \subset \mathbf{DF}(f)_\bullet$ is bounded by Lemma 3.63. Similarly, image of ext is precisely the preimage of $\mathbf{DF}(\mathrm{ev}_0(f))^{\mathfrak{c}}_\bullet \subset \mathbf{DF}(f)_\bullet$, so $\mathrm{ext}(\mathfrak{c})$ lies in the interior of the image of $\mathscr{D}eg(\mathrm{ev}_0(f))^{\mathfrak{c}}$ by Lemma 3.65. □

Lemma 3.69. *For any stack $\mathfrak{X}$ with separated inertia, any $p \in \mathfrak{X}(k)$, and any cone $\sigma \in \mathbf{DF}(\mathfrak{X}, p)_n$ with $v_0(\sigma) = f \in \mathbf{DF}(\mathfrak{X}, p)_1$, there is a canonical cone $\sigma' \in \mathbf{DF}(\mathrm{Filt}(\mathfrak{X}), f)_n$ with $(\mathrm{ev}_1)_*(\sigma') = \sigma$.*

Proof. Consider the product map $p : \Theta_k \times \Theta_k \to \Theta_k$ given in coordinates by $(t_0, t_1) \mapsto t_0 t_1$. The composition

$$\Theta_k^{n+1} \xrightarrow{p \times \mathrm{id}_{\Theta_k^{n-1}}} \Theta_k^n \xrightarrow{\sigma} \mathfrak{X}$$

can be regarded as a non-degenerate n-dimensional filtration of $f \in \mathrm{Filt}(\mathfrak{X})(k)$ lifting $\sigma \in \mathbf{DF}(\mathfrak{X}, f(1))_n$. □

The following should be interpreted as saying that the map of fans $\mathbb{T}$ is "bounded near f"

Lemma 3.70. *Let $\mathfrak{X}$ be a stack satisfying (†) with separated flag spaces. Then $\mathbb{T}$ is injective, and f is an interior point of the subset $\mathbb{T}(\mathscr{D}eg(\mathrm{ev}_0(f))^{\mathfrak{c}}) \subset \mathscr{D}eg(f(1))$. For any rational simplex $\xi : \Delta_\xi \to \mathscr{D}eg(f(1))$ mapping $x \in \Delta_\xi$ to f, the preimage $\xi^{-1}(\mathbb{T}(\mathscr{D}eg(\mathrm{ev}_0(f))^{\mathfrak{c}})) \subset \Delta_\xi$ is a finite union of sub-simplices.*

Proof. We have seen in Proposition 3.41 that separated representable maps of stacks induce injective maps on degeneration fans, so $(\mathrm{ev}_1)_* : \mathbf{DF}(\mathrm{Filt}(\mathfrak{X}), f)_\bullet \to \mathbf{DF}(\mathfrak{X}, f(1))_\bullet$ is injective. ext is injective because $\mathrm{ev}_0 \circ \mathrm{ext}$ is injective, and hence $\mathbb{T} = (\mathrm{ev}_1)_* \circ \mathrm{ext}$ is injective. First we show that for any rational simplex $\xi : \Delta_\xi \to \mathscr{D}eg(f(1))$ mapping $x \in \Delta_\xi$ to f, the preimage $\xi^{-1}(\mathbb{T}(\mathscr{D}eg(\mathrm{ev}_0(f))^{\mathfrak{c}})) \subset \Delta_\xi$ is a finite union of sub-simplices which contains an open neighborhood of x.

By barycentric subdivision of Δ_ξ centered at x, one can reduce to proving the claim when x is the 0^{th} vertex of Δ_ξ, so we may assume that for our $\xi \in \mathbf{DF}(\mathfrak{X}, f(1))_n$ we have $v_0(\xi) = f^m$ for some $m > 0$. Proposition 1.27

implies that the m^{th}-power map $\mathrm{Filt}(\mathfrak{X}) \to \mathrm{Filt}(\mathfrak{X})$ is an isomorphism on connected components, so if $\Theta_k^{n-1} \to \mathrm{Filt}(\mathfrak{X})$ is a map such that the image of the generic point admits an m^{th} root, then the entire map admits and m^{th} root. The resulting filtration $\Theta_k^n \to \mathfrak{X}$ represents the same simplex in $\mathscr{D}eg(\mathfrak{X}, f(1))$ set theoretically. We may therefore assume that $v_0(\xi) = f$.

In this case ξ lifts to $\mathbf{DF}(f)_\bullet$ by Lemma 3.69. Injectivity of ev_1 implies that for any $\xi : h_{[n]} \to \mathbf{DF}(f(1))_\bullet$ which lies in the image of $(\mathrm{ev}_1)_* : \mathbf{DF}(f)_\bullet \to \mathbf{DF}(f(1))_\bullet$, we have

$$h_{[n]} \times_{\mathbf{DF}(\mathfrak{X},f(1))_\bullet} \mathbf{DF}(\mathrm{ev}_0(f))_\bullet = h_{[n]} \times_{\mathbf{DF}(\mathrm{Filt}(\mathfrak{X}),f)_\bullet} \mathbf{DF}(\mathrm{ev}_0(f))_\bullet.$$

Lemma 3.68 therefore implies that $\xi^{-1}(\mathscr{D}eg(\mathrm{ev}_0(f))^\mathfrak{c}) \subset \Delta_\xi$ is a finite union of rational sub-simplices which contains x in its interior.

If $f \in \mathbb{T}(\mathscr{D}eg(\mathrm{ev}_0(f))^\mathfrak{c})$ were not an interior point, then one could use Lemma 3.64 to find a sequence of x_i in some rational simplex $\Delta_\xi \to \mathscr{D}eg(f(1))$ which converge to a point $x\Delta_\xi$ such that x maps to f, but no x_i maps to $\mathbb{T}(\mathscr{D}eg(\mathrm{ev}_0(f))^\mathfrak{c})$. This contradicts the fact that $\xi^{-1}(\mathbb{T}(\mathscr{D}eg(\mathrm{ev}_0(f))^\mathfrak{c}))$ contains an open neighborhood of x. Therefore the subset $\mathbb{T}(\mathscr{D}eg(\mathrm{ev}_0(f))^\mathfrak{c})$ must contain an open neighborhood of f. □

Under stronger hypotheses still, we can prove that $\mathbb{T}$ is bounded.

Lemma 3.71. *If $\mathfrak{X}$ satisfies (†) and has separated flag spaces, and if f is contained in a closed substack $\mathfrak{Y} \subset \mathrm{Filt}(\mathfrak{X})$ for which $\mathrm{ev}_1 : \mathfrak{Y} \to \mathfrak{X}$ is quasi-compact, then the injective map $\mathbb{T}$ is bounded, and f lies in the interior of the closed subset $\mathscr{D}eg(\mathrm{ev}_0(f))^\mathfrak{c} \subset \mathscr{D}eg(\mathfrak{X}, f(1))$.*

Proof. We know that the map $\mathrm{ev}_1 : \mathbf{DF}(\mathrm{Filt}(\mathfrak{X}), f)_\bullet \to \mathbf{DF}(\mathfrak{X}, f(1))_\bullet$ is bounded by Proposition 3.41, and ext is bounded by Lemma 3.68. □

3.6. **The component space $\mathscr{C}omp(\mathfrak{X})$.** We introduce a topological space which is much smaller than $|\mathbf{DF}(\mathfrak{X}, p)_\bullet|$ and plays a key role in our construction of numerical invariants below.

Definition 3.72. Let $\mathfrak{X}$ be a stack satisfying (†), and let $\xi : S \to \mathfrak{X}$ be a map from a scheme. We say that a connected component of $\mathrm{Filt}^n(\mathfrak{X})$ is *non-degenerate* if it contains a non-degenerate point, in which case every point in the component is non-degenerate by Proposition 1.25. We define a formal fan $\mathbf{CF}(\xi)_\bullet$ and the *component fan* $\mathbf{CF}(\mathfrak{X})_\bullet$ by

$$\mathbf{CF}(\mathfrak{X})_n := \{\text{non-degenerate } \alpha \in \pi_0 \mathrm{Filt}^n(\mathfrak{X})\}, \text{and}$$

$$\mathbf{CF}(\xi)_n := \{\alpha \in \pi_0(\mathrm{Flag}^n(\xi)) \text{ whose image in } \pi_0 \mathrm{Filt}^n(\mathfrak{X}) \text{ is non-degenerate}\}.$$

We define the *component space* of $\mathfrak{X}$ (respectively ξ) to be $\mathscr{C}omp(\mathfrak{X}) := \mathbb{P}(\mathbf{CF}(\mathfrak{X})_\bullet)$ (respectively $\mathscr{C}omp(\xi) = \mathbb{P}(\mathbf{CF}(\xi)_\bullet)$).

Note that the rational points of $\mathscr{C}omp(\mathfrak{X})$ correspond exactly to the set $\pi_0(\mathrm{Filt}(\mathfrak{X}))/\mathbb{N}^\times$.

Remark 3.73. For an arbitrary stack $\mathfrak{Y}$, one can define the set of connected components $\pi_0(\mathfrak{Y})$ as the set of points $|\mathfrak{Y}|$ modulo the smallest equivalence relation identifying any two points in the image of a map $S \to \mathfrak{Y}$ where S is a connected scheme. Using this one can define $\mathbf{CF}(\mathfrak{X})_\bullet$ and $\mathbf{CF}(\xi)_\bullet$ even for stacks which are not algebraic.

For any scheme S and S-point $\xi : S \to \mathfrak{X}$, the map $\mathrm{Flag}^n(\xi) \to \mathrm{Filt}^n(\mathfrak{X})$ defines a canonical map of fans $\mathbf{CF}(\xi)_\bullet \to \mathbf{CF}(\mathfrak{X})_\bullet$. Furthermore for $p \in \mathfrak{X}(k)$ we can define a map of fans $\mathbf{DF}(\mathfrak{X}, p)_\bullet \to \mathbf{CF}(p)_\bullet$ by assigning a k-point of $\mathrm{Flag}^n(p)$ to its connected component in $\pi_0(\mathrm{Flag}^n(p))$.[12]

3.6.1. *The component space of a quotient stack.* In this subsection G will denote either a split reductive group over a field k or $G = \mathrm{GL}_N$ over $\mathbb{Z}$, and X will denote a G-quasi-projective scheme over a base scheme B. We shall compute the component fan $\mathbf{CF}(X/G)_\bullet$ by considering the map $X/T \to X/G$. Using the description in Theorem 1.37 of the stack of filtered objects in X/G and X/T, we can identify the map $\mathrm{Filt}^n(X/T) \to \mathrm{Filt}^n(X/G)$ with the canonical surjective map

$$\bigsqcup_{\psi \in \mathrm{Hom}(\mathbb{G}_m^n, T)} Y_\psi / T \to \bigsqcup_{\psi \in \mathrm{Hom}(\mathbb{G}_m^n, T)/W} Y_\psi / P_\psi$$

The Weyl group W acts naturally on $\mathrm{Filt}^n(X/T)$ by the canonical identification $w \cdot Y_\psi \simeq Y_{w\psi w^{-1}}$ for $w \in W$, and the map $\mathrm{Filt}^n(X/T) \to \mathrm{Filt}^n(X/G)$ is invariant with respect to this action.

Lemma 3.74. *If G is geometrically connected, then the W-invariant map $\mathbf{CF}(X/T)_\bullet \to \mathbf{CF}(X/G)_\bullet$ induces an isomorphism*

$$\mathbf{CF}(X/T)_\bullet / W \simeq \mathbf{CF}(X/G)_\bullet.$$

Proof. Observe that $Y_\psi/T \to Y_\psi/P_\psi$ induces a bijection on connected components. This follows from the fact that P_ψ is connected, so the set of connected components of both stacks corresponds bijectively to the set of connected components of Y_ψ. The components of $\mathrm{Filt}^n(X/G)$ and $\mathrm{Filt}^n(X/T)$ which are non-degenerate correspond to components of Y_ψ for ψ which have a finite kernel, so the equivalence $\pi_0(\mathrm{Filt}^n(X/G)) \simeq \pi_0(\mathrm{Filt}^n(X/T))/W$ preserves the non-degenerate components and gives an equivalence of component fans. $\square$

Because projective geometric realization of fans commutes with colimits, and the quotient is a colimit, it follows that

$$\mathscr{C}omp(X/G) \simeq \mathscr{C}omp(X/T)/W$$

as well. Consider the set $\{(T_i \subset T, Z_i \subset X^{T_i})\}_{i \in I}$ of all sub-tori which arise as the reduced identity component of the stabilizer of some point in X along

[12] Note that the group of k-points of $\mathrm{Aut}(p)$ also acts on $\mathbf{DF}(\mathfrak{X}, p)_\bullet$ and if G is connected, then the morphism $\mathbf{DF}(\mathfrak{X}, p)_\bullet \to \mathbf{CF}(p)_\bullet$ factors through $\mathbf{DF}(\mathfrak{X}, p)_\bullet / \mathrm{Aut}(p)$. One can define a fan which is still coarser than $\mathbf{CF}(\xi)_\bullet$ whose set of n-cones is $\mathrm{im}(\mathbf{CF}(\mathfrak{X}, \xi)_n \to \pi_0 \mathrm{Filt}^n(\mathfrak{X}))$, but $\mathbf{CF}(\xi)_\bullet$ will suffice for our purposes.

with a choice of connected component Z_i of the fixed locus X^{T_i}. The index set I is finite, as can be shown by reducing to the case of a linear action of T on projective space. We therefore have

Lemma 3.75. *The fans* $\mathbf{CF}(X/T)_\bullet$ *and* $\mathbf{CF}(X/G)_\bullet$ *are bounded.*

Proof. By Lemma 3.74 it suffices to prove this for X/T. For each i, choose a finite type point in $Z_i(k_i)$. Consider the map of stacks

$$\mathfrak{X}' := \bigsqcup_i \operatorname{Spec}(k_i)/T_i \to \bigsqcup_i Z_i/T_i \to X/T.$$

Lemma 1.6 implies that $\operatorname{Grad}^n(\mathfrak{X}') \to \operatorname{Grad}^n(X/T)$ is surjective for all n, and Lemma 1.24 implies that $\operatorname{Grad}^n(-)$ and $\operatorname{Filt}^n(-)$ have the same connected components. It follows that $\mathbf{CF}(\mathfrak{X}')_\bullet \to \mathbf{CF}(X/T)_\bullet$ is surjective. The claim now follows from the observation that $\mathbf{CF}(\operatorname{Spec}(k_i)/T_i)_\bullet \simeq \mathbf{DF}(\operatorname{Spec}(k_i)/T_i, \mathrm{pt})_\bullet$ is bounded (see Lemma 3.37). $\square$

We can give an even more concise description of $\mathscr{C}omp(X/T)$. Partially order the set $\{(T_i, Z_i)\}_{i\in I}$ above by the rule

$$(T_i, Z_i) \prec (T_j, Z_j) \text{ if } T_i \subset T_j \text{ and } Z_j \subset Z_i.$$

Let $N = \operatorname{Hom}(\mathbb{G}_m, T)$ be the cocharacter lattice. For each i we consider the subgroup $N_i := \operatorname{Hom}(\mathbb{G}_m, T_i) \subset N$. Note that with our indexing convention the N_i are not distinct as subgroups of N, but we regard them as distinct abstract groups. For any $i \prec j$ we have a canonical embedding $N_i \subset N_j$. Then $|\,\mathbf{CF}(X/T)_\bullet|$ is a union of the vector spaces $(N_i)_\mathbb{R}$ along the resulting inclusions $(N_i)_\mathbb{R} \subset (N_j)_\mathbb{R}$. The proof of Lemma 3.74 actually implies the following

Lemma 3.76. $|\,\mathbf{CF}(X/T)_\bullet| \simeq \operatorname*{colim}_{i\in I} (N_i)_\mathbb{R}$ *and* $\mathscr{C}omp(X/T) \simeq \operatorname*{colim}_{i\in I}((N_i)_\mathbb{R} - \{0\})/\mathbb{R}^\times_{>0}$.

Thus $|\,\mathbf{CF}(X/T)_\bullet|$ is a finite union of real vector spaces along linear embeddings, and $|\,\mathbf{CF}(X/G)_\bullet|$ is the quotient of this space by an action of W which permutes indices and acts linearly on each vector space.

Remark 3.77. Note that the inclusions of vector spaces come from an inclusion of lattices $N_i \hookrightarrow N_j$. The image of these lattices in $\mathscr{C}omp(X/T) = \bigcup (N_i)_\mathbb{R}$ is precisely the set of integral points, and the action of W preserves integral points. The set of non-zero integral points of $\mathscr{C}omp(X/G)$, which is in bijection with the set of non-degenerate connected components of $\operatorname{Filt}(X/G)$, is the union of the image of the maps $N_i \setminus \{0\} \to \mathscr{C}omp(X/G)$.

3.6.2. *The component space of a quasi-compact stack.*

Lemma 3.78. *Let* $\mathfrak{X} = \bigcup \mathfrak{X}_i$ *be a set-theoretic disjoint union of locally closed substacks. Then the map of fans* $\bigsqcup_i \mathbf{CF}(\mathfrak{X}_i)_\bullet \to \mathbf{CF}(\mathfrak{X})_\bullet$ *is surjective.*

Proof. We know from Lemma 1.24 that $\mathrm{Grad}^n(\mathfrak{X}) \to \mathrm{Filt}^n(\mathfrak{X})$ induces a bijection on connected components, so it suffices to show that for any n, $\bigsqcup_i \mathrm{Grad}^n(\mathfrak{X}_i) \to \mathrm{Grad}^n(\mathfrak{X})$ induces a surjection on connected components. This follows from Corollary 1.7.2, which implies that this map is universally bijective. □

Corollary 3.78.1. *If $\mathfrak{X}$ is quasi-compact satisfying (†), then the fan $\mathbf{CF}(\mathfrak{X})_\bullet$ is bounded.*

Proof. Because $\mathfrak{X}$ is quasi-compact, we can find a smooth map $\mathrm{Spec}(R) \to B$ such that the base change $\mathfrak{X}_R \to \mathfrak{X}$ is surjective. By Corollary 1.7.3 the maps $\mathrm{Grad}^n_R(\mathfrak{X}_R) \to \mathrm{Grad}^n_B(\mathfrak{X})$ are surjective and hence surjective on π_0. It follows that the map $\mathbf{CF}(\mathfrak{X}_R)_\bullet \to \mathbf{CF}(\mathfrak{X})_\bullet$, where the former is computed relative to $\mathrm{Spec}(R)$ and the latter relative to B, is surjective, and we may therefore assume that $B = \mathrm{Spec}(R)$. We can write any noetherian stack with affine stabilizers as a set-theoretic union of locally closed substacks which are quotient stacks [K3, Proposition 3.5.9][HR, Proposition 8.2]. The component fan of a quotient stack over the base $\mathrm{Spec}(R)$ is bounded, by the computations above, so it follows from Lemma 3.78 that $\mathbf{CF}(\mathfrak{X})_\bullet$ is bounded. □

Remark 3.79. After developing some more sophisticated methods, we will show in Lemma 4.37 that for any quasi-compact stack $\mathfrak{X}$ satisfying (†) and any map $\xi : S \to \mathfrak{X}$ from a quasi-compact space S, the fan $\mathbf{CF}(\mathfrak{X}, \xi)_\bullet$ is bounded.

3.7. **Cohomology and functions on the component space.** We now discuss a method of constructing continuous functions on $\mathscr{D}eg(\mathfrak{X}, p)$ and $|\,\mathbf{DF}(\mathfrak{X}, p)_\bullet|$ from cohomology classes on the stack $\mathfrak{X}$ with values in some fixed coefficient ring $A \subset \mathbb{R}$. In order for our framework and results to be as flexible as possible – for instance to work over fields other than $\mathbb{C}$ – we will axiomatize the properties we need. These properties hold for many common cohomology theories. We use H^* to denote any contravariant functor from some subcategory of the homotopy category of B-stacks,[13] which must at least contain the stacks Θ^n_S for any finite type B-scheme S, to graded A-modules which satisfies the following axioms.

(1) For fields k of finite type over B, there is a canonical isomorphism $H^*(\Theta^n_k) \simeq A[u_1, \ldots, u_n]$ with u_i in cohomological degree 2. We regard the ring $A[u_1, \ldots, u_n]$ as polynomial A-valued functions on $(\mathbb{R}_{\geq 0})^n$.
(2) For any $\phi : [m] \to [n]$ in $\mathfrak{Cone}$, the restriction homomorphism $H^*(\Theta^n_k) \to H^*(\Theta^m_k)$ induced by the morphism $\Theta^m_k \to \Theta^n_k$ agrees with the restriction of polynomial functions along the corresponding inclusion $(\mathbb{R}_{\geq 0})^m \subset (\mathbb{R}_{\geq 0})^n$.

[13] By homotopy category we mean the category whose objects are stacks and whose maps are 1-morphisms of stacks up to 2-isomorphism.

(3) For any integral affine finite type B-scheme S, the composed homomorphism $H^*(\Theta^n_S) \to H^*(\Theta^n_{k(s)}) \simeq A[u_1, \ldots, u_n]$ is independent of the finite type point $s \in S$.

We regard the choice of cohomology theory $H^*(-)$ satisfying (1)-(3) as fixed throughout this paper. We sometimes use the phrase *rational cohomology classes* to refer to classes in a cohomology theory with coefficient ring $A = \mathbb{Q}$.

Example 3.80. When B is locally of finite type over $\mathbb{C}$, we may discuss the topological stack underlying the analytification of any $\mathfrak{X}$ satisfying (†). This topological stack is defined by taking a presentation of $\mathfrak{X}$ by a groupoid in schemes and then taking the analytification, which is a groupoid in topological spaces. The cohomology is then defined as the cohomology of the classifying space of this topological stack [N2]. For global quotient stacks $\mathfrak{X} = X/G$ this agrees with the equivariant cohomology $H^*_{G^{an}}(X^{an}; A)$ which agrees with $H^*_K(X^{an}; A)$ when $K \subset G$ is a maximal compact subgroup weakly homotopy equivalent to G. The properties (1)-(3) are well known in this setting.

Remark 3.81. The stack $\Theta^n = \mathbb{A}^n/\mathbb{G}_m^n$ deformation retracts onto $\mathrm{pt}/\mathbb{G}_m^n$, so standard computations in equivariant cohomology show that $H^*(\mathbb{A}^n/\mathbb{G}_m^n) \simeq A[u_1, \ldots, u_n]$, where $u_i, \ldots, u_n \in H^2$. Note that while $H^*(\mathrm{pt}/\mathbb{G}_m^n) \simeq A[u_1, \ldots, u_n]$ as well, the generators in H^2 are only canonical up to the action of $GL_n(\mathbb{Z})$ via automorphisms of $\mathrm{pt}/\mathbb{G}_m^n$. Automorphisms of Θ^n correspond to $M \in \mathrm{GL}_n(\mathbb{Z})$ such that M and M^{-1} both fix $(\mathbb{R}_{\geq 0})^n$, which implies that M is a permutation matrix. It follows that the generators of $H^2(\Theta^n)$ are canonical up to permutation. We encode this distinction in (1) by regarding $A[u_1, \ldots, u_n]$ as functions on $(\mathbb{R}_{\geq 0})^n$ rather than $\mathbb{R}^n$.

Example 3.82. When B is locally finite type over some other field, one can take H^* to be operational equivariant Chow cohomology [EG, K3]. For properties (1) and (2) see [P1]. Applying [G, Theorem 3.5] to the special $(\mathbb{G}_m)^n_k$-linear scheme $\mathbb{A}^n_k$ gives a version of the Künneth formula $H^*(\Theta^n_S) \simeq H^*(S) \otimes H^*(\Theta^n_k)$ for any finite type B-scheme S. This combined with the fact that $H^*(\mathrm{Spec}(k)) = A$ for any field k of finite type over B implies (3).

Lemma 3.83. *Let $\mathfrak{X}$ be a stack and let $\eta \in H^{2l}(\mathfrak{X})$. Then there is a unique continuous function $\hat{\eta} : |\,\mathbf{CF}(\mathfrak{X})_\bullet| \to \mathbb{R}$ defined by the property that for any $\xi \in \mathbf{CF}(\mathfrak{X})_n$ the restriction of $\hat{\eta}$ along the map $\xi : \mathbb{R}^n_{\geq 0} \to |\,\mathbf{CF}(\mathfrak{X})_\bullet|$ is the homogeneous degree l function $f^*(\eta) \in H^*(\Theta^n_k) \simeq A[u_1, \ldots, u_n]$, where $f : \Theta^n_k \to \mathfrak{X}$ is a finite type point of $\mathrm{Filt}^n(\mathfrak{X})$ lying on the connected component corresponding to ξ.*

Proof. Property (3) guarantees that $f^*(\eta) \in A[u_1, \ldots, u_n]$ only depends on the connected component on which $f \in \mathrm{Filt}^n(\mathfrak{X})$ lies. Consequently $\eta \in H^{2l}(\mathfrak{X})$ defines a function $\pi_0 \,\mathrm{Filt}^n(\mathfrak{X}) \to A[u_1, \ldots, u_n]_{\deg -1}$, which we regard as a space of real valued degree l polynomials on $\mathbb{R}^n_{\geq 0}$. The geometric realization is a colimit, so a continuous function $|F| \to \mathbb{R}$ is defined by a family of continuous functions $(\mathbb{R}_{\geq 0})^n \to \mathbb{R}$ for each $\xi \in F_n$ which is compatible with the continuous maps $(\mathbb{R}_{\geq 0})^k \to (\mathbb{R}_{\geq 0})^n$ for each morphism in $(\mathfrak{Cone}|F)$. Properties (1) and (2) give exactly such a family of continuous functions on $(\mathbb{R}_{\geq 0})^n$. $\square$

For $p \in \mathfrak{X}(k)$, we will overload notation by using $\hat{\eta}$ to denote the function on $|\,\mathbf{DF}(\mathfrak{X}, p)|$ obtained by restricting $\hat{\eta}$ along the canonical map $|\,\mathbf{DF}(\mathfrak{X}, p)| \to |\,\mathbf{CF}(\mathfrak{X})|$. A filtration $f : \Theta_k \to \mathfrak{X}$ defines an integral point in both $|\,\mathbf{DF}(\mathfrak{X}, f(1))|$ and $|\,\mathbf{CF}(\mathfrak{X})|$, the image of $1 \in \mathbb{R}_{\geq 0}$ under the rational ray corresponding to f,

$$\mathbb{R}_{\geq 0} \to |\,\mathbf{DF}(\mathfrak{X}, f(1))| \to |\,\mathbf{CF}(\mathfrak{X})|.$$

We denote this point by $f \in |\,\mathbf{DF}(\mathfrak{X}, f(1))|$ as well. Given a class $\eta \in H^{2l}(\mathfrak{X})$, we can evaluate the funtion $\hat{\eta}$ at this point,

$$\hat{\eta}(f) = \frac{1}{q^l} f^*(\eta) \in A, \tag{27}$$

where q^l is the canonical generator of the cohomology $H^*(\Theta_k; A) \simeq A[\![q]\!]$.

Remark 3.84. Operational Chow cohomology should satisfy property (3) in mixed characteristic as well, but the literature on operational Chow cohomology of stacks in mixed characteristic is not as developed. In practice, however, we will always be considering Chern classes of locally free sheaves (or complexes) on $\mathfrak{X}$. For these classes, one can check property (3) directly by relating the Chern classes of a locally free sheaf on Θ_k^n to the weights of the sheaf restricted to $(\mathrm{pt}/\mathbb{G}_m^n)_k$, so the conclusion of Lemma 3.83 still applies.

3.7.1. *Positive definite classes.* One useful notion we will make frequent use of is the following

Definition 3.85. We say that a class $b \in H^4(\mathfrak{X})$ is *positive definite* if for all non-degenerate maps $g : \mathrm{pt}/\mathbb{G}_m \to \mathfrak{X}$, the pullback $\gamma^*(b) \in H^4(\mathrm{pt}/\mathbb{G}_m) \simeq A \cdot u^2 \subset \mathbb{R}[u]$ is a positive multiple of u^2.

Note that the generator of $H^2(\mathrm{pt}/\mathbb{G}_m)$ is only canonical up to sign, but u^2 is a canonical generator for $H^4(\mathrm{pt}/\mathbb{G}_m)$, so the notion of sign is well-defined. We leave the proof of the following to the reader:

Lemma 3.86. *The following are equivalent:*

(1) $b \in H^4(\mathfrak{X})$ *is positive definite,*
(2) the function $\hat{b} : |\,\mathbf{CF}(\mathfrak{X})_\bullet| \to \mathbb{R}$ *is positive away from the cone point, and*
(3) for any $p \in \mathfrak{X}(k)$*, the function* $\hat{b} : |\,\mathbf{DF}(\mathfrak{X}, p)_\bullet| \to \mathbb{R}$ *is positive away from the cone point.*

Note that b is positive definite if and only if for every non-degenerate map $g : (\mathrm{pt}/\mathbb{G}_m)^n \to \mathfrak{X}$ the resulting quadratic form $g^*(b) \in \mathrm{Sym}^2(\mathbb{R}^n)$ is positive definite. In fact, the previous lemma shows that if the fan $\mathbf{CF}(\mathfrak{X})_\bullet$ is bounded, then there is a finite set of non-degenerate morphisms

$$g_i : (pt/\mathbb{G}_m)^{n_i} \to \mathfrak{X}, \text{ for } i = 1, \ldots, N$$

with finite kernels such that $b \in H^4(\mathfrak{X})$ is positive definite if and only if $g_i^*(b) \in H^4((\mathrm{pt}/\mathbb{G}_m)^{n_i}) \simeq \mathrm{Sym}^2(\mathbb{R}^{n_i})$ is a positive definite bilinear form for all $i = 1, \ldots, N$. In this case the set of positive definite classes (if non-empty) is the interior of a convex cone of full dimension in $H^4(X/G)$, and small perturbations of a positive definite class remain positive definite.

Remark 3.87. Let $b \in H^4(\mathfrak{X})$ be positive definite. In this case given $\gamma \in \mathbf{DF}(\mathfrak{X}, p)_2$ we can define the length of the γ as

$$\mathrm{length}(\gamma) = \arccos\left(\frac{b(\phi_{1,1}^*\gamma) - b(\phi_{1,0}^*\gamma) - b(\phi_{0,1}^*\gamma)}{2\sqrt{b(\phi_{0,1}^*\gamma) b(\phi_{1,0}^*\gamma)}}\right),$$

where $\phi_{m,n}^* : \mathbf{DF}(\mathfrak{X}, p)_2 \to \mathbf{DF}(\mathfrak{X}, p)_1$ for $m, n \geq 0$ is the map induced by the morphism in $\mathfrak{Cone}$ corresponding to the homomorphism $\mathbb{Z} \to \mathbb{Z}^2$ taking $\phi_{m,n} : 1 \mapsto (m, n)$. One can check that the formula for $\mathrm{length}(\gamma)$ is invariant under any homomorphism $\mathbb{Z}^2 \to \mathbb{Z}^2$ preserving the two positive coordinate rays in $\mathbb{R}^2$, and thus length is a well-defined function of a rational 1-simplex in $\mathscr{D}eg(\mathfrak{X}, p)$. We can then define a *spherical metric* on $\mathscr{D}eg(\mathfrak{X}, p)$ by the formula

$$d(f, g) := \inf\left\{\sum_i \mathrm{length}(\gamma^{(i)})\right\}$$

Where the infimum is taken over all piecewise linear paths, meaning sequences $\gamma^{(0)}, \ldots, \gamma^{(n)}$ of rational 1-simplices in $\mathscr{D}eg(\mathfrak{X}, p)$ such that $\phi_{1,0}^*\gamma^{(0)} = f$, $\phi_{0,1}^*\gamma^{(n)} = g$ and $\phi_{0,1}^*\gamma^{(i)} = \phi_{1,0}^*\gamma^{(i+1)}$. This is a generalization of the spherical metric discussed in [MFK, Section 2]

3.8. **A criterion for quasi-compactness of flag spaces.**

Definition 3.88. If $\mathfrak{X}$ is a stack satisfying (†), then we say $\mathfrak{X}$ *has quasi-compact flag spaces* if for any connected component $\mathfrak{Y} \subset \mathrm{Filt}(\mathfrak{X})$, the map $\mathrm{ev}_1 : \mathfrak{Y} \to \mathfrak{X}$ is quasi-compact.

Proposition 3.89. *Let $\mathfrak{X}$ be a quasi-compact algebraic stack satisfying (†) over a base algebraic space B. Let $\Phi : |\,\mathbf{CF}(\mathfrak{X})_\bullet| \to \mathbb{R}$ be a function such that for any $c \in \mathbb{R}$ the preimage of $\Phi^{-1}((-\infty, c])$ in any rational cone $\mathbb{R}^n_{\geq 0} \to |\,\mathbf{CF}(\mathfrak{X})_\bullet|$ is bounded. Then $\mathrm{ev}_1 : \mathrm{Filt}^n(\mathfrak{X}) \to \mathfrak{X}$ is quasi-compact on each connected component of $\mathrm{Filt}^n(\mathfrak{X})$.*

Proof. Because ev_1 is locally of finite type it suffices to show that the connected components of $\mathrm{Filt}^n(\mathfrak{X})$ itself are quasi-compact. The projection $\mathrm{ev}_0 : \mathrm{Filt}^n(\mathfrak{X}) \to \mathrm{Grad}^n(\mathfrak{X})$ is quasi-compact and a bijection on connected components by Lemma 1.24, so it suffices to show that the connected components of $\mathrm{Grad}^n(\mathfrak{X})$ are quasi-compact.

We first address the case $n = 1$. The function $\Phi : |\mathbf{CF}(\mathfrak{X})| \to \mathbb{R}$ defines a locally constant function on $\mathrm{Filt}(\mathfrak{X})$ and hence on $\mathrm{Grad}(\mathfrak{X})$ as well. The value on each connected component of $\mathrm{Filt}(\mathfrak{X})$ is the value of Φ at the image of $1 \in \mathbb{R}_{\geq 0}$ under the corresponding rational ray $\mathbb{R}_{\geq 0} \to |\mathbf{CF}(\mathfrak{X})|$ (See the formula (27)). By convention we let $\Phi = -\infty$ on the connected components of $\mathrm{Filt}(\mathfrak{X})$ classifying trivial filtrations (they are isomorphic to $\mathfrak{X}$). We claim that the substack of $\mathrm{Grad}(\mathfrak{X})$ on which $\Phi \leq c$ is quasi-compact.

Write $\mathfrak{X}$ as a set theoretic union $\mathfrak{X} = \bigcup \mathfrak{X}_i$ of locally closed substacks of the form X_i/GL_{n_i} for a GL_{n_i}-quasi-projective scheme. Then $\mathrm{Grad}(\mathfrak{X}) = \bigcup_i \mathrm{Grad}(\mathfrak{X}_i)$ is a set-theoretic union of locally closed substacks as well. Consider the restriction of Φ along the map $|\mathbf{CF}(\mathfrak{X}_i)| \to |\mathbf{CF}(\mathfrak{X})|$. In Lemma 3.76 we described $\mathscr{C}omp(X_i/\mathrm{GL}_{n_i})$ as a finite union of real vector spaces associated to various subgroups of the cocharacter group of $\mathbb{G}_m^{n_i}$, modulo the action of the Weyl group S_{n_i}. The hypothesis that $\Phi^{-1}((-\infty, c])$ is bounded in any rational cone of $|\mathbf{CF}(\mathfrak{X})|$ implies that there are only finitely many integral points in $|\mathbf{CF}(\mathfrak{X}_i)|$ for which $\Phi \leq c$. We know from Theorem 1.36 that $\mathrm{Grad}(\mathfrak{X}_i)$ has quasi-compact connected components, so it follows that the substack of $\mathrm{Grad}(\mathfrak{X}_i)$ on which $\Phi \leq c$ is quasi-compact. This holds for all i, so the substack of $\mathrm{Grad}(\mathfrak{X})$ on which $\Phi \leq c$ is quasi-compact as well.

The argument for $n > 1$ is very similar, so we omit it. The only difference is that one should consider the set of connected components of $\mathrm{Filt}^n(\mathfrak{X})$ such that each of the vertex maps $v_i : \mathrm{Filt}^n(\mathfrak{X}) \to \mathrm{Filt}(\mathfrak{X})$ maps to the locus where $\Phi \leq c$ and show that this subset of $\mathrm{Filt}^n(\mathfrak{X})$ is quasi-compact. □

Example 3.90. When $\mathfrak{X}$ is quasi-compact and admits a positive definite class $b \in H^4(\mathfrak{X}; \mathbb{R})$, the function $\hat{b} : |\mathbf{CF}(\mathfrak{X})| \to \mathbb{R}$ of Lemma 3.83 satisfies the hypothesis of Proposition 3.89. Hence $\mathfrak{X}$ has quasi-compact flag spaces.

4. Construction of Θ-stratifications

In this section we strengthen the main existence result, Theorem 2.7 in several ways, leading to Theorem 4.38. We introduce the class of Θ-reductive stacks, and reduce the existence of Θ-stratifications of a Θ-reductive stacks to a single boundedness hypothesis in Theorem 4.39. We formulate the existence and uniqueness of a HN filtration for $p \in \mathfrak{X}(k)$ as the problem of maximizing a certain continuous function $\mu : \mathscr{D}eg(\mathfrak{X}, p) \to \mathbb{R}$ which factors through the projection $\mathscr{D}eg(\mathfrak{X}, p) \to \mathscr{C}omp(\mathfrak{X}, p)$. The rough idea is that a maximum exists because $\mathscr{C}omp(\mathfrak{X}, p)$ is compact, and it is unique because μ is locally strictly quasi-concave, and on a Θ-reductive stack the subset of $\mathscr{D}eg(\mathfrak{X}, p)$ on which $\mu > 0$ is convex. Our main technical results are Proposition 4.21 and Proposition 4.31.

4.1. Numerical invariants and the Harder-Narasimhan problem.

In Theorem 2.7 we encoded the data of a Θ-stratification of a stack $\mathfrak{X}$ satisfying (†) as a subset $S \subset \mathrm{Irred}(\mathrm{Filt}(\mathfrak{X}))$ and a function $\mu : S \to \Gamma$ for some totally ordered set Γ. In the remainder of the paper we shall specialize to a more restrictive situation in which we can use the structures of Section 3 to analyze the Harder-Narasimhan problem.

Definition 4.1. A *numerical invariant* on a stack $\mathfrak{X}$ consists of a realizable subset $\mathfrak{U} \subset \mathscr{C}omp(\mathfrak{X})$ and a continuous real-valued function $\mu : \mathfrak{U} \to \mathbb{R}$. Given a numerical invariant we define the *stability function* $M^\mu : |\mathfrak{X}| \to \mathbb{R} \cup \{\infty\}$ as

$$M^\mu(p) = \sup\{\mu(f) \,|\, f \in \mathfrak{U} \text{ with } f(1) = p \in |\mathfrak{X}|\},$$

where we are abusing notation by using $f \in \mathfrak{U}$ to denote both a filtration f and the point in $\mathscr{C}omp(\mathfrak{X})$ corresponding to the connected component of $\mathrm{Filt}(\mathfrak{X})$ which contains f, and by letting $\mu(f)$ denote the corresponding value of μ. We say that $p \in |\mathfrak{X}|$ is *unstable* if $M^\mu(p) > 0$ and *semistable* otherwise.

Remark 4.2. One can immediately generalize this notion to that of an A-valued numerical invariant for some totally ordered $\mathbb{R}$-algebra A. One should assume that every bounded-above subset of A admits a supremum, so that M^μ takes values in $A \cup \{\infty\}$. Many of the results of this section continue to hold, with identical proofs, but for simplicity our exposition will mostly deal with the case $A = \mathbb{R}$.

The set of rational points in $\mathfrak{U}$ are dense, and we can identify this set with a subset of $\pi_0(\mathrm{Filt}(\mathfrak{X}))/\mathbb{N}^\times$. Because μ is continuous, it is uniquely determined by its restriction to rational points, so we can regard μ as a locally constant function on a collection of connected components $\mathfrak{Y} \subset \mathrm{Filt}(\mathfrak{X})$ which is invariant under the $\mathbb{N}^\times$-action. If we choose a subset $S \subset \mathrm{Irred}(\mathrm{Filt}(\mathfrak{X}))$ which is a complete set of $\mathbb{N}^\times$-orbit representatives of irreducible components of $\mathfrak{Y}$, then the numerical invariant is uniquely determined by S and the induced function $\mu : S \to \mathbb{R}$. Thus Definition 4.1 is not a new notion, but rather an instance of the data (8) which satisfies some additional hypotheses:

(1) μ is locally constant in the sense of Simplification 2.8;
(2) the image of S in $\mathscr{C}omp(\mathfrak{X})$ consists of all rational points in some realizable set $\mathfrak{U}$; and
(3) μ extends to a continuous function on $\mathfrak{U}$.

For any numerical invariant $\mu : \mathfrak{U} \to \mathbb{R}$, we can ask if μ *defines* a Θ-stratification of $\mathfrak{X}$, referring to the more general context of Section 2.2 (see in particular (10)).

Fix a numerical invariant $\mu : \mathfrak{U} \subset \mathscr{C}omp(\mathfrak{X}) \to \mathbb{R}$. For any $p \in \mathfrak{X}(k)$, the canonical continuous map $\mathscr{D}eg(p) \to \mathscr{C}omp(\mathfrak{X})$ allows us to consider the preimage $\mathfrak{U}_p \subset \mathscr{D}eg(p)$ of $\mathfrak{U}$ and restrict μ to a continuous function on $\mathfrak{U}_p$, which we also denote by μ. Note that this is compatible with the map $\mathscr{D}eg(p) \to \mathscr{D}eg(p')$ if k'/k is a field extension and $p' \in \mathfrak{X}(k')$ the k'-point induced by p. This allows us to reformulate the Harder-Narasimhan problem as follows:

Problem 4.3 (Harder-Narasimhan)**.** Given an unstable $p \in \mathfrak{X}(k)$, is there a unique rational point $f \in \mathfrak{U}_p \subset \mathscr{D}eg(\mathfrak{X}, p)$ which maximizes $\mu(f)$?

If such an f exists, it corresponds to a filtration $f : \Theta_k \to \mathfrak{X}$ of p up to composition with the ramified covering maps $(\bullet)^n : \Theta_k \to \Theta_k$. If f is still a maximizer for μ after base change to the algebraic closure $\bar{k}$, it will be a Harder-Narasimhan filtration in the sense of part (1) of Theorem 2.7.

4.1.1. *Properties of numerical invariants.*

Definition 4.4. We shall refer to a numerical invariant on $\mathfrak{X}$ as *(locally, strictly) quasi-concave* if for any field k and $p \in \mathfrak{X}(k)$ the subset

$$\mathfrak{U}_p^{\mu>0} := \{x \in \mathfrak{U}_p | \mu(x) > 0\} \subset \mathscr{D}eg(\mathfrak{X}, p)$$

is (locally) convex and the restriction of $\mu : \mathfrak{U} \to \mathbb{R}$ to $\mathfrak{U}_p^{\mu>0}$ is (strictly) quasi-concave.

This notion is useful because of the following

Corollary 4.4.1 (Uniqueness of HN filtrations)**.** *Let $\mathfrak{X}$ be a stack and let $\mu : \mathfrak{U} \subset \mathscr{C}omp(\mathfrak{X}) \to \mathbb{R}$ be a strictly quasi-concave numerical invariant, then an HN filtration of any unstable point $p \in \mathfrak{X}(k)$ is unique up to the action of $\mathbb{N}^\times$, if it exists.*

Proof. By definition the subset $\mathfrak{U}_p^{\mu>0} \subset \mathscr{D}eg(\mathfrak{X}, p)$ is convex and the function μ is strictly quasi-concave, so this follows from Lemma 3.30. □

In order to formulate the standard hypotheses for a numerical invariant, we need the following:

Definition 4.5. We will say that $f_1, f_2 \in \mathrm{Filt}(\mathfrak{X})(k)$ are *antipodal* if there is a 2-commutative diagram of the form

$$\begin{tikzcd}
\Theta_k \arrow[dr] \arrow[drr, bend left, "f_1"] & & \\
& (\mathrm{pt}/\mathbb{G}_m)_k \arrow[r] & \mathfrak{X} \\
\Theta_k \arrow[ur] \arrow[urr, bend right, "f_2"'] & &
\end{tikzcd}$$

such that the cocharacters $(\mathbb{G}_m)_k \to (\mathbb{G}_m)_k$ induced by the two maps $\Theta_k \to (\mathrm{pt}/\mathbb{G}_m)_k$ have opposite sign. Note in particular that if f_1 and f_2 are antipodal, then they are both split filtrations. We say that two points of $\mathscr{C}omp(\mathfrak{X})$ are antipodal if the corresponding connected components of $\mathrm{Filt}(\mathfrak{X})$ contain a pair of antipodal points.

Example 4.6. When $\mathfrak{X} = B\mathrm{SL}_2$, the degeneration space $\mathscr{D}eg(\mathfrak{X}, *)$ is an infinite disjoint union of points, one for each Borel subgroup of SL_2, i.e. one point for each line in $\mathbb{A}_k^2$ regarded as a two step filtration $L \subset k^2$. For any two distinct lines, there is a unique (up to multiples) one parameter subgroup of SL_2 acting with positive weight on L_1 and negative weight on L_2. This one parameter subgroup corresponds to a map $\mathrm{pt}/\mathbb{G}_m \to \mathfrak{X}$, and the maps $\Theta \to \mathfrak{X}$ corresponding to the two Borel subgroups factor through this $\mathrm{pt}/\mathbb{G}_m$. Thus any two distinct points of $\mathscr{D}eg(B\mathrm{SL}_2, *)$ are antipodal.

Definition 4.7 (Standard numerical invariant)**.** We say that a numerical invariant $\mu : \mathcal{U} \subset \mathscr{C}omp(\mathfrak{X}) \to \mathbb{R}$ is *standard* if it is locally strictly quasi-concave and $\mathcal{U}^{\mu>0}$ does not contain a pair of antipodal points.

Note that any strictly quasi-concave numerical invariant satisfies the property that $\mathcal{U}^{\mu>0}$ does not contain a pair of antipodal points, so it is automatically standard. We will see in Corollary 4.21.1 below that any standard numerical invariant on a Θ-reductive stack is strictly quasi-concave.

Given a point $p \in \mathfrak{X}(k)$ for an algebraically closed field k, the existence of a maximizer of μ on $\mathcal{U}_p \subset \mathscr{D}eg(p)$ is weaker than the existence of a HN filtration for p, because the maximum need not occur at a rational point. We will therefore often restrict our focus to numerical invariants which satisfy the following condition:

(R) For any rational simplex $\Delta \to \mathcal{U} \subset \mathscr{C}omp(\mathfrak{X})$, if the restriction of μ to Δ has a point with $\mu > 0$, then μ obtains a maximum at a rational point of Δ.

Lemma 4.8. *Let $\mu : \mathcal{U} \subset \mathscr{C}omp(\mathfrak{X}) \to \mathbb{R}$ be a numerical invariant satisfying (R), let $p \in \mathfrak{X}(k)$, and let $\bar{p} \in \mathfrak{X}(\bar{k})$ be the induced point over the algebraic closure. Then p has a HN filtration if and only if $\mu : \mathcal{U}_{\bar{p}} \subset \mathscr{D}eg(\bar{p}) \to \mathbb{R}$ obtains a maximum.*

Proof. Because the flag space $\mathrm{Flag}(p)$ is locally finite type over k and is compatible with base change, p has a HN filtration if and only if $\bar{p}$ does. Condition (R) means that μ obtains a maximum on $\mathcal{U}_{\bar{p}}$ if and only if it obtains a maximum at a rational point. □

4.1.2. *Numerical invariants from cohomology classes.*

Definition 4.9. Let $b \in H^4(\mathfrak{X};\mathbb{R})$ be positive semi-definite, and let $\ell \in H^2(\mathfrak{X};\mathbb{R})$. Then the *numerical invariant associated to ℓ and b* is the pair

$$\mathcal{U} := \left\{ x \in \mathscr{C}omp(\mathfrak{X}) \,\middle|\, \hat{b}(\tilde{x}) > 0 \right\} \quad \text{and} \quad \mu(x) = \frac{\hat{l}(\tilde{x})}{\sqrt{\hat{b}(\tilde{x})}},$$

where $\tilde{x} \in |\,\mathbf{CF}(\mathfrak{X})_\bullet|$ is some lift of $x \in \mathscr{C}omp(\mathfrak{X})$.

Note that both $\mathcal{U}$ and μ are well-defined by the homogeneity properties of $\hat{l}$ and $\hat{b}$ in Lemma 3.83. $\mathcal{U}$ is open and hence realizable, and μ is continuous on $\mathcal{U}$, so Definition 4.9 indeed defines a numerical invariant. We will almost always be interested in the case when b is positive definite.

Lemma 4.10. *The numerical invariant associated to $\ell \in H^2(\mathfrak{X};\mathbb{R})$ and $b \in H^4(\mathfrak{X};\mathbb{R})$ is locally quasi-concave, and $\mathcal{U}^{\mu>0}$ does not contain a pair of antipodal points. It is standard (Definition 4.7) if b is positive definite.*

Proof. To show local quasi-concavity, it suffices to consider the restriction of μ along the map $\mathbb{R}^n_{\geq 0} \to |\,\mathbf{DF}(\mathfrak{X},p)_\bullet| \to |\,\mathbf{CF}(\mathfrak{X})_\bullet|$ defined for a $p \in \mathfrak{X}(k)$

and $\xi \in \mathbf{DF}(p)_n$. For any $c \geq 0$, the subset where $\mu > c$ corresponds to the subset of x in $\mathbb{R}^n_{\geq 0}$ where

$$\hat{\ell}(x) - c\sqrt{\hat{b}(x)} > 0 \qquad \text{and } \hat{b}(x) > 0. \tag{28}$$

$\hat{\ell}$ is linear, and $\sqrt{\hat{b}}$ is a seminorm, so the left hand side above is a concave function. It follows that $\{x \in \mathbb{R}^n_{\geq 0} | \hat{b}(x) > 0 \text{ and } \mu(x) > c\}$ is convex for any c. Applying this to $c = 0$ shows that $\mathcal{U}_p^{\mu>0}$ is convex and hence does not contain a pair of antipodal points, and applying this to $c > 0$ shows that μ is quasi-concave on $\mathcal{U}_p^{\mu>0}$. When b is positive definite, then the function in (28) is strictly concave, which implies that $\mu > c$ on the interior of the set $\{\mu(x) \geq c\}$, hence μ is locally strictly quasi-concave. □

Remark 4.11. Note that the argument above shows that the function $\mu(x) = \hat{\ell}(\tilde{x})/\Phi(\tilde{x})$ on $\mathcal{U} = \mathscr{C}omp(\mathcal{X})$ defines a standard numerical invariant for any function $\Phi : |\,\mathbf{CF}(\mathcal{X})_\bullet| \to \mathbb{R}$ which restricts to a norm on each rational cone.

Lemma 4.12. *Any numerical invariant $\mu : \mathcal{U} \subset \mathscr{C}omp(\mathfrak{X}) \to \mathbb{R}$ associated to $\ell \in H^2(\mathfrak{X};\mathbb{Q})$ and a positive definite $b \in H^4(\mathfrak{X};\mathbb{Q})$ as in Definition 4.9 satisfies condition (R).*

Proof. Let $\Delta_\sigma \to \mathcal{U}$ be a rational simplex such that $\mu > 0$ at some point in Δ_σ. Restricted to $\Delta_\sigma = \mathbb{R}^n_{\geq 0}/\mathbb{R}^\times_{>0}$, μ has the form $\ell \cdot x/\sqrt{b(x)}$ where $\ell \in \mathbb{Q}^n$ and $b(x)$ is a rational quadratic form such that $b(x) > 0$ for nonzero $x \in \mathbb{R}^n_{\geq 0}$. We know that $\mu(x)$ achieves a maximum on $\mathbb{R}^n_{\geq 0}$ which is > 0, so the affine hyperplane $\{\ell \cdot x = 1\}$ meets the cone $\mathbb{R}^n_{\geq 0}$. Maximizing $\ell \cdot x/\sqrt{b(x)}$ on $\mathbb{R}^n_{\geq 0}$ is thus equivalent to minimizing the rational quadratic form $b(x)$ on the rational polyhedron $\mathbb{R}^n_{\geq 0} \cap \{\ell \cdot x = 1\}$.

The Kuhn-Tucker condition[14] for this convex minimization problem states that $x \in \mathbb{R}^n$ is a global minimum for this constrained optimization problem if and only if it satisfies the constraints and there is a Lagrange multiplier $\lambda \in \mathbb{R}$ and a Kuhn-Tucker multiplier $\tau \in \mathbb{R}^n_{\geq 0}$ such that

$$0 = \nabla b(x) - \tau + \lambda\ell \text{ and } \tau \cdot x = 0.$$

These two equality constraints on (x, τ, λ) combined with the equality constraint $\ell \cdot x = 0$ define a rational linear subspace of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. Thus the set of points (x, τ, λ) representing a solution of the Kuhn-Tucker condition is the intersection if a rational linear subspace with the rational polyhedral cone $\{x \geq 0, \tau \geq 0\}$, and such a set always contains a rational point if it is nonempty. □

Example 4.13. Let X be a projective over affine k-variety with a reductive group action. Let $\mathcal{L}$ be a G-linearized ample invertible sheaf on X, and let $|\bullet|$

[14] The Kuhn-Tucker condition is analogous to Lagrange multiplier equations, but for optimization problems with inequality constraints.

be a Weil-group-invariant positive definite bilinear form on the cocharacter lattice of G. Theorem 1.37 identifies a filtration f of a point $p \in X(k)$ with a one parameter subgroup $\lambda : \mathbb{G}_m \to G$ such that $q := \lim_{t\to 0} \lambda(t) \cdot p$ exists, taken up to conjugation by an element of $P_\lambda(k)$. One can choose a class $l \in H^2(\mathfrak{X};\mathbb{Q})$ and a positive definite $b \in H^4(\mathfrak{X};\mathbb{Q})$ such that

$$\mu(f) = \frac{f^* l}{\sqrt{f^* b}} = \mu(p,\lambda) = \frac{-1}{|\lambda|} \operatorname{weight}_\lambda \mathcal{L}|_q \in \mathbb{R}. \tag{29}$$

is the normalized Hilbert-Mumford numerical invariant [DH]. Thus Θ-stability agrees with GIT stability, and the Harder-Narsimhan filtration of an unstable point corresponds Kempf's canonical destabilizing one parameter subgroup [K1].

Proof. We define $l = c_1(\mathcal{L}) \in H^2(\mathfrak{X};\mathbb{Q})$, so that

$$f^* l = c_1(f^*\mathcal{L}) = -\operatorname{weight}((f^*\mathcal{L})_{\{0\}}) \cdot q,$$

which follows from the fact[15] that $f^*\mathcal{L} \simeq \mathcal{O}_\Theta(w)$ where $w = -\operatorname{weight}(f^*\mathcal{L})_{\{0\}}$. For the denominator, $|\bullet|$ can be interpreted as a class in $H^4(\mathrm{pt}/G;\mathbb{C})$ under the identification $H^*(\mathrm{pt}/G;\mathbb{C}) \simeq (\mathrm{Sym}(\mathfrak{g}^\vee))^G$, and we let b be the image of this class under the map $H^4(\mathrm{pt}/G) \to H^4(X/G)$. For a morphism $f : \Theta \to X/G$, f^*b is the pullback of the class in $H^4(\mathrm{pt}/G)$ under the composition $\Theta \to X/G \to \mathrm{pt}/G$. We therefore have $f^*b = |\lambda|^2 q^2 \in H^4(\Theta)$. $\square$

4.1.3. *Induced numerical invariants.* Given a stack $\mathfrak{X}$ satisfying (†) and a numerical invariant $\mu : \mathcal{U} \to \mathbb{R}$ with $\mathcal{U} \subset \mathscr{C}omp(\mathfrak{X})$, we define the *restriction of* μ along a map of stacks $\pi : \mathfrak{Y} \to \mathfrak{X}$ with quasi-finite inertia (for instance, a representable map) to be the function

$$\mu \circ \pi_* : (\pi_*)^{-1}(\mathcal{U}) \to \mathbb{R},$$

where $\pi_* : \mathscr{C}omp(\mathfrak{Y}) \to \mathscr{C}omp(\mathfrak{X})$ is the associated map on component spaces. We are particularly interested in the case where $\mathfrak{Y} = \mathrm{Grad}(\mathfrak{X})$ and $u : \mathrm{Grad}(\mathfrak{X}) \to \mathfrak{X}$ is the canonical forgetful map.

Lemma 4.14. *Let $\pi : \mathfrak{Y} \to \mathfrak{X}$ be a map of stacks which satisfy* (†)*, and let $\mu : \mathcal{U} \subset \mathscr{C}omp(\mathfrak{X}) \to \mathbb{R}$ be a numerical invariant which defines a (weak) Θ-stratification $\{\mathfrak{X}_{\leq c}\}_{c\geq 0}$. If $\{\mathfrak{X}_{\leq c}\}_{c\geq 0}$ induces a (weak) Θ-stratification $\{\mathfrak{Y}_{\leq c}\}_{c\geq 0}$ of $\mathfrak{Y}$, then the resulting (weak) Θ-stratification is the stratification defined by the induced invariant $\mu \circ \pi_*$.*

Proof. Recall from Equation (10) that a Θ-stratification (with locally constant μ) can be uniquely recovered from a subset $S \subset \pi_0(\mathrm{Filt}(\mathfrak{X}))$ and a map $\mu : S \to \mathbb{R}_{\geq 0}$ define a Θ-stratification of S by definine $|\mathfrak{X}_{\leq c}| = \{p \in |\mathfrak{X}|| M^\mu(p) \leq$

[15]Every invertible sheaf on Θ is of the form $\mathcal{O}_\Theta(n)$, which correspond to the free $k[t]$ module with generator in degree $-n$. Note that the isomorphism $\mathrm{Pic}(\Theta) \simeq \mathbb{Z}$ is canonical, because $\Gamma(\Theta, \mathcal{O}_\Theta(n)) = 0$ for $n > 0$ whereas $\Gamma(\Theta, \mathcal{O}_\Theta(n)) \simeq k$ for $n \leq 0$. This holds in contrast to $\mathrm{Pic}(\mathrm{pt}/\mathbb{G}_m)$. The stack $\mathrm{pt}/\mathbb{G}_m$ has an automorphisms exchanging $\mathcal{O}(1)$ and $\mathcal{O}(-1)$.

$c\}$ and $|\mathfrak{S}_c| = \{f \in |\operatorname{Filt}(\mathfrak{X})| | f$ lie on a component in S, and $\mu(f) = M^\mu(f)\}$. In our case S is a set of orbit representatives for the action of $\mathbb{N}^\times$ on

$$\pi_0(\operatorname{Filt}(\mathfrak{X}))_{\mathfrak{U}} := \{\alpha \in \pi_0(\operatorname{Filt}(\mathfrak{X})) | \alpha \in \mathfrak{U} \subset \mathscr{C}omp(\mathfrak{X})\},$$

and μ is given by the numerical invariant $\mu : \pi_0(\operatorname{Filt}(\mathfrak{X}))_{\mathfrak{U}} \to \mathbb{R}$. If $\pi : \mathfrak{Y} \to \mathfrak{X}$ induces a stratification of $\mathfrak{Y}$, then one can likewise encode this stratification by data $\mu' : S' \subset \pi_0(\operatorname{Filt}(\mathfrak{Y})) \to \mathbb{R}$. In this case S' is the preimage of S under $\operatorname{Filt}(\pi) : \operatorname{Filt}(\mathfrak{Y}) \to \operatorname{Filt}(\mathfrak{X})$, and μ' is the composition $S' \to S \to \mathbb{R}$. This is by definition the data of the induced numerical invariant of $\mathfrak{Y}$. $\square$

In particular, we have:

Corollary 4.14.1. *If a numerical invariant μ defines a Θ-stratification of $\mathfrak{X}$, then the restriction of μ to $\operatorname{Grad}(\mathfrak{X})$ along the map $u : \operatorname{Grad}(\mathfrak{X}) \to \mathfrak{X}$ defines a Θ-stratification, and this is the same as the induced Θ-stratificaton of Proposition 2.15.*

For the induced Θ-stratification of $\operatorname{Grad}(\mathfrak{X})$, many connected components do not technically contain a semistable point. Indeed the numerical invariant μ on $\mathfrak{X}$ defines a locally constant function on $\operatorname{Grad}(\mathfrak{X})$, because $\pi_0(\operatorname{Grad}(\mathfrak{X})) = \pi_0(\operatorname{Filt}(\mathfrak{X}))$. If we consider the restriction $\mu \circ u_*$ of μ to $\operatorname{Grad}(\mathfrak{X})$, then for any $p \in \operatorname{Grad}(\mathfrak{X})(k)$, $\mu(u_*(\mathfrak{c}_p)) = \mu(p)$, where $\mathfrak{c}_p$ is the canonical filtration of p. Therefore on any component of $\operatorname{Grad}(\mathfrak{X})$ on which $\mu > 0$, every point is destabilized by its canonical filtration. Moreover $M^{\mu \circ u_*}_{\operatorname{Grad}(\mathfrak{X})}(p) \geq \mu(p)$.

Definition 4.15. We say that a point $p \in \operatorname{Grad}(\mathfrak{X})$ is *graded-semistable* if $M^{\mu \circ u_*}_{\operatorname{Grad}(\mathfrak{X})}(p) = \mu(p)$, i.e. there are no filtrations of p on which the value of the numerical invariant is higher than its value on the canonical filtration of p, which is the same as the value of the numerical invariant on the split filtration $\sigma(p)$ in $\mathfrak{X}$.

If μ is the numerical invariant associated to $\ell \in H^2(\mathfrak{X})$ and $b \in H^4(\mathfrak{X})$ as in Definition 4.9, then we can define a new numerical invariant on $\operatorname{Grad}(\mathfrak{X})$ by the formula

$$\mu'(f) = \mu(u_*(f)) - \mu(f(1))$$

for any filtration $f : \Theta_k \to \operatorname{Grad}(\mathfrak{X})$. With this modified numerical invariant, a point $p \in \operatorname{Grad}(\mathfrak{X})(k)$ is graded-semistable with respect to μ if and only if it is semistable with respect to μ'. Combining this with Corollary 2.16.1, we see that if $\mathcal{Z}_c \subset \operatorname{Grad}(\mathfrak{X})$ is the union of the connected components which meet the center $\mathcal{Z}_c^{\mathrm{ss}}$ of the Θ-stratum $\mathcal{S}_c \hookrightarrow \mathfrak{X}$, then the center $\mathcal{Z}_c^{\mathrm{ss}}$ is precisely the semistable locus for the numerical invariant induced by μ' on $\mathcal{Z}_c$, which differs from μ by a constant shift by $c \in \mathbb{R}$.

This discussion of graded-semistability generalizes the inductive description of the KN stratification in geometric invariant theory given in [K2], where it is shown that the centers of the strata induced by a reductive group acting on a projective variety are themselves semistable loci for a reductive subgroup acting on a closed subvariety.

4.2. **Θ-reductive stacks.** We now come to the main class of stacks considered in this paper.

Definition 4.16. A stack $\mathfrak{X}$ is Θ-*reductive* if the morphism $\mathrm{ev}_1 : \mathrm{Filt}_B(\mathfrak{X}) \to \mathfrak{X}$ satisfies the valuative criterion for properness for all discrete valuation rings.

By the valuative criterion for properness in Definition 4.16, we mean the condition that for any valuation ring R with function field K any commutative diagram of the form

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathrm{Filt}(\mathfrak{X}) \arrow[d, "\mathrm{ev}_1"] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed] & \mathfrak{X} \end{tikzcd} ,$$

has a unique dotted arrow making the diagram commute. This is the "usual" valuative criterion for maps of schemes. In words, it states that for any family over $\mathrm{Spec}(R)$, any filtration of the generic point extends uniquely to a filtration of the family. In most of our examples, one can actually verify the valuative criterion for ev_1 for all valuation rings R, but our results only need the condition for discrete valuation rings.

Another way to state this property is that for any discrete valuation ring R over the base stack B and any map $\Theta_R - \{(0,0)\} \to \mathfrak{X}$ relative to B, where $(0,0) \in \Theta_R$ denotes the closed point, there exists a unique extension to a map $\Theta_R \to \mathfrak{X}$ relative to B. This is a relative notion for the map $\mathfrak{X} \to B$, and in particular it is smooth local on B.

Remark 4.17. We have stated Definition 4.16 for arbitrary stacks, and in general there are several different versions of the valuative criterion one might require. This subtely does not affect our discussion, because for stacks satisfying (†), we know from Lemma 1.10 that ev_1 is representable by algebraic spaces, quasi-separated, and finite type, so the various versions of the valuative criterion are equivalent [S2, Tag 0ARI].

Example 4.18. The stack $\mathfrak{X} = \mathrm{pt}/G$ is Θ-reductive for any reductive group G over a field k. Indeed, the formation of $\mathrm{Filt}(\mathrm{pt}/G)$ commutes with base change to an algebraic closure $\bar{k}/k$, so we may assume G is split. Applying Theorem 1.37, we see that the fiber of $\mathrm{ev}_1 : \mathrm{Filt}(\mathrm{pt}/G) \to \mathrm{pt}/G$ over the point is an infinite disjoint union of generalized flag varieties G/P_λ, which are proper.

Example 4.19. Let $V = \mathrm{Spec}(k[x,y,z])$ be a linear representation of $\mathbb{G}_m$ where x, y, z have weights $-1, 0, 1$ respectively, and let $X = V - \{0\}$ and $\mathfrak{X} = X/(\mathbb{G}_m)_k$. The fixed locus is the punctured line $Z = \{x = z = 0\} \cap X$, and the connected component of $\mathrm{Filt}(\mathfrak{X})$ corresponding to the cocharacter $\lambda(t) = t$ is the quotient $S/(\mathbb{G}_m)_k$ where

$$S = \{(x,y,z) | z = 0 \text{ and } y \neq 0\}$$

$S \subset X$ is not closed. Its closure contains the points where $x \neq 0$ and $y = 0$. These points would have been attracted by λ to the missing point $\{0\} \in V$. It follows that $S/(\mathbb{G}_m)_k \to X/(\mathbb{G}_m)_k$ is not proper, and hence $X/(\mathbb{G}_m)_k$ is not Θ-reductive.

Our first method for constructing new Θ-reductive stacks is a consequence of Proposition 1.19:

Corollary 4.19.1. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be an affine morphism of algebraic stacks which satisfy (†). If $\mathfrak{Y}$ is Θ-reductive then $\mathfrak{X}$ is Θ-reductive.*

Proof. Proposition 1.19 states that the canonical map $\mathrm{Filt}(\mathfrak{X}) \to \mathrm{Filt}(\mathfrak{Y}) \times_{\mathfrak{Y}} \mathfrak{X}$ is a closed immersion, and if $\mathfrak{Y}$ is Θ-reductive, then $\mathrm{Filt}(\mathfrak{Y}) \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{X}$ satisfies the valuative criterion for properness with respect to discrete valuation rings. It follows that the composition $\mathrm{Filt}(\mathfrak{X}) \to \mathfrak{X}$ satisfies the valuative criterion for properness for DVR's. □

Example 4.20. Corollary 4.19.1 implies that any stack of the form V/G, where G is a reductive group acting on an affine scheme V, is Θ-reductive.

Another source of Θ-reductive stacks arises from the theory of Θ-stability. We will show in Proposition 4.26 that the substack of semistable points in a Θ-reductive stack is Θ-reductive. We will see other examples of Θ-reductive stacks in Section 5.

4.3. **Uniqueness and recognition of HN filtrations, and the semistable locus.** In this section we describe two ways in which Θ-reductive stacks are particularly well-suited for Problem 4.3, and we show that the semistable locus in a Θ-reductive stack is again Θ-reductive.

4.3.1. *Convexity of the degeneration space and uniqueness of HN filtrations.* Let $p \in \mathfrak{X}(k)$, and consider two k-points f_1, f_2 of $\mathrm{Flag}(p)$. Let $U = \mathbb{A}^1_k - \{0\}$. We consider f_1 and f_2 as morphisms $U \times \mathbb{A}^1_k/(\mathbb{G}_m)^2_k \to \mathfrak{X}$ and $\mathbb{A}^1_k \times U/(\mathbb{G}_m)^2_k \to \mathfrak{X}$ respectively with a fixed isomorphism of their restrictions to $U \times U/(\mathbb{G}_m)^2_k \simeq \mathrm{pt}$, so we can glue them to define

$$f_1 \cup f_2 : (\mathbb{A}^2_k - \{0\})/(\mathbb{G}_m)^2_k \to \mathfrak{X}$$

And the data of the morphism $f_1 \cup f_2$ is equivalent to the data of the pair f_1, f_2. One of the key properties of Θ-reductive stacks is that any such morphism from $(\mathbb{A}^2_k - \{0\})/(\mathbb{G}_m)^2_k$ extends uniquely to $\mathbb{A}^2_k/(\mathbb{G}_m)^2_k$.[16]

Proposition 4.21. *Let $\mathfrak{X}$ be a Θ-reductive stack satisfying (†), and let $p \in \mathfrak{X}(k)$. Then any two distinct rational points in $\mathscr{D}eg(p)$ are either antipodal or connected by a unique rational 1-simplex in $\mathscr{D}eg(p)$.*

[16] We can rephrase the construction of $f_1 \cup f_2$ above more formally and more generally as the observation that restriction defines an equivalence of stacks

$$\underline{\mathrm{Map}}\left((\mathbb{A}^2 - \{0\})/\mathbb{G}_m^2, \mathfrak{X}\right) \xrightarrow{\simeq} \mathrm{Filt}(\mathfrak{X}) \times_{\mathfrak{X}} \mathrm{Filt}(\mathfrak{X}),$$

where the fiber product is taken with respect to ev_1.

We will prove Proposition 4.21 at the end of this section, but first we observe that this immediately implies the uniqueness of HN filtrations for points in Θ-reductive stacks.

Corollary 4.21.1. *Let $\mu : \mathcal{U} \subset \mathscr{C}omp(\mathfrak{X}) \to \mathbb{R}$ be a locally quasi-concave numerical invariant on a Θ-reductive stack $\mathfrak{X}$ with quasi-affine diagonal. If $\mathcal{U}^{\mu>0}$ does not contain any pair of antipodal points, then for any $p \in \mathfrak{X}(k)$ the subset $\mathcal{U}_p^{\mu>0} \subset \mathscr{D}eg(\mathfrak{X}, p)$ is convex. Hence μ is quasi-concave.*

In the context of Corollary 4.21.1, if μ is locally *strictly* quasi-concave, then it is strictly quasi-concave. Hence Corollary 4.4.1 implies that *an HN filtration of any unstable point $p \in \mathfrak{X}(k)$ must be unique up to the action of $\mathbb{N}^\times$, if it exists.*

Example 4.22. For a Θ-reductive stack $\mathfrak{X}$, any $\ell \in H^2(\mathfrak{X}; \mathbb{Q})$, and two antipodal filtrations f_1, f_2, the numbers $\hat{\ell}(f_1)$ and $\hat{\ell}(f_2)$ have opposite signs. It follows that any numerical invariant associated to cohomology classes as in Definition 4.9 satisfies the hypothesis of Corollary 4.21.1.

The technical heart of the proof of Proposition 4.21 is the following lemma.

Lemma 4.23. *Let $f : \Theta^2 \to \mathfrak{X}$ be a morphism such that $\ker(\mathbb{G}_m^2 \to \operatorname{Aut} f(0))$ has positive dimensional kernel. Then either the two pointed maps $\Theta \to \mathfrak{X}$ induced by the restriction $f|_{\mathbb{A}^2 - \{0\}/\mathbb{G}_m^2}$ are antipodal, or there is a commutative diagram of the form*

$$\begin{tikzcd} \Theta^2 \arrow[r] \arrow[dr] & \Theta^2 \arrow[r, "f"] & \mathfrak{X} \\ & \Theta \arrow[ur] & \end{tikzcd}$$

where $\Theta^2 \to \Theta^2$ is of the form $(z_1, z_2) \mapsto (z_1^a, z_2^b)$ for some $a, b > 0$, and $\Theta^2 \to \Theta$ maps $(1,1)$ to 1.

Proof. As $G = \ker(\mathbb{G}_m^2 \to \operatorname{Aut} f(0))$ is a positive dimensional subgroup, it contains a one dimensional torus G_0 of rank 1. This subgroup is determined by a vector $[a, b]$ in the cocharacter lattice of $\mathbb{G}_m^2$. We prove the claim in three cases:

Case 1: $ab < 0$

By permuting the two coordinates we may assume that $a > 0$ and $b < 0$. Then composing with the non-degenerate map $\Theta^2 \to \Theta^2$ defined by (z_1^a, z_2^{-b}), the new map $\Theta^2 \to \mathfrak{X}$ has a subgroup G_0 corresponding to the vector $[1, -1]$ in the cocharacter lattice of G_0. Let us assume then that $a = 1$ and $b = -1$. So that G_0 is precisely the kernel of the homomorphism $\phi : \mathbb{G}_m^2 \to \mathbb{G}_m$ given by $(z_1, z_2) \mapsto z_1 z_2$.

Consider the morphism $\pi : \Theta^2 \to \Theta$ defined by $(z_1, z_2) \mapsto z_1 z_2$, which is equivariant with respect to ϕ. We claim that the pullback functor π^* induces an isomorphism

$$\pi^* : \operatorname{Perf}(\Theta) \xrightarrow{\simeq} \mathcal{C} := \left\{ F^\bullet \in \operatorname{Perf}(\Theta^2) \,\middle|\, F^\bullet|_{\{0\}} \text{ has weight 0 w.r.t. } G_0 \right\} \tag{30}$$

The proof uses the main structure theorem of [HL1]. The one parameter subgroup $\lambda(t) = z_1^t z_2^{-t}$ generating G_0 defines a KN-stratum $Y = \{(z_1, z_2) | z_2 = 0\} \subset \mathbb{A}^2$, with fixed locus $Z = \{(0,0)\} \in \mathbb{A}^2$. According to the main theorem of [HL1], the restriction functor $\mathcal{C} \to \mathrm{Perf}(\Theta^2 \setminus (Y/\mathbb{G}_m^2)) \simeq \mathrm{Perf}(\mathbb{A}^1_{z_1}/\mathbb{G}_m)$ is a fully faithful functor of symmetric monoidal ∞-categories. Furthermore the composition

$$(\mathbb{A}^1_{z_1} \times \{1\})/\mathbb{G}_m \to \mathbb{A}^2/\mathbb{G}_m^2 \xrightarrow{\pi} \mathbb{A}^1/\mathbb{G}_m$$

is an isomorphism of stacks. It follows that the functor $\pi^* : \mathrm{Perf}(\Theta) \to \mathcal{C}$ is an equivalence.

It follows that any symmetric monoidal functor $\mathrm{Perf}(\mathfrak{X}) \to \mathrm{Perf}(\Theta^2)$ landing in the subcategory $\mathcal{C}$ factors uniquely as a symmetric monoidal functor through $\pi^* : \mathrm{Perf}(\Theta) \to \mathrm{Perf}(\Theta^2)$. The same argument applies to the categories APerf and APerf^{cn}. Hence the Tannakian formalism [L3] implies that the morphism $f : \Theta^2 \to \mathfrak{X}$ must factor uniquely through $\pi : \Theta^2 \to \Theta$.

Case 2: $ab > 0$

We will show that two points of $\mathrm{Filt}(\mathfrak{X})$ determined by $f|_{\mathbb{A}^2 - \{0\}/\mathbb{G}_m^2}$ are antipodal. Choose a generator for G_0 of the form $[a, b]$ with $a, b > 0$. Again we use the results of [HL1]. We can consider Θ^2 to consist of a single KN-stratum with one parameter subgroup $\lambda(t) = z_1^{at} z_2^{bt}$, and $Z = \{(0)\} \in \mathbb{A}^2$. In this case the baric structure on the KN-stratum Θ^2 shows that the pullback along the projection $\pi : \Theta^2 \to \mathrm{pt}/\mathbb{G}_m^2$ induces an equivalence

$$\pi^* : \{F^\bullet \in \mathrm{Perf}(\mathrm{pt}/\mathbb{G}_m^2) \text{ with } \lambda\text{-weight} 0\} \xrightarrow{\simeq} \{F^\bullet \in \mathrm{Perf}(\Theta^2) | F^\bullet|_Z \text{ has } \lambda\text{-weight } 0\}$$

The subcategory of $\mathrm{Perf}(\mathrm{pt}/\mathbb{G}_m^2)$ can be identified with the $\mathrm{Perf}(\mathrm{pt}/\mathbb{G}_m)$ via the pullback functor along the morphism $\phi : \mathrm{pt}/\mathbb{G}_m^2 \to \mathrm{pt}/\mathbb{G}_m$, where ϕ is the group homomorphism $(z_1, z_2) \mapsto z_1^{-b} z_2^a$. As in the previous case, this argument extends to the categories APerf and APerf^{cn} as well, so the tannakian formalism implies that the map $\Theta^2 \to \mathfrak{X}$ factors through the composition $\Theta^2 \to \mathrm{pt}/\mathbb{G}_m^2 \xrightarrow{\phi} \mathrm{pt}/\mathbb{G}_m$. Note that ϕ maps the two factors of $\mathbb{G}_m^2$ to opposite cocharacters of $\mathbb{G}_m$ (up to positive multiple), so we have exhibited that the two pointed maps are antipodal.

Case 3: $ab = 0$

In this case, we may assume, by permuting coordinates, that $a = 0$, so that G_0 is generated by the one parameter subgroup $\lambda(t) = z_2^t$. Let $Z = \mathbb{A}^1_{z_1} \times \{0\} \subset \mathbb{A}^2$, and note that this subvariety is equivariant with respect to $\mathbb{G}_m^2$ and fixed by λ. Any $F^\bullet \in \mathrm{Perf}(Z/\mathbb{G}_m^2)$ decomposes canonically into a direct sum of objects on which λ acts with fixed weight. Because any $\mathbb{G}_m^2$ equivariant open subset containing (0) contains all of Z, semicontinuity implies that if $F^\bullet|_{\{0\}}$ is concentrated in weight 0 with respect to λ, then it is concentrated in weight 0 everywhere on Z. Therefore, if $F^\bullet \in \mathrm{Perf}(\Theta^2)$ and

$F^\bullet|_{\{0\}}$ has weight 0 with respect to λ, then $F^\bullet|_Z$ has weight 0 with respect to λ as well.

If we consider the projection $\pi : \Theta^2 \to \Theta$ given by $(z_1, z_2) \mapsto z_1$, then the counit of adjunction is an isomorphism $\mathrm{id}_{\mathrm{Perf}(\Theta)} \simeq R\pi_* L\pi^*$. It follows that π^* is fully faithful. Its image is exactly the full subcategory $\{F^\bullet \in \mathrm{Perf}(\Theta^2) | F|_Z \text{ has } \lambda\text{-weight } 0\}$, which we have seen is precisely the subcategory $\{F^\bullet \in \mathrm{Perf}(\Theta^2) | F^\bullet|_{\{0\}} \text{ has } \lambda\text{-weight } 0\}$. As in the previous cases, the Tannakian formalism implies that $f : \Theta^2 \to \mathfrak{X}$ factors uniquely through π.

$\square$

Proof of Proposition 4.21. Fix two maps $f_1, f_2 : \Theta_k \to \mathfrak{X}$. Then finding a morphism $g : \Theta_k^2 \to \mathfrak{X}$ along with an isomorphism of the restriction of g to $(\mathbb{A}_k^2 - \{0\})/\mathbb{G}_m^2$ with $f_1 \cup f_2$ is equivalent to finding a lift in the diagram

$$\begin{tikzcd}
(\mathbb{A}_k^1 - \{0\})/\mathbb{G}_m \simeq \operatorname{Spec} k \arrow[r] \arrow[d] & \underline{\mathrm{Map}}(\Theta, \mathfrak{X}) \arrow[d, "\mathrm{ev}_1"] \\
\mathbb{A}_k^1/\mathbb{G}_m \arrow[r, "f_1"] \arrow[ur, dashed] & \mathfrak{X}
\end{tikzcd}$$

where the left vertical morphism is the inclusion of the point $\{1\}$, and the top horizontal morphism classifies the morphism $f_2 : \Theta_k \to \mathfrak{X}$.

After restricting f_1 along $\mathbb{A}_k^1 \to \mathbb{A}_k^1/\mathbb{G}_m$, the valuative criterion for properness for the morphism ev_1 guarantees the existence and uniqueness of a lift compatible with the given map $\mathbb{A}^1 - \{0\} \to \underline{\mathrm{Map}}(\Theta, \mathfrak{X})$. Because ev_1 is separated, the restriction of this lift to $\mathbb{G}_m \times \mathbb{A}_k^1$ along either morphism in the groupoid Equation (3) is uniquely determined by its restriction to $\mathbb{G}_m \times (\mathbb{A}_k^1 - \{0\})$. This provides descent datum satisfying the cocycle condition for the lift $\mathbb{A}_k^1 \to \underline{\mathrm{Map}}(\Theta, \mathfrak{X})$. Hence there exists a unique lift $\Theta_k \to \underline{\mathrm{Map}}(\Theta, \mathfrak{X})$.

In order to complete the proof, we must show that if f_1 and f_2 are non-degenerate and distinct, then the unique extension of $f_1 \cup f_2$ to Θ_k^2 is also non-degenerate. Because f_1 and f_2 are non-degenerate, the only point at which the maps of stabilizer groups can have a positive dimensional kernel is the origin, and Lemma 4.23 implies that this can not happen if f_1 and f_2 are distinct, unless they are antipodal. $\square$

4.3.2. *The recognition theorem for HN filtrations.* It might arise that one wishes to verify that a candidate filtration $f : \Theta_k \to \mathfrak{X}$ of a point $f(1) \in \mathfrak{X}(k)$ is the HN filtration.

Theorem 4.24 (Recognition theorem)**.** *Let $\mathfrak{X}$ be a stack satisfying (†), let $\mu : \mathfrak{U} \subset \mathscr{C}omp(\mathfrak{X}) \to \mathbb{R}$ be a standard numerical invariant, and let $f : \Theta_k \to \mathfrak{X}$ be a filtration in $\mathfrak{X}$ with $\mu(f) > 0$. If f is a HN filtration, then* $\mathrm{ev}_0(f) \in \mathrm{Grad}(\mathfrak{X})(k)$ *is graded-semistable (See Definition 4.15). Conversely, if μ is strictly quasi-concave and* $\mathrm{ev}_0(f)$ *is graded-semistable, then f is an HN filtration of $f(1)$.*

We will need the following observation:

Lemma 4.25. *Let $\mathfrak{X}$ be a stack satisfying (†), and let $f \in \mathrm{Filt}(\mathfrak{X})(k)$. Then the two compositions agree*

$$\mathbf{DF}(\mathrm{ev}_0(f))^{\mathfrak{c}}_{\bullet} \underset{u_*}{\overset{\mathbb{T}}{\rightrightarrows}} \mathbf{DF}(f(1))_{\bullet} \longrightarrow \mathbf{CF}(\mathfrak{X})_{\bullet}\ ,$$

where u_* is the map induced by the forgetful map $u : \mathrm{Grad}(\mathfrak{X}) \to \mathfrak{X}$, and $\mathbb{T}$ is the transfer map of Theorem 3.60.

Proof. Considering the concrete description of the map $\mathbb{T}$ of (25) following its definition, for any cone $\xi \in \mathbf{DF}(\mathrm{ev}_0(f))^{\mathfrak{c}}_n$ we have a map $\mathrm{ext}(\xi) : \Theta_k \times \Theta^n_k \to \mathfrak{X}$ whose restriction to $\{t_0 = 0\}$ is a map $(\mathrm{pt}/\mathbb{G}_m)_k \times \Theta^n_k \to \mathfrak{X}$ representing ξ, and the restriction to $\{t_0 = 1\}$ represents $\mathbb{T}(\xi)$. We regard the restriction of $\mathrm{ext}(\xi)$ to $\mathbb{A}^1_k \times \Theta^n_k \to \mathfrak{X}$ as a map $\mathbb{A}^1_k \to \mathrm{Filt}^n(\mathfrak{X})$ which maps $\{0\}$ to the filtration underlying the cone $u_*(\xi)$ and maps $\{1\}$ to the the filtration underlying the cone $\mathbb{T}(\xi)$. Hence these two filtrations lie on the same connected component of $\mathrm{Filt}^n(\mathfrak{X})$, which implies that the two maps take ξ to the same cone in $\mathbf{CF}(\mathfrak{X})_\bullet$. □

Proof of Theorem 4.24. Let μ' denote the restriction of μ to $\mathrm{Grad}(\mathfrak{X})$, and let f' be a filtration of $\mathrm{ev}_0(f) \in \mathrm{Grad}(\mathfrak{X})(k)$ for which μ' is defined and $\mu'(f') > \mu'(\mathfrak{c})$. The filtration f' is not antipodal to $\mathfrak{c}$, because it lies over $\mathcal{U}^{\mu>0}$. Let $\Delta^1_\sigma \to \mathscr{D}eg(\mathrm{Grad}(\mathfrak{X}), \mathrm{ev}_0(f))$ be the canonical rational 1-simplex with $v_0(\sigma) = \mathfrak{c}$ and $v_1(\sigma) = f'$ from Theorem 3.61. By hypothesis $\mu'(f') > \mu'(\mathfrak{c}) = \mu(f)$ and μ' is locally strictly quasi-concave, so the maximum of μ' restricted to Δ_σ occurs either at f' or at some point in the interior of Δ_σ, and μ' must be strictly increasing in a neighborhood of $\mathfrak{c} \in \Delta_\sigma$.

By Theorem 3.61 we may choose a small subinterval of Δ_σ containing v_0 and lying in $\mathscr{D}eg(\mathrm{ev}_0(f))^{\mathfrak{c}}$ on which μ' is strictly increasing. The map $\mathbb{T}$ identifies this interval with a rational 1-simplex in $\mathscr{D}eg(\mathfrak{X}, f(1))$ with $v_0 = f$, and Lemma 4.25 implies that under this identification the restriction of $\mu' : \mathscr{D}eg(\mathrm{ev}_0(f)) \to \mathbb{R}$ corresponds to restriction of $\mu : \mathscr{D}eg(f(1))$. Hence we have found a rational 1-simplex in $\mathbf{DF}(\mathfrak{X}, f(1))$ for which $v_0 = f$ and along which μ is strictly increasing, which shows that f is not a HN filtration. The contrapositive: if f is a HN filtration, then $\mathrm{ev}_0(f)$ is graded-semistable.

Conversely, μ is strictly quasi-concave, replace k with a field extension if necessary and let $f' : \Theta_k \to \mathfrak{X}$ be the HN filtration of $f(1)$. Assume that $\mu(f') > \mu(f)$. Again f and f' can not be antipodal, because $\mu > 0$ on both filtrations, hence by Proposition 4.21 there is a unique rational 1-simplex $\Delta_\sigma \to \mathscr{D}eg(\mathfrak{X}, f(1))$ with $v_0(\sigma) = f$ and $v_1(\sigma) = f'$. This rational 1-simplex must be contained in $\mathcal{U}_p$ because $\mathcal{U}$ is locally convex. By the same argument as in the previous paragraph, μ must be strictly increasing in a neighborhood of v_0. Again by Theorem 3.61 we can identify a small subinterval of Δ_σ containing v_0 with a rational 1-simplex in $\mathscr{D}eg(\mathrm{ev}_0(f))^{\mathfrak{c}}$

with $v_0 = \mathfrak{c}$. The function μ is strictly increasing on this interval, so $\mathrm{ev}_0(f)$ is not graded-semistable. $\square$

We have the following immediate consequence:

Corollary 4.25.1. *Let $\mathfrak{X}$ be a stack satisfying (†), and let $\mu : \mathfrak{U} \subset \mathscr{C}omp(\mathfrak{X}) \to \mathbb{R}$ be a standard numerical invariant. If $f : \Theta_k \to \mathfrak{X}$ is a HN filtration, then*

$$M^\mu(\mathrm{gr}(f)) = \mu(f).$$

Hence the semi-continuity property (5) of Theorem 2.7 holds for μ.

Proof. By $\mathrm{gr}(f)$ we mean $u(\mathrm{ev}_0(f))$, where $u : \mathrm{Grad}(\mathfrak{X}) \to \mathfrak{X}$ is the forgetful map. We know from Theorem 4.24 that $\mathrm{ev}_0(f)$ is graded-semistable. By definition this means that $M^\mu_{\mathfrak{X}}(\mathrm{gr}(f))$ is the value of μ on the canonical filtration of $\mathrm{gr}(f)$, which is the image of the canonical filtration of $\mathrm{ev}_0(f) \in \mathrm{Grad}(\mathfrak{X})$ under u. The filtration f degenerates to the canonical filtration of $\mathrm{gr}(f)$, so the value of μ on the canonical filtration of $\mathrm{gr}(f)$ is $\mu(f)$. $\square$

4.3.3. *The semistable locus of a Θ-reductive stack is Θ-reductive.* The proof of the following is conceptually similar to that of Theorem 4.24.

Proposition 4.26. *Let $\mathfrak{X}$ be a Θ-reductive stack satisfying (†). For any $\ell \in H^2(\mathfrak{X};\mathbb{R})$, let $\mathfrak{X}^{\mathrm{ss}} \subset \mathfrak{X}$ be the subfunctor classifying maps $S \to \mathfrak{X}$ which map every point of S to semistable point of $\mathfrak{X}$. Then $\mathfrak{X}^{\mathrm{ss}}$ is Θ-reductive.*

Remark 4.27. This proposition is stated for semistablity as determined by a class $\ell \in H^2(\mathfrak{X};\mathbb{R})$, because its proof does not make any reference to a class $b \in H^4(\mathfrak{X};\mathbb{R})$. However, the result holds with essentially the same proof for any standard numerical invariant.

Lemma 4.28. *Let $\mathfrak{X}$ be a Θ-reductive stack satisfying (†), and let $\ell \in H^2(\mathfrak{X};\mathbb{R})$. If $f : \Theta_k \to \mathfrak{X}$ is a filtration such that $f(1)$ is ℓ-semistable, then $f(0)$ is ℓ-semistable if and only if $f^*(\ell) = 0 \in H^2(\Theta_k;\mathbb{R})$.*

Proof. The only if direction follows from the observation that $f^*(\ell) = (f')^*(\ell) = -(f'')^*(\ell) \in H^2(\Theta_k;\mathbb{R})$, where f' is the canonical split filtration of $f(0) = \mathrm{gr}(f)$ and f'' is the canonical split filtration coming from the opposite $\mathbb{G}_m$ action on $\mathrm{gr}(f)$. It follows that if $f^*(\ell) \neq 0$, then either f' or f'' is a destabilizing filtration of $f(0)$.

Conversely, say $f : \Theta_k \to \mathfrak{X}$ is a filtration with $f^*(\ell) = 0$, and assume there exists a destabilizing filtration $g : \Theta_k \to \mathfrak{X}$ of $f(0)$. The locally finite type algebraic space $\mathrm{Flag}(f(0))$ has an action of $\mathbb{G}_m$ via the homomorphism $\lambda : (\mathbb{G}_m)_k \to \mathrm{Aut}(f(0))$, and $g_0 := \lim_{t\to 0} t \cdot g \in \mathrm{Flag}(f(0))$ exists because $\mathfrak{X}$ is Θ-reductive and hence $\mathrm{Flag}(f(0))$ satisfies the valuative criterion for properness. By construction g_0 is fixed by the $\mathbb{G}_m$-action on $\mathrm{Flag}(f(0))$ and thus canonically corresponds to a non-degenerate filtration $g_0 : \Theta_k \to \mathrm{Grad}(\mathfrak{X})$ of the point $\mathrm{ev}_0(f) \in \mathrm{Grad}(\mathfrak{X})(k)$. Note that $g_0^*(\ell) = g^*(\ell) > 0 \in H^2(\Theta_k, \mathbb{R})$, because g_0 lies on the same connected component of $\mathrm{Flag}(f(0))$. Hence g_0, regarded as a point in $\mathscr{D}eg(\mathrm{Grad}(\mathfrak{X}), \mathrm{ev}_0(f))$, is not antipodal to $\mathfrak{c}$, because $\mathfrak{c}^*(\ell) = f^*(\ell) = 0$.

We now apply Theorem 3.61 to obtain a canonical rational 1-simplex $\Delta^1 \to \mathscr{D}eg(\mathrm{ev}_0(f))$ mapping v_0 to $\mathfrak{c}$ and v_1 to g_0. This corresponds to a cone $\sigma : \mathbb{R}^2_{\geq 0} \to |\,\mathbf{DF}(\mathrm{ev}_0(f))_\bullet|$ for which the point $(1,0)$ corresponds to the filtration $\mathfrak{c}$ and $(0,1)$ corresponds to g_0. The restriction of the class $\ell \in H^2(\mathfrak{X};\mathbb{R})$ along the forgetful map $u : \mathrm{Grad}(\mathfrak{X}) \to \mathfrak{X}$ defines a function $u^*(\ell)^\wedge : |\,\mathbf{DF}(\mathrm{Grad}(\mathfrak{X}), \mathrm{ev}_0(f))_\bullet| \to \mathbb{R}$ by Lemma 3.83 and the discussion following it. $u^*(\ell)^\wedge$ restricts to a linear function on this cone $\sigma : \mathbb{R}^2_{\geq 0} \to |\,\mathbf{DF}(\mathrm{ev}_0(f))_\bullet|$ whose value is $\mathfrak{c}^*(\ell) = 0$ at $(1,0)$ and $g_0^*(\ell) > 0$ at $(0,1)$. In particular, this implies that $u^*(\ell)^\wedge$ is strictly positive at every point of $\mathbb{R}^2_{\geq 0} \setminus \mathbb{R}_{\geq 0} \cdot (1,0)$.

Now by Theorem 3.61, a subcone $\sigma' \subset \mathbb{R}^2_{\geq 0}$ spanned by $(1,0)$ and $(1-\epsilon,\epsilon)$ for some $0 < \epsilon \ll 0$ maps to $|\,\mathbf{DF}(\mathrm{ev}_0(f))^{\mathfrak{c}}_\bullet| \subset |\,\mathbf{DF}(\mathrm{ev}_0(f))_\bullet|$, and hence is identified under the map $\mathbb{T}$ with a cone in $|\,\mathbf{DF}(\mathfrak{X}, f(1))|$ mapping $(1,0)$ to f. Furthermore, by Lemma 4.25 the restriction of $\hat{\ell} : |\,\mathbf{DF}(f(1))_\bullet| \to \mathbb{R}$ along the map $\mathbb{T}$ agrees with the map $u^*(\ell)^\wedge : |\,\mathbf{DF}(\mathrm{ev}_0(f))^{\mathfrak{c}}_\bullet| \to \mathbb{R}$ discussed above. It follows that $\hat{\ell}$ is strictly positive on the cone $\mathbb{T}(\sigma') : \mathbb{R}^2_{\geq 0} \to |\,\mathbf{DF}(f(1))_\bullet|$ outside of the ray $\mathbb{R}_{\geq 0} \cdot (1,0)$. In particular, there is some filtration f' of $f(1)$ for which $(f')^*(\ell) > 0$, and hence $f(1)$ is not semistable. □

Proof of Proposition 4.26. It suffices to show that for any discrete valuation ring R and map $f : \Theta_R \to \mathfrak{X}$ such that $f(\Theta_R \setminus (0,0)) \subset |\mathfrak{X}^{\mathrm{ss}}|$, $f(0,0) \in |\mathfrak{X}^{\mathrm{ss}}|$ as well. Consider the restriction $f_K : \Theta_K \to \mathfrak{X}$, where K is the fraction field of R. Because $f_K(1)$ is semistable, we must have $f_K^*(\ell) = 0 \in H^2(\Theta_K;\mathbb{R})$ by Lemma 4.28. Because the values of $f^*(\ell)$ is locally constant in a family of filtrations, we must have $f_k^*(\ell) = 0$ as well, where f_k is the restriction of f to the residue field k of R. Applying Lemma 4.28 now implies that $f_k(0) = f(0,0)$ is semistable as well. □

4.4. **Boundedness conditions for the existence of HN filtrations.** Our first method for establishing the existence of HN filtrations is fairly unsophisticated. Nevertheless it can clarify the question. For any $p \in \mathfrak{X}(k)$ and $\mathcal{U} \subset \mathscr{C}omp(\mathfrak{X})$, let $\mathcal{U}_{\mathscr{C}omp(p)} \subset \mathscr{C}omp(\mathfrak{X}, p)$ denote the preimage of $\mathcal{U}$ under the canonical map $\mathscr{C}omp(\mathfrak{X}, p) \to \mathscr{C}omp(\mathfrak{X})$.

Lemma 4.29. *Let $\mu : \mathcal{U} \subset \mathscr{C}omp(\mathfrak{X}) \to \mathbb{R}$ be a numerical invariant on a stack $\mathfrak{X}$, let $p \in \mathfrak{X}(k)$ be an unstable point, and let $\bar{p} \in \mathfrak{X}(\bar{k})$ be the extension of p to an algebraic closure $k \subset \bar{k}$. Then μ obtains a maximum on $\mathscr{D}eg(\bar{p})$ if and only if the following boundedness condition is satisfied:*

(B1) $\exists$ a bounded sub-fan $F_\bullet \subset \mathbf{CF}(\mathfrak{X}, p)_\bullet$ such that $\mathbb{P}(F_\bullet) \cap \mathcal{U}_{\mathscr{C}omp(p)} \subset \mathbb{P}(F_\bullet)$ is closed and $\forall x \in \mathcal{U}_{\mathscr{C}omp(p)}$ with $\mu(x) > 0$, there is another point $y \in \mathbb{P}(F_\bullet) \cap \mathcal{U}_{\mathscr{C}omp(p)}$ with $\mu(y) \geq \mu(x)$.

Furthermore, this is equivalent to the existence of an HN filtration if μ satisfies (R).

Proof. Note that $\mathscr{D}eg(\mathfrak{X}, \bar{p}) \to \mathscr{C}omp(\mathfrak{X}, p)$ is surjective, and therefore so is $\mathcal{U}_p \to \mathcal{U}_{\mathscr{C}omp(p)}$. It follows that μ obtains a maximum on the former if and only if it obtains a maximum on the latter. The only if direction of the claim follows immediately from choosing $F_\bullet$ to be the subfan generated by a rational simplex in $\mathcal{U}_p \subset \mathscr{C}omp(\mathfrak{X}, p)$ containing a maximizer of μ. The if direction follows from the fact that μ is continuous and $\mathbb{P}(F_\bullet) \cap \mathcal{U}_{\mathscr{C}omp(p)}$ can be covered by a finite disjoint union of closed subsets of standard n-simplices of various dimensions, because $F_\bullet$ is bounded. It follows that μ must obtain a maximum at some point of $\mathbb{P}(F_\bullet) \cap \mathcal{U}_{\mathscr{C}omp(p)}$. The "furthermore" claim is Lemma 4.8. $\square$

In the context of Theorem 2.7, however, one needs to show additionally that only finitely many HN types appear in a family over a quasi-compact scheme. When $\mathfrak{X}$ is a stack satisfying (†) which has quasi-compact flag spaces, there are several ways in which one might strengthen (B1). One such strengthening is the following:

(B1+) There exists a collection of cones $\{\sigma_i \in \mathbf{CF}(\mathfrak{X})_{n_i}\}_{i \in I}$ such that:
- (a) for any map from a finite type scheme $\xi : S \to \mathfrak{X}$, at most finitely many σ_i lift to $\mathbf{CF}(\mathfrak{X}, \xi)_\bullet$; and
- (b) for any finite type point $p \in \mathfrak{X}(k)$ and $x \in \mathscr{C}omp(\mathfrak{X}, p)$ with $\mu(x) > 0$, there is another point $y \in \mathscr{C}omp(\mathfrak{X}, p)$ with $\mu(y) \geq \mu(x)$ which lies in a lift of some σ_i to $\mathscr{C}omp(\mathfrak{X}, p)$.

Using Lemma 4.29 it is fairly straightforward to show that if (B1+) holds, then every unstable finite type point of $\mathfrak{X}$ admits an HN filtration, and the "local finiteness" condition (4) of Theorem 2.7 holds, and we leave the proof to the reader.

Example 4.30. The primary example of (B1+) holding for a non quasi-compact $\mathfrak{X}$ arises for the moduli of vector bundles on a smooth projective curve. We will identify in Proposition 5.15 cones $\sigma_i \in \mathbf{CF}(\mathfrak{X})_{n_i}$ corresponding to components of $\mathrm{Filt}^{n_i}(\mathfrak{X})$ containing filtrations of length n_i whose corresponding rank-degree sequence in the upper half plane is convex. These cones satisfy (B1+). The fact that finitely many σ_i lift to $\mathbf{CF}(\mathfrak{X}, \xi)_\bullet$ follows from a Lemma of Grothendieck bounding the slopes of subbundles occuring in a bounded family of vector bundles [HL5].

Condition (B1+) is basically a bookkeeping condition. It has been difficult to develop general principles which are non-tautological and can be used to establish the existence and local finiteness of HN filtrations. One provisional result is the following:

Proposition 4.31. *Let $\mathfrak{X}$ be a stack satisfying (†). Let $\mu : \mathcal{U} \subset \mathscr{C}omp(\mathfrak{X}) \to \mathbb{R}$ be a numerical invariant satisfying (R), where $\mathcal{U} \subset \mathscr{C}omp(\mathfrak{X})$ is closed. Then the existence of an HN filtration for every unstable finite type point of $\mathfrak{X}$ and the "HN-boundedness" condition $(4')$ of Theorem 2.7 combined are equivalent to the following condition:*

(B2) For any map from a finite type affine scheme $\xi : T \to \mathfrak{X}$, $\exists$ a quasi-compact substack $\mathfrak{X}' \subset \mathfrak{X}$ such that $\forall$ finite type points $p \in T(k)$ and

$f \in \mathrm{Flag}(p)$ with $\mu(f) > 0$, there is another filtration $f' \in \mathrm{Flag}(p)$ with $\mu(f') \geq \mu(f)$ and $\mathrm{gr}(f') \in \mathfrak{X}'$.

Remark 4.32. Note that condition (B2) holds automatically when $\mathfrak{X}$ is quasi-compact. In fact, when $\mathfrak{X}$ is quasi-compact, the proof of Proposition 4.31 actually establishes the existence of HN filtrations for all points (not necessarily finite type) and the analog of the HN-boundedness condition for HN filtrations of all points.

We shall prove Proposition 4.31 after establishing several lemmas.

Lemma 4.33. *Let $\mathfrak{X} \to \mathfrak{Y}$ be any smooth map of stacks which satisfy (†). If $\mathrm{Grad}(\mathfrak{X}) \to \mathrm{Grad}(\mathfrak{Y})$ is surjective, then so is $\mathrm{Filt}(\mathfrak{X}) \to \mathrm{Filt}(\mathfrak{Y})$.*

Proof. Lemma 3.54 implies that given a point of $f \in |\mathrm{Filt}(\mathfrak{Y})|$ and a lift of $\mathrm{ev}_0(f) \in \mathrm{Grad}(\mathfrak{Y})$ to $\mathrm{Grad}(\mathfrak{X})$, one can lift f to $\mathrm{Filt}(\mathfrak{Y})$ by solving an infinite sequence of infinitesimal lifts along the map $\mathfrak{X} \to \mathfrak{Y}$. Therefore the fact that $\mathfrak{X} \to \mathfrak{Y}$ is formally smooth implies that $\mathrm{Filt}(\mathfrak{X}) \to \mathrm{Filt}(\mathfrak{Y})$ is surjective. □

Lemma 4.34. *Let $\mathfrak{X}$ be a quasi-compact stack satisfying (†). Then there is an affine scheme X with an action of $\mathbb{G}_m^n$ for some $n \geq 0$ along with a smooth representable surjective map $X/\mathbb{G}_m^n \to \mathfrak{X}$ such that the induced map $\mathrm{Filt}^r(X/\mathbb{G}_m^n) \to \mathrm{Filt}^r(\mathfrak{X})$ is smooth, representable, and surjective for all r.*

Proof. By Lemma 4.33 it suffices to find a smooth surjective map $X/\mathbb{G}_m^n \to \mathfrak{X}$ such that $\mathrm{Grad}^r(X/\mathbb{G}_m^n) \to \mathrm{Grad}^r(\mathfrak{X})$ is surjective. Because $\mathfrak{X}$ is quasi-compact we may assume without loss of generality that the base B is quasi-compact, and choose a smooth surjective map $\mathrm{Spec}(R) \to B$. Corollary 1.7.3 implies that $\mathrm{Grad}^r(\mathfrak{X}_R) \to \mathrm{Grad}^r(\mathfrak{X})$ is smooth and surjective. Furthermore for any stack $\mathfrak{Y}$ over $\mathrm{Spec}(R)$, Corollary 1.7.4 implies the equivalence of mapping stacks $\mathrm{Grad}^r_B(\mathfrak{Y}) \simeq \mathrm{Grad}^r_{\mathrm{Spec}(R)}(\mathfrak{Y})$. We may therefore replace $\mathfrak{X}$ with $\mathfrak{X}_R$ and prove the claim under the assumption that $B = \mathrm{Spec}(R)$.

For any stack of the form $\mathfrak{X} = X/\mathrm{GL}_n$ for a quasi-separated quasi-compact GL_n-space X, the map $X/\mathbb{G}_m^n \to X/\mathrm{GL}_n$ suffices, by the explicit computation of $\mathrm{Grad}(\mathfrak{X})$ relative to the scheme $\mathrm{Spec}(R)$ in Theorem 1.36. In general, we may assume that $\mathfrak{X}$ is reduced, because $\mathrm{Grad}^r(\mathfrak{X}^{\mathrm{red}}) \to \mathrm{Grad}^r(\mathfrak{X})$ is a surjective closed immersion. Let us stratify $\mathfrak{X}$ by locally closed substacks $\mathfrak{X}_i$ of the form X_i/GL_{n_i} where X_i is quasi-affine (See [K3, Proposition 3.5.9] when the base is a field and [HR, Proposition 8.2] in general). Then Corollary 1.7.1 implies that for all i, $\mathrm{Grad}^r(\mathfrak{X}_i) \simeq \mathrm{Grad}^r(\mathfrak{X}) \times_{\mathfrak{X}} \mathfrak{X}_i$. So we have a stratification by locally closed substacks

$$\mathrm{Grad}^r(\mathfrak{X}) = \bigcup_i \mathrm{Grad}^r(\mathfrak{X}_i).$$

We let Y_i be a disjoint union of open affine $\mathbb{G}_m^{n_i}$-equivariant subschemes which cover X_i. Note that $\mathrm{Grad}^r(Y_i/\mathbb{G}_m^{n_i}) \to \mathrm{Grad}^r(X_i/\mathbb{G}_m^{n_i})$ is surjective. Each Y_i is finite type over $\mathrm{Spec}(R)$ by hypothesis, so we may apply Theorem B.2 to the smooth map $Y_i/\mathbb{G}_m^{n_i} \to X_i/\mathrm{GL}_{n_i} \simeq \mathfrak{X}_i$. The result is an affine $\mathbb{G}_m^{n_i}$-scheme U_i

and a smooth map $U_i/\mathbb{G}_m^{n_i} \to \mathfrak{X}$ such that the projection $(U_i/\mathbb{G}_m^{n_i})\times_{\mathfrak{X}}\mathfrak{X}_i \to \mathfrak{X}_i$ factors through an étale surjection $(U_i/\mathbb{G}_m^{n_i}) \times_{\mathfrak{X}} \mathfrak{X}_i \to Y_i/\mathbb{G}_m^{n_i}$. It follows from Corollary 1.7.1 that the composition

$$\operatorname{Grad}^r(U/\mathbb{G}_m^{n_i}) \times_{\mathfrak{X}} \mathfrak{X}_i \to \operatorname{Grad}^r(Y_i/\mathbb{G}_m^{n_i}) \to \operatorname{Grad}^r(X_i/\mathbb{G}_m^{n_i}) \to \operatorname{Grad}^r(\mathfrak{X}_i)$$

is surjective. Choosing n larger than n_i for all i, we let $X = \bigsqcup_i U_i \times \mathbb{G}_m^{n-n_i}$, and the resulting map $X/\mathbb{G}_m^n \to \mathfrak{X}$ satisfies the claim. $\square$

It will be convenient to further reduce matters to a quotient stack over a scheme:

Lemma 4.35. *Let $\mathfrak{X}$ be a quasi-compact stack satisfying (†). For any smooth map $\operatorname{Spec}(R) \to B$ whose image contains the image of $\mathfrak{X} \to B$, one can construct a stack $X/\mathbb{G}_m^n \to \operatorname{Spec}(R)$, where X is an affine R-scheme of finite type, and a smooth representable surjective map $X/\mathbb{G}_m^n \to \mathfrak{X}$ over B such that the induced map $\operatorname{Filt}^r_R(X/\mathbb{G}_m^n) \to \operatorname{Filt}^r_B(\mathfrak{X})$ is smooth, representable, and surjective for all $r > 0$.*

Proof. Consider the smooth representable map $X/\mathbb{G}_m^n \to \mathfrak{X}$ over B from Lemma 4.34. If we let $X'/\mathbb{G}_m^n = (X/\mathbb{G}_m^n) \times_B \operatorname{Spec}(R)$, then $\operatorname{Filt}_R(X'/\mathbb{G}_m^n) \simeq \operatorname{Spec}(R) \times_B \operatorname{Filt}_B(X/\mathbb{G}_m^n)$ by Lemma 1.3, so $\operatorname{Filt}^r_R(X'/\mathbb{G}_m^n) \to \operatorname{Filt}_B(X/\mathbb{G}_m^n)$ is smooth and surjective as well. $\square$

Lemma 4.36. *Let $\mathfrak{X}$ be a quasi-compact stack satisfying (†). For any map from a quasi-compact space $\xi : S \to \mathfrak{X}$ and any union of connected components $Y \subset \operatorname{Flag}(\xi)$, there is a quasi-compact subspace $Y' \subset Y$ such that*

$$\operatorname{ev}_1(Y') = \operatorname{ev}_1(Y) \subset |S|.$$

Proof. Let $X/\mathbb{G}_m^n \to \mathfrak{X}$ be a smooth surjective representable map such that $\operatorname{Filt}_R(X/\mathbb{G}_m^n) \to \operatorname{Filt}_B(\mathfrak{X})$ is surjective, as constructed in Lemma 4.35. Let $\xi' : S' \to X/\mathbb{G}_m^n$ be the base change of ξ along the map $\pi : X/\mathbb{G}_m^n \to \mathfrak{X}$. Because π is representable, we know that S' is a space. We can therefore consider the commutative (non-cartesian) diagram

$$\begin{tikzcd} \operatorname{Flag}_R(\xi') \arrow[r] \arrow[d, "\operatorname{ev}_1"] & \operatorname{Flag}_B(\xi) \arrow[d, "\operatorname{ev}_1"] \\ S' \arrow[r, "\pi"] & S \end{tikzcd}$$

where the horizontal maps are smooth and surjective. If we denote the preimage of $Y \subset \operatorname{Flag}_B(\xi)$ as $Y' \subset \operatorname{Flag}_R(\xi')$, then $\pi(\operatorname{ev}_1(Y')) = \operatorname{ev}_1(Y)$.

It therefore suffices to prove the claim for a stack of the form $X/\mathbb{G}_m^n$ over an affine scheme $\operatorname{Spec}(R)$. The claim is now a consequence of the explicit description of the stack of filtrations $\operatorname{Filt}_R(X/\mathbb{G}_m^n)$ given in Theorem 1.36. The connected components of $\operatorname{Filt}(X/\mathbb{G}_m^n)$ are indexed by a one parameters subgroup $\lambda : \mathbb{G}_m \to \mathbb{G}_m^n$ and a connected component $Z \subset X^\lambda$. The connected components of $\operatorname{Filt}(X/\mathbb{G}_m^n)$ are $Y_{Z,\lambda}/\mathbb{G}_m^n$, where $Y_{Z,\lambda} \hookrightarrow X$ is the locally closed attracting locus for the connected component $Z \subset X^\lambda$. Note that

ev_1 is a locally closed immersion on each connected component, and despite $\mathrm{Filt}(X/\mathbb{G}_m^n)$ having infinitely many connected components, there are only finitely many locally closed substacks of $X/\mathbb{G}_m^n$ arising in this way.

Now any map $\xi : S \to X/\mathbb{G}_m^n$ can be presented as $S'/\mathbb{G}_m^n \to X/\mathbb{G}_m^n$ for some $\mathbb{G}_m^n$-equivariant map $S' \to X$. The connected components of $\mathrm{Flag}(\xi)$ are of the form $Y'/\mathbb{G}_m^n$, where Y' is a connected component of $S' \times_X Y_{Z,\lambda}$. There are finitely many locally closed subschemes of S arising in this way, so given an infinite collection of connected components of $\mathrm{Flag}(\xi)$, finitely many suffice to cover their image in S. □

Corollary 4.36.1. *If $\mathfrak{X}$ is a quasi-compact stack satisfying (†), then for any map from a quasi-compact space $\xi : S \to \mathfrak{X}$ and any union of connected components $Y \subset \mathrm{Flag}(\xi)$, $\mathrm{ev}_1(|Y|) \subset |S|$ is constructible.*

Proof. This is an immediate consquence of Lemma 4.36 and the fact that ev_1 is locally of finite type. □

Lemma 4.37. *Let $\mathfrak{X}$ be a quasi-compact stack satisfying (†). Then for any map $\xi : S \to \mathfrak{X}$ from a quasi-compact space S, the component fan $\mathbf{CF}(\xi)_\bullet$ is bounded.*

Proof. From the base change properties of mapping stacks, Lemma 1.3, we may replace $\mathfrak{X}$ with $\mathfrak{X} \times_B S$ and therefore assume that $B = S$ and the map $\xi : S \to \mathfrak{X}$ is a section of the structure map $\mathfrak{X} \to S$. Let $\pi : X/\mathbb{G}_m^n \to \mathfrak{X}$ be a smooth surjective representable map such that $\mathrm{Filt}^r(X/\mathbb{G}_m^n) \to \mathrm{Filt}^r(\mathfrak{X})$ is surjective, as constructed in Lemma 4.34. Let $\xi' : S' \to X/\mathbb{G}_m^n$ be the base change of ξ along the π. Because π is representable, we know that S' is a space, and as in the proof of Lemma 4.36 we have a smooth surjection $\mathrm{Flag}^r(\xi') \to \mathrm{Flag}^r(\xi)$ for all r. It follows that $\mathbf{CF}(\xi')_\bullet \to \mathbf{CF}(\xi)_\bullet$ is surjective, so we may assume that $\mathfrak{X} = X/\mathbb{G}_m^n$ over a base space S and that ξ is a section of the map $X/\mathbb{G}_m^n \to S$.

We use the explicit description from Theorem 1.36:

$$\mathrm{Flag}^r(\xi) = \bigsqcup_{\psi \in \mathrm{Hom}(\mathbb{G}_m^r, \mathbb{G}_m^n)} \xi^{-1}(Y_\psi).$$

Note that each connected component of $\mathrm{Flag}^r(\xi)$ is a locally closed subspace of S under the projection $\mathrm{Flag}^r(\xi) \to S$. Also, because finitely many locally closed subspaces arise as blades $Y_\psi \hookrightarrow X$ associated to some homomorphism $\psi : \mathbb{G}_m^r \to \mathbb{G}_m^n$, only finitely many locally closed subspaces of S arise as connected components of $\mathrm{Flag}^r(\xi)$. It follows that there is a finite set of geometric points $p_i \in S(\bar{k}_i), i = 1, \ldots, N$ such that

$$\bigsqcup_i \mathbf{DF}(X/\mathbb{G}_m, \xi(p_i))_\bullet \to \mathbf{CF}(X/\mathbb{G}_m^n, \xi)_\bullet$$

is surjective. The explicit computation of Lemma 3.37 that the left hand side is bounded, so the right hand side is bounded as well. □

Proof of Proposition 4.31. Let $\xi : T \to \mathfrak{X}$ be a family, with T a quasi-compact space. We may assume, by enlarging $\mathfrak{X}'$ if necessary, that ξ factors through $\mathfrak{X}'$. The hypothesis (B2) states that for the purposes of finding HN filtrations of unstable points, it suffices to consider filtrations in the open substack $\mathrm{Filt}(\mathfrak{X}') \subset \mathrm{Filt}(\mathfrak{X})$. In particular, we may assume from this point forward that $\mathfrak{X}$ is quasi-compact. In this case Lemma 4.37 implies that (B1) holds for any point $p \in |\mathfrak{X}|$, not just finite type points, by letting $F_\bullet = \mathbf{CF}(\mathfrak{X}, p)_\bullet$. Hence every unstable point has a HN filtration.

We shall use Noetherian induction to show that there is a quasi-compact subspace $Y \subset \mathrm{Flag}(\xi)$ which contains an HN filtration for each unstable point of T, thereby verifying the HN-boundedness condition of Theorem 2.7. We consider the class of closed subsets of T

$$\mathcal{C} = \left\{ Z \subset |T| \;\middle|\; \begin{array}{c} \exists \text{ quasi-compact open subset } Y \subset |\mathrm{Flag}(\xi)| \\ \text{containing an HN filtration for every point in } Z \end{array} \right\}$$

Our goal is to show that $|T| \in \mathcal{C}$. For the inductive argument it suffices to consider an irreducible $Z \in \mathcal{C}$, because $\mathcal{C}$ is closed under union of subsets. So we consider a closed irreducible subset $Z \subset |T|$, and assume that $Z' \in \mathcal{C}$ for every proper closed subset $Z' \subsetneq Z$.

Let $U \subset |\mathrm{Flag}(\xi)|$ denote the union of all connected components which contain an HN filtration for some point in Z. By Corollary 4.36.1 then $\mathrm{ev}_1(U) \cap Z \subset |T|$, the set of unstable points in Z, is constructible. If its closure is a proper subset of Z, we can apply the inductive hypothesis to conclude that $Z \in \mathcal{C}$.

Otherwise, we may assume that $\mathrm{ev}_1(U) \cap Z$ is dense in Z. Let $V \subset |\mathrm{Flag}(\xi)|$ be a quasi-compact connected open subset containing an HN filtration of the generic point of Z, which we know exists because $\mathfrak{X}$ is quasi-compact. Let $U^{\mu > \mu(V)} \subset U$ be the union of all connected components on which $\mu > \mu(V)$. We know from Corollary 4.36.1 that the image $\mathrm{ev}_1(U^{\mu > \mu(V)}) \cap Z$ is constructible, and it does not contain the generic point because $M^\mu = \mu(V)$ at the generic point of Z. It follows that

$$Z' := \overline{Z \setminus \mathrm{ev}_1(V)} \cup \overline{\mathrm{ev}_1(U^{\mu > \mu(V)}) \cap Z} \subset Z$$

is a closed proper subset. The inductive hypothesis provides a quasi-compact open subspace $Y \subset |\mathrm{Flag}(\xi)|$ containing an HN filtration for each point of Z', and by construction V contains an HN filtration for every point of $Z \setminus Z'$. Thus $Y \cup V$ is a quasi-compact open subset containing an HN filtration for every point of Z, and $Z \in \mathcal{C}$. □

4.5. **Necessary and sufficient conditions for the existence of Θ-stratifications.** For reference, let us reformulate Theorem 2.7 by incorporating the simplifications which we have established above. It applies, in particular, to any numerical invariant associated to a class $\ell \in H^2(\mathfrak{X}; \mathbb{Q})$ and a positive definite class $b \in H^4(\mathfrak{X}; \mathbb{Q})$.

Theorem 4.38. *Let $\mathfrak{X}$ be a stack satisfying (†), and let $\mu : \mathfrak{U} \subset \mathscr{C}omp(\mathfrak{X}) \to \mathbb{R}$ be a standard numerical invariant satisfying (R) and for which $\mathfrak{U} \subset \mathscr{C}omp(\mathfrak{X})$ is closed. Then μ defines a weak Θ-stratification if and only if it satisfies*

(i) the uniqueness part of the HN-property (1) of Theorem 2.7,
(ii) the simplified HN-specialization property (2') of Theorem 2.7,
(iii) the condition (B2) of Proposition 4.31.

Proof. We verify the hypotheses of Theorem 2.7. The open strata property holds because μ is locally constant (see Simplification 2.8). The HN-boundedness condition combined with the existence of HN filtrations for finite type points is equivalent to the condition (B2) by Proposition 4.31. All that is left to verify is the semi-continuity condition, namely that if $f : \Theta_k \to \mathfrak{X}$ is a HN filtration, then $M^\mu(f(0)) = \mu(f)$. This is shown in Corollary 4.25.1. □

When $\mathfrak{X}$ is Θ-reductive, the necessary and sufficient conditions for the existence of a Θ-stratification simplify further:

Theorem 4.39. *Let $\mathfrak{X}$ be a Θ-reductive stack satisfying (†). Let $\mu : \mathfrak{U} \subset \mathscr{C}omp(\mathfrak{X}) \to \mathbb{R}$ be a standard numerical invariant satsifying (R) and for which $\mathfrak{U} \subset \mathscr{C}omp(\mathfrak{X})$ is closed. Then μ defines a weak Θ-stratification if and only if (B2) holds.*

Proof. By Proposition 4.31, the condition (B2) implies the existence of a HN filtration for any finite type point of $\mathfrak{X}$, and the uniqueness up to the action of $\mathbb{N}^\times$ follows from Corollary 4.21.1. The HN-specialization property holds automatically for Θ-reductive stacks (see Remark 2.10), so the result follows from Theorem 4.38. □

Example 4.40. Theorem 4.39 is stated for a standard numerical invariant, but all of the examples of standard numerical invariants we consider in this paper are associated to a class $\ell \in H^2(\mathfrak{X}; \mathbb{Q})$ and a positive definite $b \in H^4(\mathfrak{X}; \mathbb{Q})$. In this case $\mathfrak{U} = \mathscr{C}omp(\mathfrak{X})$ is clearly closed, and μ is standard by Lemma 4.10 and satisfies (R) by Lemma 4.12. So all of the conditions of Theorem 4.39 hold automatically. Note also that the condition (B2) holds tautologically if $\mathfrak{X}$ is quasi-compact. So the theorem implies that if $\mathfrak{X}$ is Θ-reductive and quasi-compact, any numerical invariant associated to $\ell \in H^2(\mathfrak{X}; \mathbb{Q})$ and a positive definite $b \in H^4(\mathfrak{X}; \mathbb{Q})$ defines a Θ-stratification of $\mathfrak{X}$.

We can formulate other conditions which guarantee the existence of Θ-stratifications for Θ-reductive stacks, although the hypotheses are less general than for the previous corollary. The following two corollaries are inspired by the theory of Bridgeland stability, and specifically [T4].

Corollary 4.40.1. *Let $\mathfrak{X}$ be a Θ-reductive stack satisfying (†), let μ be a standard numerical invariant with $\mathfrak{U} \subset \mathscr{C}omp(\mathfrak{X})$ closed and satisfying condition (R), and assume that*

(1) *for any map from a finite type scheme* $\xi : S \to \mathfrak{X}$*, the value of* $M^\mu(\xi(p))$ *is bounded above for finite type points* $p \in |S|$*, and*
(2) *for any* $c \in \mathbb{R}$*, the set of finite type graded-semistable points of* $\mathrm{Grad}(\mathfrak{X})$ *(in the sense of Definition 4.15) for which* $\mu < c$ *is bounded (up to the action of* $\mathbb{N}$ *on* $\mathrm{Grad}(\mathfrak{X})$*).*

Then μ *defines a weak* Θ*-stratification.*

Proof. We know from Theorem 4.24 that a filtration f of a finite type point of $\mathfrak{X}$ is an HN filtration if and only if $\mathrm{ev}_0(f) \in \mathrm{Grad}(\mathfrak{X})$ is semistable. So the conditions above imply that for any map from a finite type scheme $\xi : S \to \mathfrak{X}$, the points in $\mathrm{Grad}(\mathfrak{X})$ which arise as the associated graded of an HN filtration of an unstable finite type point of S is bounded, up to the action of $\mathbb{N}$ on $\mathrm{Grad}(\mathfrak{X})$. It follows that it suffices to consider filtrations in some quasi-compact open substack $\mathfrak{X}' \subset \mathfrak{X}$. This verifies the condition (B2), so Theorem 4.39 applies. □

The second formulation is the following

Corollary 4.40.2. *Let* $\mathfrak{X}$ *be a* Θ*-reductive stack satisfying* (†)*, let* μ *be a standard numerical invariant with* $\mathcal{U} \subset \mathscr{C}omp(\mathfrak{X})$ *closed and satisfying condition (R), and assume that*

(1) *for any map from a finite type scheme* $\xi : S \to \mathfrak{X}$*, the value of* $M^\mu(\xi(p))$ *is bounded above for finite type points* $p \in |S|$*, and*
(2) *for any* $c \in \mathbb{R}$*, the set of finite type points of* $\mathfrak{X}$ *for which* $M^\mu(p) \le c$ *is bounded.*

Then μ *defines a weak* Θ*-stratification.*

Proof. These conditions also imply condition (B2), because for any map from a finite type scheme $\xi : S \to \mathfrak{X}$, one can choose an upper bound N for $M^\mu(-)$ on $|S|$, and then choose a quasi-compact open substack $\mathfrak{X}' \subset \mathfrak{X}$ which contains all points for which $M^\mu(p) \le c$. For any unstable point of S, it suffices to look for a HN filtration in $\mathfrak{X}'$. □

4.6. **Proper maps and polynomial valued invariants.** In this section we study stacks which admit a map $\pi : \mathfrak{X} \to \mathfrak{Y}$ which is proper and representable, where $\mathfrak{Y}$ is Θ-reductive and quasi-compact. If X is G-projective then X/G is a stack of this kind, so this situation generalizes the situation typically encountered in geometric invariant theory. The result will be a generalization of Theorem 4.39 for stacks of this kind, Theorem 4.44. The methods of this section also lead to results on "perturbation" of numerical invariants which generalize Theorem 4.54.

We will restrict our attention in this subsection to cohomology theories which come equipped with a natural "Chern class" group homomorphism $c_1 : \mathrm{Pic}(\mathfrak{X}) \to H^2(\mathfrak{X}; A)$ for any algebraic stack $\mathfrak{X}$. By choosing the scaling appropriately, we may normalize c_1 so that under the canonical isomorphism $H^*(\Theta_k; A) \simeq A[u]$ we have $c_1(\mathcal{O}_{\Theta_k}\langle 1\rangle) = u$, where $\mathcal{O}_{\Theta_k}\langle 1\rangle$ is the unique invertible sheaf whose fiber at $\{0\}/\mathbb{G}_m$ has weight -1.

4.6.1. *Polynomial valued numerical invariants.* It will be convenient for this section to work with numerical invariants valued in the polynomial ring $\mathbb{R}[\epsilon]$, which we regarded as a totally ordered ring by saying $\phi(\epsilon) \leq \psi(\epsilon)$ if $\phi(r) \leq \psi(r)$ for all $0 < r \ll 1$. Numerical invariants of this form arise from a positive definite element $b \in H^4(\mathfrak{X};\mathbb{Q})$ and $\ell = \ell_0 + \epsilon\ell_1 + \epsilon^2\ell_2 + \cdots \in H^2(\mathfrak{X};\mathbb{Q})[\epsilon]$ via the same formula as before

$$\mu(f) = \frac{f^*(\ell)}{\sqrt{f^*(b)}} = \frac{f^*(\ell_0)}{\sqrt{f^*(b)}} + \epsilon\frac{f^*(\ell_1)}{\sqrt{f^*(b)}} + \cdots \in \mathbb{R}[\epsilon].$$

Note that any set of polynomials of bounded degree has a supremum. More importantly for any compact space C, any map $\mu : C \to \mathbb{R}[\epsilon]$ whose coefficient functions are continuous and whose image has bounded degree attains a maximum in C. One can show this by inductively defining smaller and smaller non-empty closed subsets $C = C_{-1} \supset C_0 \supset C_1 \supset \cdots$ where $C_i = \{x \in C_{i-1} | \epsilon^i \text{ coefficient of } \mu(x) \text{ is maximal}\}$. Then μ is maximized on C_n if n is larger than the degree of any $\mu(x)$. This property guarantees that the results of the previous sections on existence and uniqueness of HN filtrations generalize verbatim for numerical invariants of this form. For further discussion, see Section 5.4.

4.6.2. *Numerically positive classes.* We will make use of the following weak notion of a *numerically positive* class $\ell \in H^2(\mathfrak{X};\mathbb{Q})[\epsilon]$ relative to a representable map $\mathfrak{X} \to \mathfrak{Y}$. We consider all commutative diagrams of the form

$$\begin{tikzcd} \mathbb{P}^1_k/\mathbb{G}_m \arrow[r, "\phi"] \arrow[d] & \mathfrak{X} \arrow[d] \\ \mathrm{pt}/\mathbb{G}_m \arrow[r] & \mathfrak{Y} \end{tikzcd} , \qquad (31)$$

where $\mathbb{G}_m$ acts non-trivially on $\mathbb{P}^1_k$ and the induced map $\mathbb{P}^1_k/\mathbb{G}_m \to \mathfrak{X} \times_{\mathfrak{Y}} (\mathrm{pt}/\mathbb{G}_m)$ is finite. We define the point $0 \in \mathbb{P}^1_k$ to be the limit $\lim_{t\to 0} t \cdot x$ for a general point $x \in \mathbb{P}^1_k$, and we let $\infty \in \mathbb{P}^1_k$ be the second fixed point. We can restrict $\phi^*(\ell)$ to $H^2(\{0\}/\mathbb{G}_m;\mathbb{Q})[\epsilon] \simeq \mathbb{Q}[\epsilon] \cdot u$ and $H^2(\{\infty\}/\mathbb{G}_m;\mathbb{Q})[\epsilon] \simeq \mathbb{Q}[\epsilon] \cdot u$, and we say that ℓ is numerically positive (respectively NEF) if $\phi^*(\ell)_\infty - \phi^*(\ell)|_0 > 0$ (respectively ≥ 0) for any diagram of the form (31).

There is of course a more common notion of numerically positive classes in singular cohomology or operational Chow cohomology: one requires that for any map $\mathrm{Spec}(A) \to \mathfrak{Y}$, the restriction of ℓ to the algebraic space $\mathfrak{X}_A$ is numerically positive in the sense that its restriction to any proper curve in $\mathfrak{X}_A$ has positive degree. A numericaly positive (respectively NEF) class in this more common sense is also numerically positive (respectively NEF) in the weaker sense above, because the difference $\phi^*(\ell)_\infty - \phi^*(\ell)|_0$ gives the degree of the invertible sheaf $\mathcal{L}$ on $\mathbb{P}^1/\mathbb{G}_m$ for which $\phi^*(\ell) = c_1(\mathcal{L})$. We have chosen our notion of positivity because it is the minimal condition needed for the discussion that follows, and it avoids a detour into the theory of degree.

4.6.3. *HN filtrations.* Consider a proper representable map $\pi : \mathfrak{X} \to \mathfrak{Y}$. For any $x \in \mathfrak{X}(k)$, we use the homeomorphism $\pi_* : \mathscr{D}eg(\mathfrak{X}, x) \xrightarrow{\simeq} \mathscr{D}eg(\mathfrak{Y}, \pi(x))$ to identify functions on $\mathscr{D}eg(\mathfrak{X}, x)$ with functions on $\mathscr{D}eg(\mathfrak{Y}, \pi(x))$. Note, however, that $\mathscr{D}eg(\mathfrak{X}, x)$ has fewer rational 1-simplices than $\mathbf{DF}(\mathfrak{Y}, \pi(x))_\bullet$, so the condition of a function on $\mathscr{D}eg(\mathfrak{Y}, \pi(x))$ being quasi-concave is stronger than being quasi-concave on $\mathscr{D}eg(\mathfrak{X}, x)$.

Lemma 4.41. *Let $b \in H^4(\mathfrak{Y}; \mathbb{Q})$ be positive definite, and let $\ell \in H^2(\mathfrak{X}; \mathbb{Q})$ be NEF relative to π. Then for any $x \in \mathfrak{X}(k)$ the function*

$$\mu(f) = \frac{\hat{\ell}(f)}{\sqrt{\widehat{\pi^*(b)}(f)}}$$

on $\mathfrak{U}^{\mu>0}_{\pi(x)} \subset \mathscr{D}eg(\mathfrak{Y}, \pi(x))$ is strictly quasi-concave.

Proof. Given two filtrations f_1 and f_2 of x, we know from the proof of Proposition 4.21 that the resulting composition $f_1 \cup f_2 : (\mathbb{A}^2 - \{0\})/\mathbb{G}_m^2 \to \mathfrak{X} \to \mathfrak{Y}$ extends uniquely to a map $\mathbb{A}^2_k/\mathbb{G}^2_m \to \mathfrak{Y}$, because $\mathfrak{Y}$ is Θ-reductive. Consider the diagram

$$\begin{tikzcd} S/\mathbb{G}_m^2 \arrow[r] \arrow[dr] & X/\mathbb{G}_m^2 \arrow[r] \arrow[d, "\pi'"] & \mathfrak{X} \arrow[d, "\pi"] \\ & \mathbb{A}^2_k/\mathbb{G}^2_m \arrow[r] & \mathfrak{Y} \end{tikzcd},$$

in which the square is Cartesian, and S is the normalization of the strict transform of the section of π' over $(\mathbb{A}^2 - \{0\})/\mathbb{G}_m^2$ defined by $f_1 \cup f_2$. By hypothesis X is proper over $\mathbb{A}^2_k/\mathbb{G}^2_m$ and hence so is S. So S is a normal toric surface, the map $S \to \mathbb{A}^2_k$ is a proper toric map, and the map $S \to X$ is finite by construction.

Now consider the degeneration space $|\,\mathbf{DF}(S/\mathbb{G}^2_m, \mathbf{1}^2)_\bullet|$, which we canonically identify with $\mathbb{R}^2_{\geq 0}$ via the homeomorphism

$$|\,\mathbf{DF}(S/\mathbb{G}^2_m, \mathbf{1}^2)_\bullet| \to |\,\mathbf{DF}(\mathbb{A}^2_k/\mathbb{G}^2_m, \mathbf{1}^2)_\bullet| \simeq \mathbb{R}^2_{\geq 0}.$$

The restriction of the function $\hat{\ell}$ on $|\,\mathbf{DF}(\mathfrak{X}, x)|$ to $|\,\mathbf{DF}(S/\mathbb{G}^2_m, \mathbf{1}^2)_\bullet|$ is a piecewise linear function on $\mathbb{R}^2_{\geq 0}$ that is linear on each cone of the toric fan of S. The NEF condition on $\hat{\ell}$ implies that $\hat{\ell}$ is concave (see [F2]). Once one has established that $\hat{\ell}$ is concave, the argument in the proof of Lemma 4.10 which shows that μ is strictly quasi-concave applies verbatim. ☐

Remark 4.42. The function $\mu(f)$ does not descend to a numerical invariant on $\mathfrak{Y}$. For example, there can be multiple points $x_1, x_2 \in \mathfrak{X}(k)$ mapping to the same point $y \in \mathfrak{Y}(k)$, but the resulting functions on $\mathscr{D}eg(\mathfrak{Y}, y)$ need not agree.

Corollary 4.42.1. *Let $b \in H^4(\mathfrak{Y}; \mathbb{Q})$ be positive definite and let $\ell \in H^2(\mathfrak{X}; \mathbb{Q})[\epsilon]$ be NEF relative to $\mathfrak{Y}$. Then any $x \in \mathfrak{X}(k)$ such that $M^\mu(x) > 0$ has a HN filtration which is unique up to ramified covers of Θ.*

Proof. Let $\bar{x}$ denote the extension of the point x to an algebraic closure $\bar{k}$. We must show that the function μ obtains a unique maximum on $\mathscr{D}eg(\mathfrak{X},\bar{x})$, and we do this by regarding it as a function on $\mathscr{D}eg(\mathfrak{Y},\pi(\bar{x}))$. The function μ satisfies (B1) because $\mathscr{C}omp(\mathfrak{X})$ is bounded by Corollary 3.78.1, so to show the existence of a maximizer it suffices to show that the continuous function μ obtains a maximum on any rational simplex Δ^n. As discussed above, this follows from the compactness of Δ^n and the fact that the coefficients of $\mu(f) \in \mathbb{R}[\epsilon]$ depend continuously on $f \in \Delta^n$.

Now assume there are two distinct points with the same maximal value of $\mu = \phi(\epsilon)$. Then the previous lemma implies that along the unique rational 1-simplex joining the points, μ must vanish to the same order in ϵ, and $\phi(\epsilon)$ must have the maximal leading order coefficient along this 1-simplex. This contradicts the strict quasi-concavity of the coefficients functions of $\mu(f) \in \mathbb{R}[\epsilon]$ established in Lemma 4.41, so any maximum must be unique. $\square$

Example 4.43 (Generalization of Kempf's theorem)**.** In the language of this paper, Kempf's theorem in [K1] states that for a stack of the form $\mathfrak{X} = \mathrm{Spec}(A)/G$ with G reductive and a closed substack $\mathfrak{Z} \subset \mathfrak{X}$, for any point $x \in \mathfrak{X}(k)$ whose closure meets $\mathfrak{Z}$, there is a canonical filtration f of x such that $\mathrm{gr}(f) \in \mathfrak{Z}$. We can generalize this to any Θ-reductive stack which admits a positive definite class $b \in H^4(\mathfrak{X};\mathbb{Q})$ as follows: Let $\mathfrak{X}' = \mathrm{Bl}_{\mathfrak{Z}}(\mathfrak{X})$ and let $\ell = c_1(\mathcal{O}_{\mathfrak{X}'}(1))$. Then ℓ is NEF relative to the projection $\mathfrak{X}' \to \mathfrak{X}$, and for the resulting function μ on $\mathscr{D}eg(\mathfrak{X},x)$, $\mu(f) = 0$ if $\mathrm{gr}(f) \notin \mathfrak{Z}$ and $\mu(f) > 0$ otherwise. Corollary 4.42.1 guarantees that if $x \in \mathfrak{X}$ admits some filtration such that $\mathrm{gr}(f) \in \mathfrak{Z}$, then there is a unique filtration maximizing μ with $\mathrm{gr}(f) \in \mathfrak{Z}$. Kempf's theorem follows from this and the Hilbert-Mumford criterion, which says that for $x \in \mathrm{Spec}(A)/G$, if the closure of x meets a closed substack $\mathfrak{Z} \subset \mathrm{Spec}(A)/G$ then there is some filtration of x with $\mathrm{gr}(f) \in \mathfrak{Z}$.

4.6.4. *Θ-stratifications.* We now state our most general theorem on the existence of Θ-stratifications. We consider a proper representable map of stacks, $\pi : \mathfrak{X}' \to \mathfrak{X}$, where $\mathfrak{X}'$ and $\mathfrak{X}$ satisfy (†). Let $b \in H^4(\mathfrak{X};\mathbb{Q})$ be positive definite and let $\ell \in H^2(\mathfrak{X};\mathbb{Q})[\epsilon]$ be such that the associated numerical invariant $\mu : \mathfrak{U} \subset \mathscr{C}omp(\mathfrak{X}) \to \mathbb{R}[\epsilon]$ defines a weak Θ-stratification of $\mathfrak{X}$. Recall that we are assuming we are given a natural Chern class homomorphism $c_1 : \mathrm{Pic}(\mathfrak{X}) \to H^2(\mathfrak{X};\mathbb{Q})$ as discussed above.

Theorem 4.44. *If $\mathfrak{X}^{\mathrm{ss}}$ is quasi-compact and Θ-reductive, then for any $m \in H^2(\mathfrak{X}';\mathbb{Q})[\epsilon]$ which is numerically positive relative to π, the classes*

$$\ell' := \pi^*(\ell) + \epsilon^k m \qquad \text{and} \qquad \pi^*(b)$$

define a weak Θ-stratification of $\mathfrak{X}'$, where k is any integer larger than the ϵ-degree of ℓ. Furthermore, the preimage of each stratum in $\mathfrak{X}$ is a union of strata in $\mathfrak{X}'$, and for any point $x' \in \mathfrak{X}'$ such that $\pi(x')$ is unstable, the image of an HN filtration of x' is an HN filtration of $\pi(x')$.

We will need the following structure theorem. For a reduced rational curve C over k with non-trivial $\mathbb{G}_m$-action and two fixed points, we denote the two fixed points as $\{0\} = \lim_{t\to 0} t\cdot x$ and $\{\infty\} = \lim_{t\to\infty} t \cdot x$ for a general point x.

Lemma 4.45. *Let R be a discrete valuation ring and let $\Sigma \to \mathbb{A}^1_R$ be a $\mathbb{G}_m$-equivariant birational proper $\mathbb{G}_m$-equivariant map from a reduced algebraic space Σ. Then the fiber $\Sigma_{(0,0)} = C_1 \cup \cdots \cup C_n$ is a connected union of rational curves, each with a non-trivial $\mathbb{G}_m$-action with two fixed points. They are ordered such that $\forall i$ the point $\infty \in C_i$ meets the point $0 \in C_{i+1}$, the point $0 \in C_1$ meets the strict transform of $\operatorname{Spec}(R) \times \{0\} \hookrightarrow \mathbb{A}^1_R$, and the point $\infty \in C_n$ meets the strict transform of $\mathbb{A}^1_k \hookrightarrow \mathbb{A}^1_R$.*

Proof. Applying the construction of [S2, Tag 0BHM], which can be carried out $\mathbb{G}_m$-equivariantly, one can construct an equivariant map $\Sigma' \to \Sigma$ relative to $\mathbb{A}^1_R$, where Σ' is constructed via a sequence of blow ups at closed $\mathbb{G}_m$-invariant points starting with $\mathbb{A}^1_R$. It follows that these blow ups must take place in the fiber over $(0, 0)$, hence $\Sigma' \to \Sigma$ is birational and surjective. The claim can now be reduced to the equivalent claim for Σ'. This is an explicit computation which we leave to the reader. □

Proof of Theorem 4.44. The numerical invariant μ' on $\mathfrak{X}'$ associated to the classes ℓ' and b decomposes as a sum $\mu' = \mu_\ell \circ \pi_* + \epsilon^k \mu_m$, where $\pi_* : \mathscr{C}omp(\mathfrak{X}') \to \mathscr{C}omp(\mathfrak{X})$ is the induced map. We first describe the HN filtration of every unstable point, which establishes the HN-property, then we apply Theorem 4.38 by verifying the HN-specialization condition and (B2). The behavior is different for points over $\mathfrak{X}^{ss}$ versus over the unstable strata of $\mathfrak{X}$, so we treat the two cases separately.

Case $p \in \mathfrak{X}'(\bar{k})$ lies over $\mathfrak{X}^{ss}$:

Let $\mathfrak{Y} = \pi^{-1}(\mathfrak{X}^{ss}) \subset \mathfrak{X}'$. Note that because $\pi(p) \in \mathfrak{X}^{ss}$, we have $\mu_\ell \circ \pi_*(f) \leq 0$ for any filtration of p. If $\mu_\ell \circ \pi_*(f) < 0$, then $\mu'_\ell(f) < 0$ as well, because by construction $\mu_\ell \circ \pi_*(f)$ dominates $\epsilon^k \mu_m(f)$ if it is non-zero. It follows that any destabilizing filtration of p must have $\mu \circ \pi_*(f) = 0$, which implies that $f : \Theta_k \to \mathfrak{X}$ lands in the open substack $\mathfrak{X}^{ss}$ – this is an application of Theorem 3.60 and the argument of Theorem 4.24, and details will appear in [AHLH]. In particular p is ℓ'-unstable in $\mathfrak{X}$ if and only if it is ℓ'-unstable in $\mathfrak{Y}$. Applying Corollary 4.42.1 to the class $\ell' \in H^2(\mathfrak{Y}; \mathbb{Q})[\epsilon]$, which is relatively positive and hence relatively NEF for the projection $\mathfrak{Y} \to \mathfrak{X}^{ss}$, we have that every unstable point on $\mathfrak{Y}$ has a unique (up to ramified covers) HN filtration *as a point in* $\mathfrak{Y}$, and by the previous discussion this is the unique HN filtration of p in $\mathfrak{X}$ with respect to μ'.

Case $p \in \mathfrak{X}'(\bar{k})$ lies over $\mathfrak{X}^{us}$:

Because $\pi : \mathfrak{X}' \to \mathfrak{X}$ is proper, we know that $\pi_* : \mathscr{D}eg(\mathfrak{X}', p) \to \mathscr{D}eg(\mathfrak{X}, \pi(p))$ is a homeomorphism. In particular the HN filtration $f : \Theta_k \to \mathfrak{X}$ of $\pi(p)$ with respect to μ_ℓ on $\mathfrak{X}$ lifts uniquely to a filtration f' of p. We claim that this f' is the unique (up to ramified covers) HN filtration of p. Indeed, $f' \in \mathscr{D}eg(\mathfrak{X}', p)$ is the unique maximizer for $\mu_\ell \circ \pi_*$ by hypothesis, and because the value of $\mu_\ell \circ \pi_*$ is non-zero it dominates $\epsilon^k m$, so it is the unique maximizer for μ' on $\mathscr{D}eg(\mathfrak{X}', p)$ as well.

Verifying the condition (B2):

Given any map from a finite type B-scheme $\xi : S \to \mathfrak{X}'$, one can find a quasi-compact open substack $\mathcal{W} \subset \mathfrak{X}$ containing the image of the composition $S \to \mathfrak{X}' \to \mathfrak{X}$, and it follows from our analysis above that it suffices to look for a HN filtration of any finite type point of S in the quasi-compact substack $\pi^{-1}(\mathcal{W}) \subset \mathfrak{X}'$.

Verifying the HN-specialization condition:

Because $\mathfrak{X}'$ is not Θ-reductive, this property does not hold automatically. Let R be a DVR with fraction field K and residue field k, and let ξ : $\operatorname{Spec}(R) \to \mathfrak{X}'$ be a map whose generic point is unstable. We must show that given a HN filtration $f_K : \Theta_K \to \mathfrak{X}'$ of ξ_K we have $\mu'(f_K) \leq M^{\mu'}(\xi_k)$, and when equality holds f_K extends to a filtration of ξ. We know by hypothesis that this is the case for the composition $\zeta := \xi \circ \pi : \operatorname{Spec}(R) \to \mathfrak{X}' \to \mathfrak{X}$ and the numerical invariant μ_ℓ. The analysis above shows that if strict inequality holds $M^{\mu_\ell}(\zeta_K) < M^{\mu_\ell}(\zeta_k)$ then strict inequality holds upstairs as well.

There are two remaining cases, which we address simultaneously. If $M^{\mu_\ell}(\zeta_K) = M^{\mu_\ell}(\zeta_k) = 0$, then the hypothesis that $\mathfrak{X}^{\mathrm{ss}}$ is Θ-reductive implies that the composition $\pi \circ f_K : \Theta_K \to \mathfrak{X}$, a filtration of ζ_K, extends uniquely to a filtration $f : \Theta_R \to \mathfrak{X}$ of ζ. If $M^{\mu_\ell}(\zeta_K) = M^{\mu_\ell}(\zeta_k) > 0$, then the composition $\pi \circ f_K$ is a HN filtration of ζ_K, and the HN-specialization condition of Theorem 4.38 implies the existence and uniqueness of a filtration $f : \Theta_R \to \mathfrak{X}$ of ζ extending f_K.

The filtration f_K along with the map $\xi : \operatorname{Spec}(R) \to \mathfrak{X}'$ determine a lift of $f|_{\Theta_R \setminus \{(0,0)\}}$ to $\mathfrak{X}'$. Taking the stict transform of the resulting rational section of $\Theta_R \times_{\mathfrak{X}} \mathfrak{X}' \to \Theta_R$ gives us a commutative diagram

$$\begin{tikzcd} \Sigma/\mathbb{G}_m \arrow[r] \arrow[d] & \mathfrak{X}' \arrow[d, "\pi"] \\ \Theta_R \arrow[r] & \mathfrak{X} \end{tikzcd},$$

where Σ is a 2-dimensional reduced and irreducible algebraic space, the $\mathbb{G}_m$-equivariant map $\Sigma \to \mathbb{A}^1_R$ is proper and birational and an isomorphism away from the special point $(0,0) \in \mathbb{A}^1_R$, and the induced map $\Sigma/\mathbb{G}_m \to \Theta_R \times_{\mathfrak{X}} \mathfrak{X}'$ is finite.

By Lemma 4.45 the fiber $\Sigma_{(0,0)}$ consists of a chain of rational curves with non-trivial $\mathbb{G}_m$ action on each component, with all components spinning in the same direction. Label the fixed points $x_0, \ldots, x_n$ in the fiber $\Sigma_{(0,0)}$ in order, where x_0 meets the strict transform of $\operatorname{Spec}(R) \times \{0\} \hookrightarrow \mathbb{A}^1_R$ and x_n meets the strict transform of $\mathbb{A}^1_k \hookrightarrow \mathbb{A}_R$. m is positive relative to the map $\mathfrak{X}' \to \mathfrak{X}$, which by definition implies that for every graded point $\{x_i\}/\mathbb{G}_m \to \mathfrak{X}'$ induced from the map $\Sigma/\mathbb{G}_m \to \mathfrak{X}'$, the value of $\epsilon^k \mu_m$ is *strictly increasing* in i. The value of $\mu_\ell \circ \pi_*$ is constant in i, so the value of μ' on each graded point of $\{x_i\}/\mathbb{G}_m$ is strictly increasing in i. The strict transform of $f_k : \Theta_k \to \Theta_R \to \mathfrak{X}$ to $\mathfrak{X}'$ is a filtration of ξ_k in $\mathfrak{X}'$ whose associated graded is the graded point $\{x_n\}/\mathbb{G}_m \to \mathfrak{X}'$, and $\mu'(f_K)$ is the value of μ' on the graded point $\{x_0\}/\mathbb{G}_m \to \mathfrak{X}'$. It follows that $\mu'(f_K) \leq M^{\mu'}(\xi_k)$. If equality holds then $x_0 = x_n$, in which case $\Sigma \to \mathbb{A}^1_R$ is an isomorphism, and the map $\Theta_R \simeq \Sigma/\mathbb{G}_m \to \mathfrak{X}'$ is a filtration of ξ extending f_K. □

Example 4.46. We will see in Section 5 that the Harder-Narasimhan stratification of the moduli $\mathcal{M}^{\mathcal{F}}$ of torsion-free coherent sheaves on a projective scheme is a Θ-stratification defined by a positive definite class in H^4 and a class in H^2. Theorem 4.44 allows one to construct weak Θ-stratifications on any stack which is projective over $\mathcal{M}^{\mathcal{F}}$.

The formulation of this theorem includes many more specific situations. For instance the most common application of Theorem 4.44 occurs when $\ell = 0$, in which case the statement becomes:

Corollary 4.46.1. *Let $\mathfrak{Y}$ be a quasi-compact Θ-reductive stack satisfying (†), and let $\mathfrak{X} \to \mathfrak{Y}$ be a proper representable map. Then for any positive definite $b \in H^4(\mathfrak{Y};\mathbb{Q})$ and any $\ell \in H^2(\mathfrak{X};\mathbb{Q})[\epsilon]$ which is numerically positive relative to $\mathfrak{Y}$, the resulting numerical invariant $\mu : \mathcal{U} \subset \mathscr{C}omp(\mathfrak{X}) \to \mathbb{R}[\epsilon]$ defines a weak Θ-stratification of $\mathfrak{X}$.*

This of course overlaps with Theorem 4.39 in the case where $\mathfrak{X} \to \mathfrak{Y}$ is the identity map.

Example 4.47 (Projective-over-affine geometric invariant theory)**.** When G is a reductive group, X is a scheme which is projective over its affinization $Y := \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$, and $\mathcal{O}_X(1)$ is a G-equivariant ample invertible sheaf, then we consider $\ell = c_1(\mathcal{O}_X(1)) \in H^2(X/G;\mathbb{Q})$. A choice of Weyl group invariant norm on the cocharacter lattice of the maximal torus of G defines a positive definite class in $H^4(\mathrm{pt}/G;\mathbb{Q})$ whose pullback we denote by $b \in H^4(X/G;\mathbb{Q})$. Then the weak Θ-stratification induced by the numerical invariant associated to ℓ and b is the Kempf-Ness stratification of geometric invariant theory [K2].

Example 4.48 (Bialynicki-Birula stratification)**.** Let X be a proper quasi-compact and quasi-separated algebraic space with an actions of a reductive group G, and let $A \subset G$ be a central torus. Then for any cocharacter $\lambda : \mathbb{G}_m \to A$, the Bialynicki-Birula stratification of X with respect to λ is G-equivariant, and it is a Θ-stratification of X/G. We can construct this stratification using Corollary 4.46.1 as follows: We choose an invariant integral inner product on the cocharacter lattice of a maximal torus of G, and regard it as a positive definite class $b \in H^4(\mathrm{pt}/G;\mathbb{Q})$. We let χ be the rational character of G dual to λ with respect to this form. Then the class $\ell = c_1(\mathcal{O}_X \otimes \chi) \in H^2(X/G;\mathbb{Q})$ is relatively NEF with respect to the projection $X/G \to \mathrm{pt}/G$. Assume that X/G admits a class $\ell' \in H^2(X/G;\mathbb{Q})$ which is positive relative to pt/G.

Then the Bialynicki-Birula stratification of X/G with respect to λ is the Θ-stratification defined by the numerical invariant associated to $\ell + \epsilon\ell'$ and b, as in Corollary 4.46.1. The HN filtration of every point $x \in X(k)$ is given by the one parameter subgroup $\lambda : \mathbb{G}_m \to G$, and it does not depend on ℓ'. The semistable locus is empty.

Remark 4.49. In the previous example, the positive class $\ell' \in H^2(X/G;\mathbb{Q})$ does not effect the HN filtration of any point, but it is used to order the strata. The existence of a positive class is necessary in order to have a Θ-stratification. Consider a non-trivial action of $\mathbb{G}_m$ on $\mathbb{P}^1_k$. Let X be the $\mathbb{G}_m$-scheme obtained by taking two copies of $\mathbb{P}^1$ and identify 0 in the first copy with ∞ in the second, and vice versa. Then let ℓ be the first Chern class of the equivariant invertible sheaf which is $\mathcal{O}_X$ twisted by a non-trivial character of $\mathbb{G}_m$. Then ℓ is NEF relative to $\mathrm{pt}/\mathbb{G}_m$, so every point has a HN filtration by Corollary 4.42.1. But the Bialynicki-Birula strata of X can not be ordered by closure containment and thus do not form a Θ-stratification.

Remark 4.50. In Corollary 4.46.1, the stratification induced by $\ell \in H^2(\mathfrak{X};\mathbb{Q})[\epsilon]$ does not agree with the stratification induced by $\ell(r) \in H^2(\mathfrak{X};\mathbb{Q})$ for small rational $0 < r \ll 1$, but we expect that the $\ell(r)$ stratification is set-theoretically constant for sufficiently small r and refines the ℓ-stratification.

4.7. Main theorem of GIT: good moduli spaces and Θ-stratifications.

In this section we reformulate the main theorem of geometric invariant theory from the perspective of Θ-stability. Recall from [A] the following

Definition 4.51. A *good moduli space* for an algebraic stack $\mathfrak{X}$ is a map $q : \mathfrak{X} \to Y$, where Y is an algebraic space, such that $q_* : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(Y)$ is exact and the canonical map is an equivalence $\mathcal{O}_Y \simeq q_*\mathcal{O}_\mathfrak{X}$

The basic example of a good moduli space morphism is the GIT quotient map

$$\mathrm{Spec}(R)/G \to \mathrm{Spec}(R)//G := \mathrm{Spec}(R^G),$$

where G is a linearly reductive group over a field k and R is a k-algebra with a G-action. The main results of [A] show that many of the useful properties of GIT quotients are consequences of the simple Definition 4.51.

Recently, it has been shown in [AHR1] that good moduli spaces are étale locally modeled by GIT quotients:

Theorem 4.52. [AHR1, Theorem 2.9] *Let $\mathfrak{X}$ be a locally noetherian algebraic stack over an algebraically closed field k. Let $\pi : \mathfrak{X} \to Y$ be a good moduli space such that π is finite type with affine diagonal. If $x \in \mathfrak{X}(k)$ is a closed point, then there exists an affine scheme* $\mathrm{Spec}(A)$ *with an action of*

$G_x := \mathrm{Aut}(x)$ and a cartesian diagram

$$\begin{tikzcd} \mathrm{Spec}(A)/G_x \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\ \mathrm{Spec}(A)/\!/G_x \arrow[r] & Y \end{tikzcd}$$

such that $\mathrm{Spec}(A)/\!/G_x \to Y$ is an étale neighborhood of $\pi(x)$.

Using the slightly strengthened version of the slice theorem, Theorem B.1, one can show that the statement holds as written without the hypothesis that $\mathfrak{X}$ is defined over a field. This result establishes an interesting relationship between Θ-reductive stacks and good moduli spaces.

Lemma 4.53. *Let $\mathfrak{X}$ be a locally noetherian algebraic stack. Let $\pi : \mathfrak{X} \to Y$ be a good moduli space such that π is finite type with affine diagonal. Then $\mathfrak{X}$ is Θ-reductive.*

Proof. We wish to show that $\mathrm{ev}_1 : \mathrm{Filt}(\mathfrak{X}) \to \mathfrak{X}$ satisfies the valuative criterion for properness. We invoke Theorem 4.52 to obtain an étale cover $Y' \to Y$ such that Y' is a disjoint union of affine schemes and $\mathfrak{X}' := Y' \times_Y \mathfrak{X}$ is a disjoint union of stacks of the form $\mathrm{Spec}(A)/G$ where G is linearly reductive. Then Corollary 1.30.1 implies that $\mathrm{ev}_1 : \mathrm{Filt}(\mathfrak{X}') \to \mathfrak{X}'$ is the base change of $\mathrm{ev}_1 : \mathrm{Filt}(\mathfrak{X}) \to \mathfrak{X}$ along the map $\mathfrak{X}' \to \mathfrak{X}$. Because the valuative criterion for properness is étale local over the target, it suffices to show that $\mathfrak{X}'$ is Θ-reductive, which is a consequence of Corollary 4.19.1. □

In fact, in [AHLH] we will prove a converse of this theorem: a Θ-reductive stack satisfying some additional hypotheses admits a good moduli space. We will also show that if $\mathfrak{X}$ is Θ-reductive, then so is $\mathfrak{X}^{\mathrm{ss}}$. Thus the notion of a Θ-reductive stack plays an important role in understanding the structure of the semistable locus as well as the construction of a Θ-stratification.

Let us use Theorem 4.52 to reformulate the main theorem of geometric invariant theory:

Theorem 4.54 (Main theorem of GIT)**.** *Let $\mathfrak{Y}$ be a locally noetherian algebraic stack, and let $q : \mathfrak{Y} \to Y$ be a good moduli space such that q is finite type with affine diagonal. Let $\pi : \mathfrak{X} \to \mathfrak{Y}$ be a projective representable map.*

(1) For any $\ell \in H^2(\mathfrak{X};\mathbb{Q})$ which is positive relative to π and any positive definite $b \in H^4(\mathfrak{Y};\mathbb{Q})$, the numerical invariant μ associated to ℓ and b in Definition 4.9 defines a weak Θ-stratification of $\mathfrak{X}$. This weak Θ-stratification is compatible with base change along a map $Y' \to Y$ in the sense that the stratification defined by the restriction of ℓ and b to $\mathfrak{X}' := \mathfrak{X} \times_Y Y'$ agrees with the induced stratification of Lemma 2.14.

(2) If furthermore $\ell = c_1(\mathcal{L})$ for some relatively ample $\mathcal{L} \in \mathrm{Pic}(\mathfrak{X})$, then
(a) A point $p \in |\mathfrak{X}|$ is μ-semistable if and only if $\exists$ a map $\mathrm{Spec}(A) \to Y$, a point $p' \in |\mathfrak{X}_A|$, and a section $\sigma \in \Gamma(\mathfrak{X}_A, \mathcal{L}|_{\mathfrak{X}_A})$ such that p' maps to p and $\sigma(p') \neq 0$.

(b) *The canonical map* $\mathfrak{X}^{\mathrm{ss}} \to \underline{\mathrm{Proj}}_Y \left(\bigoplus_{n \geq 0} q_*(\pi_*(\mathcal{L}^n)) \right)$ *is a good moduli space, and*

(c) *The center of each stratum admits a good moduli space which is also projective over* Y*.*

(3) *(Perturbation Lemma,* [T1, Lemma 1.2]*) If* $\mathfrak{X}' \to \mathfrak{X}$ *is a projective representable map,* $\mathcal{M} \in \mathrm{Pic}(\mathfrak{X}')$ *is ample relative to* $\mathfrak{X}$*, and* $\mathcal{L} \in \mathrm{Pic}(\mathfrak{X})$ *is ample relative to* $\mathcal{Y}$*, then the* $c_1(\mathcal{L}) + rc_1(\mathcal{M})$ *stratification of* $\mathfrak{X}'$ *is set-theoretically independent of* r *for small rational* $0 < r \ll 1$ *and refines the preimage of the* $c_1(\mathcal{L})$ *stratification of* $\mathfrak{X}$*.*

Proof. The claim in (1) that μ defines a Θ stratification follows from Theorem 4.39, because $\mathfrak{X}$ is Θ-reductive by Lemma 4.53. The fact that the Θ-stratification is compatible with base change along a map $Y' \to Y$ follows from the claim that for any $p' \in \mathfrak{X}'(k)$ with image $p \in \mathfrak{X}(k)$: the induced map of formal fans is an isomorphism

$$\mathbf{DF}(\mathfrak{X}', p')_\bullet \to \mathbf{DF}(\mathfrak{X}, p)_\bullet. \tag{32}$$

This would imply that p' is semistable if and only if p is semistable, and the HN filtration of p' is the HN filtration of p.

For *any* map from a stack to an algebraic space $\pi : \mathfrak{X} \to Y$, and any $\mathbb{Z}^n$-weighted filtration $\Theta_k^n \to \mathfrak{X}$ of $p \in \mathfrak{X}(k)$, the composition $\Theta_k^n \to \mathfrak{X} \to Y$ factors uniquely through the projection $\Theta_k^n \to \mathrm{Spec}(k)$, by Lemma 1.29. If we let $q = \pi(p) \in Y(k)$ and $\mathfrak{X}_q := \pi^{-1}(q) \to \mathfrak{X}$, then $p : \mathrm{Spec}(k) \to \mathfrak{X}$ factors canonically through $\mathfrak{X}_q$, and Corollary 1.30.1 implies that the both commutative squares in the following diagram are cartesian:

$$\begin{tikzcd} \mathrm{Flag}^n(p) \arrow[r] \arrow[d] & \mathrm{Filt}^n(\mathfrak{X}_q) \arrow[r] \arrow[d, "\mathrm{ev}_1"] & \mathrm{Filt}^n(\mathfrak{X}) \arrow[d, "\mathrm{ev}_1"] \\ \mathrm{Spec}(k) \arrow[r, "p"] & \mathfrak{X}_q \arrow[r] & \mathfrak{X} \end{tikzcd}$$

In particular $\mathbf{DF}(\mathfrak{X}, p)_\bullet \simeq \mathbf{DF}(\mathfrak{X}_q, p)_\bullet$ canonically. In the setting of our map $\mathfrak{X}' \to \mathfrak{X}$, if we let $q \in Y(k)$ denote the image of p and $q' \in Y'(k)$ denote the image of p', then the canonical map $\mathfrak{X}_{q'} \to \mathfrak{X}_q$ is an equivalence, hence (32) is an equivalence.

The claims in (2) now follow from the main results of geometric invariant theory. By (1) we know that Θ-stability of a point can be evaluated étale locally over Y, and Theorem 4.52 implies that étale locally over Y, $\mathfrak{X}$ is isomorphic to a quotient of the form $\mathrm{Spec}(A)/G$ for some linearly reductive group G, where Θ-stability agrees with the usual Hilbert-Mumford numerical criterion for semistability. The claim (3) can also be checked étale locally over Y, and therefore reduces to the classical statement [T1, Lemma 1.2]. $\square$

The perturbation lemma can be regarded as a concise formulation of the theory of variation of GIT quotient [DH]. Using this same method of passing to an étale cover of the good moduli space of $\mathfrak{X}$, one may transport many of the other structures from this theory into our intrinsic setting. For instance, one may study the decomposition of $NS(\mathfrak{X})_{\mathbb{R}}$ into cells corresponding to equivalences class, where $\ell \sim \ell'$ if $\mathfrak{X}^{\ell-\mathrm{ss}} = \mathfrak{X}^{\ell'-\mathrm{ss}}$ [DH, Definition 3.4.5].

5. Example: moduli of objects in derived categories

Using different language, it is a classical result that the Harder-Narasimhan stratification of the moduli of coherent sheaves on a projective scheme is a Θ-stratification. See [N1] for a recent thorough treatment. The stack of coherent sheaves is a local quotient stack – it can be approximated by quotients of larger and larger Quot schemes by a general linear group. It is shown in [HK] that in a suitable sense the Harder-Narasimhan stratification of the stack of coherent sheaves is a colimit of the usual GIT stratifications on these Quot schemes. So the theory of Θ-stability is not necessary in this case.

In this section we use our methods to construct a Θ-stratification on a stack which is not known to be a local quotient stack, the stack of torsion-free objects in the heart of a Bridgeland stability condition. We do not make any original contributions to the theory of slope stability and Bridgeland stability in abelian categories. We simply show that slope stability and classical Harder-Narasimhan theory can be reformulated as Θ-stability on the stack of objects in the heart of a t-structure.

5.1. Recollections on the moduli of objects in derived categories.

Let k be a field and let X be a projective scheme over k. We denote the bounded derived category of coherent sheaves on X by $\mathrm{D}^b(X)$. In [L1, Definition 2.1.8], Lieblich defines the stack $\mathcal{D}^b_{pug}(X)$ of "universally gluable relatively perfect complexes on X" and shows that it is an algebraic stack locally of finite type over k. Recall that for any k-algebra R, a complex $E \in \mathrm{D}_{qc}(X_R)$ is perfect relative to R if the functor $E \otimes_R (-) : \mathrm{D}_{qc}(R\text{-Mod}) \to \mathrm{D}_{qc}(X_R)$ has bounded cohomological amplitude in the usual t-structure (see [S2, Tag 0DHZ]). If $\mathcal{A} \subset \mathrm{D}^b(X)$ is the heart of a t-structure, Polischuk and Abramovich define in [AP, P3] a moduli functor for each $v \in \mathcal{N}(X)$ which assigns to any finite type k-scheme S the groupoid of "flat families in $\mathcal{A}$,"

$$\left\{ F \in \mathrm{D}^b(X \times S) | \forall \text{ closed points } s \in S, F|_{X_s} \in \mathcal{A} \right\}. \tag{33}$$

Because the stack $\mathcal{D}^b_{pug}(X)$ is locally finite type and the notion of flat familiy is local for the fppf topology, one could use this notion to implicitly define a substack of $\mathcal{D}^b_{pug}(X)$. We will need a more explicit description of the moduli functor on non-finite-type schemes in order to study families over discrete valuation rings, so we will discuss a moduli functor which is defined on all k-algebras and is equivalent to (33) for finite type k-algebras.

Definition 5.1. Given a t-structure on $\mathrm{D}^b(X)$ and a k-algebra R, the *induced* t-structure on $\mathrm{D}_{qc}(X_R)$ is the unique t-structure for which

$$\mathrm{D}_{qc}(X_R)^{\leq 0} := \left(\begin{array}{c} \text{smallest full subcategory of } \mathrm{D}_{qc}(X_R) \\ \text{containing } R \boxtimes E, \forall E \in \mathrm{D}^b(X)^{\leq 0} \text{ which is closed} \\ \text{under small colimits and extensions} \end{array} \right)$$

and $\mathrm{D}_{qc}(X_R)^{\geq 0}$ is the category of $E \in \mathrm{D}_{qc}(X_R)$ such that $\mathrm{Hom}(F, E) = 0, \forall F \in \mathrm{D}_{qc}(X_R)^{\leq 0}$.

Proof. The category $\mathrm{D}_{qc}(X_R)$ is a presentable stable ∞-category and the category $\mathrm{D}^b(X)^{\leq 0}$ is essentially small, so by [L4, Proposition 1.4.4.11] there is an accesible t-structure on $\mathrm{D}_{qc}(X_R)$ whose subcategory of connective objects is $\mathrm{D}_{qc}(X_R)^{\leq 0}$, and because this t-structure is accessible both truncation functors $\tau^{\leq n}$ and $\tau^{\geq n}$ preserve filtered colimits [L4, Proposition 1.4.4.13]. □

Note that R generates $\mathrm{D}_{qc}(R\text{-Mod})^{\leq 0}$ under colimits and $\mathrm{D}^b(\mathfrak{X})^{\leq 0}$ generates $\mathrm{D}_{qc}(X)^{\leq 0}$ under colimits, so $M \otimes E \in \mathrm{D}_{qc}(X_R)^{\leq 0}$ for any $M \in \mathrm{D}_{qc}(R\text{-Mod})^{\leq 0}$ and $E \in \mathrm{D}_{qc}(X)^{\leq 0}$. This is simply an ∞-categorical version of the construction in [P3], and when R is finitely generated the induced t-structure on X_R constructed above agrees with the previous constructions of induced t-structures by [AP, Theorem 2.7.2] and [P3, Theorem 3.3.6].

Definition 5.2 (Moduli functor)**.** Given a t-structure on $\mathrm{D}^b(X)$ and a k-algebra R, we say that a complex $E \in \mathrm{D}_{qc}(X_R)$ is *R-flat* if $E \otimes_R M \in \mathrm{D}_{qc}(X_R)^{\heartsuit}, \forall M \in R\text{-Mod}$.[17] We define the moduli functor of *flat families of objects in $\mathcal{A}$* which assigns to an affine scheme $\mathrm{Spec}(R)$:

$$\mathcal{M}(R) = \left\{ E \in \mathrm{D}^b(X_R) \text{ perfect relative to } R \text{ and } R - \text{flat} \right\}$$

Remark 5.3. If the t-structure on $\mathrm{D}^b(X)$ is bounded with respect to the usual t structure τ, then $\mathrm{D}_{qc}(X_R)^{\leq_\tau n} \subset \mathrm{D}_{qc}(X_R)^{\leq n+m}$ for some m, where the former denotes the t-structure induced by τ and the latter denotes the usual t-structure, and likewise $\mathrm{D}_{qc}(X_R)^{\geq_\tau n} \subset \mathrm{D}_{qc}(X_R)^{\geq_\tau n-m}$ for some m. In this case any R-flat object $E \in \mathrm{D}_{qc}(X_R)$ also has finite tor amplitude in the usual t-structure, so the moduli functor can be described as $E \in \mathrm{D}_{qc}(X_R)$ which are pseudo-coherent and R-flat.

In order to analyze this moduli functor, let us gather some properties of the induced t-structure on $\mathrm{D}_{qc}(X_R)$.

Lemma 5.4. *For any ring map $R \to S$, the induced map $\phi : X_S \to X_R$ has the following properties with respect to the t-structure we have constructed:*

(1) $\phi^* : \mathrm{D}_{qc}(X_R) \to \mathrm{D}_{qc}(X_S)$ is right t-exact,
(2) $\phi_* : \mathrm{D}_{qc}(X_S) \to \mathrm{D}_{qc}(X_R)$ is t-exact,

[17] As in the rest of the paper, the tensor product denote the derived tensor product $(-) \otimes^L (-)$.

(3) *any* $E \in \mathrm{D}_{qc}(X_S)$ *lies in* $\mathrm{D}_{qc}(X_S)^\heartsuit$ *(respectively* $\mathrm{D}_{qc}(X_S)^{\leq 0}$ *or* $\mathrm{D}_{qc}(X_S)^{\geq 0}$*) if and only if* $\phi_*(E)$ *does,*
(4) *if* $R \to S$ *is flat then* ϕ^* *is exact, and*
(5) *if* $R \to S$ *is faithfully flat then* $E \in \mathrm{D}_{qc}(X_R)$ *lies in the heart if and only if* $\phi^*(E)$ *does.*

Proof. The first claim is immediate from the definitions, because $\phi^*(R \boxtimes E) = S \boxtimes E$ and ϕ^* commutes with colimits because it has a right adjoint ϕ_*. Furthermore, the adjunction between ϕ^* and ϕ_* implies that ϕ_* must map the right orthogonal complement $\mathrm{D}_{qc}(X_S)^{>0}$ to $\mathrm{D}_{qc}(X_R)^{>0}$. Furthermore, under the equivalence of stable ∞-categories $\mathrm{D}_{qc}(X_S) \simeq \mathrm{Mod}_{\mathcal{O}_{X_R} \boxtimes S}(\mathrm{D}_{qc}(X_R))$, colimits in the former are colimits in $\mathrm{D}_{qc}(X_R)$ along with their induced $\mathcal{O}_{X_R} \boxtimes R$-module structure. It follows that ϕ_* commutes with all small colimits. Note that for $E \in \mathrm{D}_{qc}(X)^{\leq 0}$ we have $\phi_*(S \boxtimes E) \in \mathrm{D}$ because $S \in \mathrm{D}_{qc}(R\text{-Mod})^{\leq 0}$. This combined with the fact that ϕ_* commutes with colimits implies that $\mathrm{D}_{qc}(X_S)^{\leq 0} \subset (\phi_*)^{-1}(\mathrm{D}_{qc}(X_R)^{\leq 0})$, which implies (2). (3) follows formally from the fact that ϕ_* is t-exact and conservative. (4) follows from (2) and the fact that $\phi_*(\phi^*(E)) \simeq S \otimes_R E$, so if $R \to S$ is flat then S is a filtered colimit of free R-modules. (5) follows from (4) and the fact that ϕ^* is conservative. □

Note as a consequence that if $p : X_R \to X$ is the projection, then we have

$$\mathrm{D}_{qc}(X_R)^{[a,b]} = \{E \in \mathrm{D}_{qc}(X_R) | p_*(E) \in \mathrm{D}_{qc}(X_R)^{[a,b]}\},$$

where $\mathrm{D}_{qc}(\mathfrak{X}_R)^{[a,b]} = \mathrm{D}_{qc}(X_R)^{\leq b} \cap \mathrm{D}_{qc}(X_R)^{\geq a}$. Futhermore, the fpqc locality of the construction of the t-structure on $\mathrm{D}_{qc}(X_R)$ established in Lemma 5.4 immediately implies the following:

Corollary 5.4.1. $\mathcal{M}$ *is a stack for the fpqc topology on the category of affine schemes.*

It follows that one can define $\mathcal{M}(Y)$ for any algebraic k-stack Y using descent to a smooth cover by affine schemes. We can make this slightly more explicit with the following:

Corollary 5.4.2. *For any algebraic k-stack Y, there is a canonical t-structure induced on* $X_Y = X \times_{\mathrm{Spec}(k)} Y$ *in which* $\mathrm{D}_{qc}(X_Y)^{\leq 0}$ *(respectively* $\mathrm{D}_{qc}(X_Y)^{\geq 0}$*) is the full subcategory of complexes E such that for any smooth map* $\mathrm{Spec}(R) \to Y$ *we have* $E|_{X_R} \in \mathrm{D}_{qc}(X_R)^{\leq 0}$ *(respectively* $\mathrm{D}_{qc}(X_R)^{\geq 0}$*). It suffices to check if a complex lies in* $\mathrm{D}_{qc}(X_R)^{\leq 0}$ *or* $\mathrm{D}_{qc}(X_R)^{\geq 0}$ *after restricting to a particular smooth cover of Y by affine schemes.*

Proof. Given a smooth hypercover $Y_\bullet \to Y$ be a simplicial scheme $Y_\bullet$ where each Y_n is a disjoint union of affines, hyperdescent for quasi-coherent sheaves gives an equivalence of stable ∞-categories

$$\mathrm{D}_{qc}(X_Y) = \mathrm{Tot}_n \, \mathrm{D}_{qc}(X_{Y_n})$$

One can check using the description of $R\operatorname{Hom}$ in the totalization that $R\operatorname{Hom}_{X_Y}(E,F) = \operatorname{Tot}(R\operatorname{Hom}(E|_{X_{Y_n}}, F|_{X_{Y_n}})$ is coconnective whenever $E \in \mathrm{D}_{qc}(X_Y)$ restricts to a connective object on each X_{Y_n} and F restricts to a coconnective object on each X_{Y_n}. Exactness under flat pullback allows one to construct an exact triangle $\tau^{\leq 0}(E) \to E \to \tau^{>0}(E) \to$ for any $E \in \operatorname{Tot}_n \mathrm{D}_{qc}(X_{Y_n})$, hence we have our t-structure. □

It follows from this corollary that $\mathcal{M}(Y)$ is the groupoid of flat and relatively perfect objects in $\mathrm{D}^b(X_Y)$ for any k-scheme Y. For concreteness, however, we will continue to work over affine schemes without loss of generality.

Lemma 5.5. *For $E \in \mathrm{D}_{qc}(X_R)$, the following are equivalent:*

(1) E is R-flat, i.e. $E \otimes_R M \in \mathrm{D}_{qc}(X)^\heartsuit, \forall M \in R\text{-Mod}$,
(2) $\phi^*(E) \in \mathrm{D}_{qc}(X_S)^\heartsuit$ for any map $\phi : X_S \to X_R$ induced by a map of k-algebras $R \to S$.
(3) $E \otimes_R (R/I) \in \mathrm{D}_{qc}(X)^\heartsuit$ for all finitely generated ideals $I \subset R$, and

Furthermore, if R is Noetherian and $E \in \mathrm{D}^b(X_R)$, then this is equivalent to

(4) $E_{R/\mathfrak{m}} \in \mathrm{D}_{qc}(X_{R/\mathfrak{m}})^\heartsuit$ for all maximal ideals $\mathfrak{m} \subset R$.

Proof. Note that by the previous lemma $\phi^*(E) \in \mathrm{D}_{qc}(X_S)$ lies in the heart if and only if $\phi_*(\phi^*(E)) = E \otimes_R S \in \mathrm{D}_{qc}(X_R)^\heartsuit$, so (1) $\Rightarrow$ (2) $\Rightarrow$ (3) tautologically. To show that (3) $\Rightarrow$ (1), one may apply the standard argument of i) reducing to showing that for any injective map of finitely generated R-modules $K \hookrightarrow M$ the map $E \otimes_R K \to E \otimes_R M$ is injective with respect to the t-structure on $\mathrm{D}_{qc}(X_R)$, ii) further reducing to the case where $M = R^{\oplus n}$, then iii) reducing by induction to the case where $M = R$. The argument in [S2, Tag 00HD], for instance, applies verbatim using only the fact that $(-) \otimes (-)$ commutes with filtered colimits on each side and $(-) \otimes M$ is right t-exact for all $M \in R$-Mod.

Finally, we show that when R is Noetherian and $E \in \mathrm{D}^b(X_R)$ then (4) implies (3): Note that (3) is equivalent to the condition that $E|_{\operatorname{Spec}(R/I)} \in \mathrm{D}_{qc}(X_{R/I})^\heartsuit$ for all ideals $I \subset R$ by Lemma 5.4. First assume that R is a complete local ring with unique maximal ideal $\mathfrak{m}$. It suffices to show that for any ideal $I \subset R$, if $E \in \mathrm{D}^b(X_{R/I})^{\leq 0}$ is such that $E|_{R/\mathfrak{m}} \in \mathrm{D}_{qc}(X_{R/\mathfrak{m}})^\heartsuit$, then $E \in \mathrm{D}_{qc}(X_{R/I})^\heartsuit$. Because R/I is again complete with unique maximal ideal $\mathfrak{m}/I \subset R/I$, we will simplify notation by replacing R/I with R. Considering the exact triangle

$$E \otimes_R \mathfrak{m}^n/\mathfrak{m}^{n+1} \to E \otimes_R (R/\mathfrak{m}^{n+1}) \to E \otimes_R (R/\mathfrak{m}^n) \to$$

shows that $E \otimes_R (R/\mathfrak{m}^n) \in \mathrm{D}_{qc}(X_R)^\heartsuit$ for all $n \geq 1$. The Grothendieck existence theorem for the stable ∞-category $\mathrm{D}^b(X_R)$ implies that $E = \operatorname{holim}_n E \otimes_R (R/\mathfrak{m}^n) \in \mathrm{D}_{qc}(X_R)$. Cofiltered homotopy limits are left t-exact with respect to any accessible t-structure on a presentable stable ∞-category, so the fact that $E \otimes_R (R/\mathfrak{m}^n) \in \mathrm{D}_{qc}(X_R)^\heartsuit$ for all n implies $E \in \mathrm{D}_{qc}(X_R)^{\geq 0}$ and hence $E \in \mathrm{D}_{qc}(X_R)^\heartsuit$.

Now for an arbitrary Noetherian ring R, consider the map

$$S := \bigcup_{\mathfrak{m}\subset R} \operatorname{Spec}(\hat{R}_{\mathfrak{m}}) \to \operatorname{Spec}(R),$$

where the union is taken over all maximal ideals and $\hat{R}_{\mathfrak{m}}$ denotes the completion of R with respect to $\mathfrak{m}$. Because R is noetherian this map is faithfully flat. The previous argument shows that if condition (4) holds for $E \in \mathrm{D}^b(X_R)$, then $E|_S$ is flat over S. This combined with observation (5) of Lemma 5.4 implies that $E \otimes_R M \in \mathrm{D}_{qc}(X_R)^\heartsuit$ for any $M \in R$-Mod, hence E is flat over R. □

Part (4) of Lemma 5.5 shows that the moduli functor of Definition 5.2 agrees with the moduli functor of [AP] defined over finite type k-schemes. The following depends on the key technical result of [AP, P3], and is a slight generalization of what is stated in those papers:

Proposition 5.6. *Assume the t-structure on $\mathrm{D}^b(X)$ is noetherian, and let R be an algebra which is essentially of finite type over k. Then the truncation functors on $\mathrm{D}_{qc}(X_R)$ preserve $\mathrm{D}^b(X_R)$, and the induced t-structure on $\mathrm{D}^b(X_R)$ is noetherian.*

Lemma 5.7. *Let $R \to S$ be a localization of noetherian k-algebras. Then $\phi^* : \mathrm{D}^b(X_S) \to \mathrm{D}^b(X_R)$ is essentially surjective.*

Proof. We may work with the usual t-structure on $\mathrm{D}^b(-)$. The key observation is that because $R \to S$ is a localization, for any $E \in \mathrm{QCoh}(X_S)$ we have $\phi^*(\phi_*(E)) = \phi_*(E) \otimes_R S \simeq E$, where $\phi_*(E)$ is just E regarded as a quasi-coherent sheaf on X_R. It follows that for any bounded complex of quasi-coherent sheaves $E^\bullet$ on X_S with coherent homology sheaves, the pushforward $\phi_*(E^\bullet)$ is a complex of quasi-coherent sheaves on X_R whose restriction to X_S is $E^\bullet$.

From this point, a fairly well known descending induction argument allows one to replace $\phi_*(E^\bullet)$ with a bounded coherent subcomplex $F^\bullet \subset \phi_*(E^\bullet)$ such that the induced map $F^\bullet \otimes_R S \to \phi_*(E^\bullet) \otimes_R S \simeq E^\bullet$ is a quasi-isomorphism. For instance, the proof of [P3, Lemma 2.3.1] applies verbatim once one establishes the following:

Claim: For any surjective map $F \to G$ in $\mathrm{QCoh}(X_R)$ such that $G \otimes_R S \in \mathrm{QCoh}(X_S)$ is coherent, there is a coherent subsheaf $F' \subset F$ such that $F' \otimes_R S \to G \otimes_R S$ is surjective.

This claim follows as usual by writing F as a filtered union $F = \bigcup_\alpha F_\alpha$ of coherent subsheaves $F_\alpha \subset F$, then observing that because the category of coherent sheaves on X_R is noetherian one of these $F_\alpha \otimes_R S$ must surject onto $G \otimes_R S$. □

Proof of Proposition 5.6. The case of a smooth k-algebra is [AP, Theorem 2.6.1, Theorem 2.7.2], and the case of a general k-algebra of finite type in [P3, Theorem 3.3.6]. Here we simply extend this observation to the case where S is a localization $R \to S$ of a k-algebra R of finite type. We already know from Lemma 5.4 that $\phi^* : \mathrm{D}_{qc}(X_R) \to \mathrm{D}_{qc}(X_S)$ is t-exact. It therefore suffices to show that $\phi^* : \mathrm{D}^b(X_R) \to \mathrm{D}^b(X_S)$ is essentially surjective, which is Lemma 5.7. By the same logic, a sequence of surjective maps $E_1 \to E_2 \to \cdots$ in $\mathrm{D}^b(X_S)^\heartsuit$ can be lifted to a descending chain of surjections in $\mathrm{D}^b(X_R)^\heartsuit$, so it must stabilize and hence the t-structure on $\mathrm{D}^b(X_S)$ is noetherian. $\square$

Corollary 5.7.1. *For any field extension K/k, the subcategory $\mathrm{D}^b(X_K) \subset \mathrm{D}_{qc}(X_K)$ is preserved by the truncation functors for the induced t-structure and thus inherets a t-structure. If K has finite transcendence degree over k then $\mathrm{D}^b(X_K)^\heartsuit$ is noetherian.*

Proof. Any $E \in \mathrm{D}^b(X_K)$ is relatively perfect, so by [L1, Proposition 2.2.1] we can find a finitely generated k-subalgebra $R \subset K$ and an $E' \in \mathrm{D}^b(X_R)$ such that $E \simeq E' \otimes_R$. The extension $R \subset K$ must be flat because K contains the field of fractions of R, so the restriction functor $\mathrm{D}_{qc}(X_R) \to \mathrm{D}_{qc}(X_K)$ is t-exact. It follows that all truncations of E are the restriction of truncations of E', hence they lie in $\mathrm{D}^b(X_K)$. If K has finite transcendence degree then we can arrange that K is the field of fractions of R, so by Proposition 5.6 the heart is noetherian. $\square$

Corollary 5.7.2 (Open heart property)**.** *Let R be a finite type k-algebra and let $E \in \mathrm{D}^b(X_R)$. The set of prime ideals*

$$U := \left\{ \mathfrak{p} \in \mathrm{Spec}(R) \middle| E|_{R_\mathfrak{p}} \in \mathrm{D}^b(X_{R_\mathfrak{p}})^\heartsuit \right\}$$

is open, and it contains those primes for which $E|_{\kappa(\mathfrak{p})} \in \mathrm{D}_{qc}(X_{\kappa(\mathfrak{p})})^\heartsuit$, where $\kappa(\mathfrak{p})$ denotes the field of fractions of $R/\mathfrak{p}$.

Proof. Because restriction along the map $X_{R_\mathfrak{p}} \to X_R$ is t-exact, the subset U is the complement of the image under the projection $X_R \to \mathrm{Spec}(R)$ of the closed subsets $\mathrm{Supp}(\tau^{<0}(E))$ and $\mathrm{Supp}(\tau^{>0}(E))$. Therefore $\mathrm{Spec}(R) \setminus U$ is closed because the projection $X_R \to \mathrm{Spec}(R)$ is proper. The fact that if $E|_{\kappa(\mathfrak{p})} \in \mathrm{D}_{qc}(X_{\kappa(\mathfrak{p})})^\heartsuit$ then $E|_{R_\mathfrak{p}} \in \mathrm{D}^b(X_{R_\mathfrak{p}})^\heartsuit$ is part (4) of Lemma 5.5. $\square$

Now let us recall the generic flatness property:

Definition 5.8 (Generic flatness [AP])**.** A t-structure on $\mathrm{D}^b(X)$ has the *generic flatness* property if given a finite type integral k-algebra R with fraction field K and an object $E \in \mathrm{D}^b(X_R)$ such that $E_K \in \mathrm{D}^b(X_K)^\heartsuit$, there is an $f \in R$ such that $E|_{\mathrm{Spec}(R_f)} \in \mathrm{D}^b(X_{R_f})$ is flat, where $R_f = R[f^{-1}]$.

This is not exactly how generic flatness was formulated in [AP, Problem 3.5.1], but over a field of characteristic 0 this is equivalent to the definition in [AP] by the following:

Lemma 5.9. *If $\mathrm{char}(k) = 0$, then generic flatness is equivalent to the following condition: for every smooth k-algebra R and every $E \in \mathrm{D}^b(X_R)^\heartsuit$, there is a dense open subset $U \subset \mathrm{Spec}(R)$ such that $E|_U$ is flat.*

Proof. Note that the restriction $\mathrm{D}^b(X_R) \to \mathrm{D}^b(X_K)$ is t-exact, so if $E \in \mathrm{D}^b(X_R)^\heartsuit$ then so is E_K, and generic flatness implies the condition in the lemma. For the other direction assume the condition of the lemma, and consider an integral k-algebra R and $E \in \mathrm{D}^b(X_R)$. If $E_K \in \mathrm{D}^b(X_K)^\heartsuit$, then by Corollary 5.7.2 we can find a dense open $U \subset \mathrm{Spec}(R)$ such that $E|_U \in \mathrm{D}^b(X_U)^\heartsuit$. By generic smoothness for reduced k-algebras we can pass to a smaller open subset $U'' \subset U \subset \mathrm{Spec}(R)$ which is smooth over k. The condition of the lemma implies that there is an open $\mathrm{Spec}(R_f) \subset U''$ such that $E|_{\mathrm{Spec}(R_f)}$ is flat, hence we have generic flatness. □

In the context of Example 5.23, where $\mathcal{A} = \mathrm{Coh}_{\leq d}$, generic flatness is classical. Furthermore, in [T4, Lemma 4.7, Proposition 3.18], Toda shows that the generic flatness property holds for a set of algebraic stability conditions which is dense (up to the action of $\tilde{\mathrm{GL}}_2^+(2,\mathbb{R})$) in the connected component of the space of Bridgeland stability conditions on $K3$ surfaces over $\mathbb{C}$ constructed and studied in [B+2]. More recently, Piyaratne and Toda establish the generic flatness property for certain Bridgeland stability conditions on 3-folds in [PT].

Proposition 5.10. *Given a noetherian t-structure on $\mathrm{D}^b(X)$ which satisfies the generic flatness condition, the stack $\mathcal{M}$ of flat families in $\mathrm{D}^b(X)^\heartsuit$ is an open substack of $\mathcal{D}^b_{pug}(X)$, hence it is algebraic and locally of finite type over k with affine diagonal.*[18]

Proof. Because any object in the heart of a t-structure is gluable, we know from Lemma 5.5 that any complex $E \in \mathcal{M}(R)$ is universally gluable, hence $\mathcal{M}$ is a substack of $\mathcal{D}^b_{pug}(X)$. We now claim that if the generic flatness property holds and $E \in \mathrm{D}^b(X_R)$ is perfect relative to R, then the set of prime ideals

$$U := \left\{ \mathfrak{p} \in \mathrm{Spec}(R) \,\middle|\, E|_{R/\mathfrak{p}} \in \mathrm{D}_{qc}(X_{R/\mathfrak{p}})^\heartsuit \right\}$$

is open. There is a subalgebra $R' \subset R$ of finite type over k and a relatively perfect complex $E' \in \mathrm{D}^b(X_{R'})$ such that $E = E' \otimes_{R'} R$ [L1, Proposition 2.2.1], so it suffices to assume that R is finite type. We will show by noetherian induction that $Z := \mathrm{Spec}(R) \setminus U$ is closed. A simple inductive argument reduces one to the case where R is integral, so we will assume this. The property $E|_{R/\mathfrak{p}} \notin \mathrm{D}_{qc}(X_{R/\mathfrak{p}})^\heartsuit$ is closed under specialization by part (4) of Lemma 5.5, so if K is the field of fractions of R and $E_K \notin \mathrm{D}^b(X_K)^\heartsuit$ then $Z = \emptyset$. On the other hand, if $E|_K \in \mathrm{D}^b(X_K)^\heartsuit$ then by generic flatness we know that there is an f such that $E|_{R_f}$ is flat, and hence $Z \subset \mathrm{Spec}(R/(f)) \subset \mathrm{Spec}(R)$ which is closed by the inductive hypothesis. □

Remark 5.11. For a Bridgeland stability condition $(Z, \mathcal{A})$, the methods of [T4, PT] show that the stack of semistable objects of a given numerical

[18] The fact that $\mathcal{D}^b_{pug}(X)$ has an affine diagonal is not established in [L1], but this follows from a formal argument which applies to any moduli of objects in a k-linear category with finite dimensional Hom spaces. See [S2, Tag 0DPW].

K-theory class is an open substack of $\mathcal{D}^b_{pug}(X)$ whenever $(Z, \mathcal{A})$ is sufficiently close to a stability condition with noetherian heart which satisfies the generic flatness property.

5.1.1. *Filtrations in* $\mathcal{M}$. Let R be a k-algebra. We regard $\mathbb{Z}^n$ as the objects of a category in which there is a unique morphism $(w_1, \ldots, w_n) \to (w'_1, \ldots, w'_n)$ if $w_i \geq w'_i$ for all i. The Reese construction (see Proposition 1.1) gives an equivalence of ∞-categories

$$\mathrm{D}_{qc}(\Theta^n_R \times X) \simeq \mathrm{Fun}(\mathbb{Z}^n, \mathrm{D}_{qc}(X_R)).$$

More explicitly, given a quasi-coherent complex on $X \times \Theta^n_R$, we may regard it as a complex of $\mathbb{Z}^n$-graded quasi-coherent modules over $\mathcal{O}_{X_R}[t_1, \ldots, t_n]$, where t_i has multi-degree $(0, \ldots, 0, -1, 0, \ldots, 0)$ with -1 in the i^{th} coordinate. Pushing forward along the map $X \times \Theta^n_R \to X \times (\mathrm{pt}/\mathbb{G}_m)^n_R$, which corresponds to forgetting the $\mathcal{O}_{X_R}[t_1, \ldots, t_n]$-module structure, we have an isomorphism $E \simeq \bigoplus_{w \in \mathbb{Z}^n} E_w$, where the quasi-coherent complex E_w is the summand in weight w. Multiplication by the monomial $t^v = t_1^{v_1} \cdots t_v^{v_n}$ gives a map $E_{w+v} \to E_w$ for all $w \in \mathbb{Z}^n$.

Now consider a functor $\mathbb{Z}^n \to \mathrm{D}_{qc}(X_R)^\heartsuit$, corresponding to objects $E_w \in \mathrm{D}_{qc}(X_R)^\heartsuit$ and maps $E_w \to E_{w'}$ for $w \geq w'$. We call such a functor a $\mathbb{Z}^n$-weighted filtration if all of the maps $E_w \to E_{w'}$ are injective. We denote the i^{th} standard basis vector in $\mathbb{Z}^n$ by e_i. By definition $\mathrm{gr}(E_\bullet)$ is the $\mathbb{Z}^n$-graded object for which

$$\mathrm{gr}_w(E_\bullet) := \mathrm{coker}(\bigoplus_i E_{w+e_i} \to E_w).$$

For any w and any ordered tuple $i_1 < \ldots < i_m$, let us denote the map by

$$\delta^p_{i_1 < \ldots < i_m} : E_{w+e_{i_1}+\cdots+e_{i_m}} \to E_{w+e_{i_1}+\cdots+e_{i_{p-1}}+e_{i_{p+1}}+\cdots+e_{i_m}}.$$

Usind this notation we define the augmented Koszul complex in $\mathrm{D}_{qc}(X_R)^\heartsuit$

$$0 \to E_{w+e_1+\cdots+e_n} \to \cdots \to \bigoplus_{i_1<i_2} E_{w+e_{i_1}+e_{i_2}} \to \bigoplus_{i_1} E_{w+e_{i_1}} \to E_w \to \mathrm{gr}_w(E_\bullet) \to 0, \quad (34)$$

where the differential is given by $\sum_{i_1<\cdots<i_m} \sum_{p=1}^n (-1)^{p+1} \delta^p_{i_1<\cdots<i_m}$. We define the Koszul complex $\mathcal{K}^\bullet_E$ to be the complex above without the last term, i.e. without the $\mathrm{gr}_w(E_\bullet)$ term on the right. So there is a canonical map from the totalization $\mathrm{Tot}^\oplus(\mathcal{K}^\bullet_E) \to \mathrm{gr}_w(E)$ in $\mathrm{D}_{qc}(X_R)$, and the augmented Koszul complex is the cone of this map.

Lemma 5.12. *Let R be a k-algebra. Under the Reese construction above,* $\mathrm{Filt}^n(\mathcal{M})(R) = \mathrm{Map}(\Theta^n_R, \mathcal{M})$ *is naturally isomorphic to the groupoid of $\mathbb{Z}^n$-weighted filtrations $\{E_w\}_{w \in \mathbb{Z}^n}$ in $\mathrm{D}_{qc}(X_R)^\heartsuit$ such that*

(1) for all $w \in \mathbb{Z}^n$, $\mathrm{gr}_w(E_\bullet)$ is R-flat and relatively perfect over R,
(2) $\exists N \gg 0$ such that $E_{w_1,\ldots,w_n} = 0$ if $w_i > N$ for any i,

(3) $\exists N \ll 0$ *such that* $E_{w_1,\ldots,w_n} \to E_{w'_1,\ldots,w'_n}$ *is an isomorphism if* $w_i < N$ *for any* i*, and*

(4) for all $w \in \mathbb{Z}^n$ *the augmented Koszul complex* (34) *is exact.*

The fiber of a map $f : \Theta^n_R \to \mathcal{M}$ *at* $(1,\ldots,1)$ *corresponds to* $\operatorname{colim}_w E_w = E_{w'} \in \mathrm{D}^b(X_R)$ *for* $w' \ll 0$*, and the fiber at* $(0,\ldots,0)$ *corresponds to* $\bigoplus_w \mathrm{gr}_w(E_\bullet)$ *in* $\mathrm{D}^b(X_R)$*.*

Remark 5.13. Conditions (1) and (4) together should be interpreted as the condition that if $E \in \mathrm{D}_{qc}(X \times \Theta^n_R)$ is the complex corresponding to the $\mathbb{Z}^n$ weighted filtration $\{E_w\}_{w\in\mathbb{Z}^n}$, then each weight summand of

$$E \otimes_{\Theta^n_R} \mathcal{O}_{\mathrm{Spec}(R)\times\{0,\ldots,0\}} \in \mathrm{D}_{qc}(X \times (\mathrm{pt}/\mathbb{G}^n_m)_R)$$

is flat and relatively perfect over R. This graded object is then $\bigoplus_{w\in\mathbb{Z}^n} \mathrm{gr}_w(E)$. Therefore (1) and (4) are equivalent to the condition that the (derived) restriction of E along the closed immersion $(\{0\}/\mathbb{G}_m)^n_R \hookrightarrow \Theta^n_R$ is flat, and its weight summands with respect to $(\mathbb{G}_m)^n_R$ are relatively perfect over R.

Proof. First we show that given a filtration $\{E_w\}_{w\in\mathbb{Z}^n}$ *satisfying these conditions, the corresponding object* $E \in \mathrm{D}_{qc}(X \times \Theta^n_R)^\heartsuit$ *is flat and relatively perfect over* Θ^n_R*:*

Consider the restriction of E along the closed immersion $i_n : \Theta^{n-1}_R \times (\mathrm{pt}/\mathbb{G}_m) = \{t_n = 0\} \hookrightarrow \Theta^n_R$, and let F_r for $r \in \mathbb{Z}$ denote the summand of $(i_n)^*(E)$ concentrated in weight r with respect to the last factor of $\mathrm{pt}/\mathbb{G}_m$. Regarding $i_n^*(E)$ as a $\mathbb{Z}^n$-graded object in $\mathrm{D}_{qc}(X_R)$, we weight w piece is

$$\mathrm{Cone}(E_{w_1,\ldots,w_{n-1},w_n+1} \xrightarrow{t_n} E_{w_1,\ldots,w_n}),$$

and conditions (2) and (3) imply that this vanishes for $w_n \ll 0$ or $w_n \gg 0$. It follows that $F_r \neq 0$ for only finitely many $r \in \mathbb{Z}$. If r_{max} denotes the largest r for which $F_r \neq 0$, then we have an exact triangle

$$F_{r_{max}} \otimes k[t_n]\langle r_{max}\rangle \to E \to E' \to$$

where the left object denotes the pullback along the projection $\Theta^n_R \to \Theta^{n-1}_R$ along the last copy of Θ followed by a twist by a character of $\mathbb{G}_m$ so that $(i_n)^*(F_{r_{max}} \otimes k[t_n]\langle r_{max}\rangle)$ is concentrated in weight r_{max} with respect to the last $\mathbb{G}_m$. This map is uniquely determined by requiring that it is an isomorphism on highest weight pieces (with respect to the last factor of $\mathrm{pt}/\mathbb{G}_m$) after restriction along i_n. In particular if F'_r denote the weight summands of $i_n^*(E')$, then $F'_r = F_r$ for $r \neq r_{max}$ and $F'_{r_{max}} = 0$. It follows that E can be constructed as an iterated extension of objects of the form $F \otimes k[t_n]\langle r\rangle$ for certain $F \in \mathrm{D}_{qc}(X \times \Theta^{n-1}_R)$ and $r \in \mathbb{Z}$.

We argue by induction on n that E is flat and relatively perfect over Θ^n_R. Because flat and relatively perfect complexes are closed under extension, pullback, and tensoring by an invertible sheaf on the base (which is what the shift by a character of $\mathbb{G}_m$ is doing), it suffices to show that the F_r constructed above are flat and relatively perfect. The base case is when $n = 1$, in this case (4) implies that the restriction $(i_1)^*(E)$ lies in $\mathrm{D}_{qc}(X \times (\mathbb{G}_m)_R)^\heartsuit$ and then the weight r pieces are just $\mathrm{gr}_r(E)$, which are flat and relatively perfect by (1).

For $n > 1$, consider $\bigoplus_r F_r\langle r\rangle \simeq i_n^*(E)$. As remarked above conditions (1) and (4) are equivalent to the condition that $E|_{(\mathrm{pt}/\mathbb{G}_m)^n_R} \in \mathrm{D}_{qc}(X \times (\mathrm{pt}/\mathbb{G}_m)^n_R)$ is flat and its weight summands are relatively perfect. It follows that (1) and (4) still hold for each F_r because the closed immersion $(\mathrm{pt}/\mathbb{G}_m)^n_R \hookrightarrow \Theta^n_R$ factors through $\Theta^{n-1}_R \times (\mathrm{pt}/\mathbb{G}_m)$. The weight $(w_1, \ldots, w_{n-1})$ component of F_r, regarded as a graded object in $\mathrm{D}_{qc}(X_R)$ is the cone of the map $E_{w_1,\ldots,w_{n-1},r+1} \to E_{w_1,\ldots,w_{n-1},r}$, so conditions (2) and (3) imply that this cone is 0 if $w_i \ll 0$ or $w_i \gg 0$ for any i. It follows that F_r satisfies (1)-(4), and hence by the inductive hypothesis F_r is flat and relatively perfect over Θ^{n-1}_R.

Second, we show that for $E \in \mathrm{D}_{qc}(X \times \Theta^n_R)$ which is flat and relatively perfect over Θ^n_R the corresponding functor $\{E_w\}_{w \in \mathbb{Z}^n}$ is a filtration satisfying (1)-(4):

As remarked above, conditions (1) and (4) together are equivalent to requiring that the restriction of E along the inclusion $(\mathrm{pt}/\mathbb{G}_m)^n_R \hookrightarrow \Theta^n_R$ is flat and has relatively perfect weight summands with respect to $\mathbb{G}_m^n$. This is clearly satisfied if E is relatively perfect and flat.[19] Next we claim that (2) and (3) hold for *any* relatively perfect and universally gluable $E \in \mathrm{D}^b(X \times \Theta^n_R)$. First by Artin approximation for relatively perfect and universally gluable complexes [L1, Proposition 2.2.1], we can find a finitely generated sub-algebra $R' \subset R$ and a universally gluable and relatively perfect $E' \in \mathrm{D}^b(X \times \Theta^n_{R'})$ along with an isomorphism $E \simeq E' \otimes_{R'} R$. We may therefore assume that R is noetherian.

We now claim that for a noetherian k-agebra R, conditions (2) and (3) apply for any $E \in \mathrm{D}^b(X \times \Theta^n_R)$. Indeed because pushforward to $X \times (\mathrm{pt}/\mathbb{G}_m)^n_R$ commutes with taking homology in the usual t-structure, we can identify the graded summands $H^i(E)_w \simeq H^i(E_w)$ in the usual t-structure for $w \in \mathbb{Z}^n$. So it suffices to prove the claim for $E \in \mathrm{Coh}(X \times \Theta^n_R)$. In this case there are graded coherent sheaves $F_0, F_1 \in \mathrm{Coh}(X \times (\mathrm{pt}/\mathbb{G}_m)^n_R)$ such that

$$E = \mathrm{coker}(F_1 \otimes k[t_1, \ldots, t_n] \to F_0 \otimes k[t_1, \ldots, t_n]),$$

where $F_i \otimes k[t_1, \ldots, t_n]$ denotes the pullback of F_i along the projection $X \times \Theta^n_R \to X \times (\mathrm{pt}/\mathbb{G}_m)^n_R$. Conditions (2) and (3) hold for the weight w summands of $F_i \otimes k[t_1, \ldots, t_n]$ for $i = 0, 1$, and it follows that these conditions hold for the E_w as well. $\square$

Example 5.14. When $n = 2$ in the previous lemma, the conditions (1) and (4) amount to the requirement that all maps $E_{v+1,w} \to E_{v,w}$ and

[19] Note that in fact $E|_{(\mathrm{pt}/\mathbb{G}_m)^n_R}$ will have finitely many non-zero graded piece for only finitely many w. This is close to but slightly weaker than (2) and (3).

$E_{v,w+1} \to E_{v,w}$ are injective, and that every square

$$\begin{tikzcd} E_{v+1,w+1} \arrow[r] \arrow[d] & E_{v+1,w} \arrow[d] \\ E_{v,w+1} \arrow[r] & E_{v,w} \end{tikzcd}$$

is a fiber product.

For clarity of notation we will use underscored letters to denote vectors $\underline{w}_1, \underline{w}_2, \ldots \in \mathbb{Z}^n$. As above we say $\underline{w}_1 \leq \underline{w}_2$ if the inequality holds for every entry. Given a flat and relatively perfect $E \in D_{qc}(\Theta_R^n \times X)$, let us denote $E_{-\infty} = E_w$ for $w \ll 0$. Then forgetting the $\mathcal{O}_{X_R}[t_1, \ldots, t_n]$ module structure, we have an injective map of $\mathbb{Z}^n$-graded objects in $\mathrm{D}_{qc}(X_R)$

$$E \hookrightarrow E_{-\infty} \otimes_R R[t_1^{\pm}, \ldots, t_n^{\pm}].$$

So a $\mathbb{Z}^n$-weighted filtration is determined by an object $E_{-\infty} \in \mathcal{M}(\mathrm{Spec}(R))$ and an $\mathcal{O}_{X_R}[t_1, \ldots, t_n]$-submodule of $\mathcal{O}_{X_R}[t_1^{\pm}, \ldots, t_n^{\pm}] \otimes E_{-\infty}$ of the form

$$E = \sum \mathcal{O}_{X_K}[t_1, \ldots, t_n] \cdot t^{-\underline{w}_i} \otimes E_i \subset E_{-\infty} \otimes_K K[t_1^{\pm}, \ldots, t_n^{\pm}] \tag{35}$$

for some finite collection of subobjects $E_i \subset E_{-\infty}$ with the property that $E_i \subset E_j$ whenever $\underline{w}_i \geq \underline{w}_j$, and which satisfy the conditions of Lemma 5.12. Over a field, a filtration is non-degenerate if and only if the weights $w \in \mathbb{Z}^n$ for which $\mathrm{gr}_w(E_\bullet) \neq 0$ span $\mathbb{Q}^n$.

Any map $\phi : \Theta_K^p \to \Theta_K^n$ is encoded by an $n \times p$ matrix with nonnegative integer entries A, and we have

$$\phi^*(E) = \sum \mathcal{O}_X[t_1, \ldots, t_p] \cdot t^{-A^\tau \underline{w}_i} \otimes E_i \subset E \otimes_K K[t_1^{\pm}, \ldots, t_p^{\pm}]$$

where A^τ denotes the transpose matrix, and we are regarding $\underline{w}_i$ as a column vector. In particular for any point in $a \in \mathbb{R}^n_{\geq 0}$ with integer coefficients, regarded as a $n \times 1$ matrix, the pullback $\phi^*(E)$ under the corresponding map $\phi : \Theta_K \to \Theta_K^n$ is the $\mathbb{Z}$-weighted filtration

$$E \supset \cdots \supset E'_v \supset E'_{v+1} \supset \cdots, \text{ where } E'_v = \sum_{i | a^\tau \underline{w}_i \geq v} E_i.$$

The object (35) automatically satisfies conditions (2) and (3) of Lemma 5.12, but it does not necessarily define a flat $\mathcal{O}_{X_R}[t_1, \ldots, t_n]$ module if $n > 1$. However for any descending non-weighted filtration $E_1 \supsetneq E_2 \supsetneq E_3 \supsetneq \cdots \supsetneq E_n \neq 0$ in $\mathrm{D}^b(X_R)^\heartsuit$ whose associated graded complexes are flat and relatively perfect over $\mathrm{Spec}(R)$, and for any weight sequence $\underline{w}_1 \leq \underline{w}_2 \leq \cdots \leq \underline{w}_n$ we define the canonical $\mathbb{Z}^n$-weighted filtration $\sigma(E_\bullet, \underline{w}_\bullet) \in \mathrm{D}^b(X \times \Theta_R^n)^\heartsuit$ as the object (35). We leave it as an exercise to the reader to check conditions (1) and (4), i.e. that the totalization of the Koszul complex is flat and relatively perfect over $\mathrm{Spec}(R)$ for each weight $\underline{w}$. More precisely, $\mathrm{Tot}^\oplus(\mathcal{K}^\bullet_{\sigma(E_\bullet, \underline{w}_\bullet)})$ in weight $\underline{w}_i$ is quasi-isomorphic to $\mathrm{gr}_{\underline{w}_i}(\sigma(E_\bullet, \underline{w}_\bullet)) = E_i/E_{i+1}$ for $i = 1, \ldots, n$, and $\mathrm{Tot}^\oplus(\mathcal{K}^\bullet_{\sigma(E_\bullet, \underline{w}_\bullet)})$ is exact in every other weight. In particular the filtration is non-degenerate if and only if $\underline{w}_1, \ldots, \underline{w}_n$ span $\mathbb{Q}^n$. We use the same notation to denote the corresponding non-degenerate map $\sigma(E_\bullet, \underline{w}_\bullet) : \Theta_R^n \to \mathcal{M}$.

Proposition 5.15. *Let K/k be a field extension, and let $E \in \mathcal{M}(\mathrm{Spec}(K))$. The rational simplices corresponding to the weighted filtrations $\sigma(E_\bullet, \underline{w}_\bullet)$, for all finite length filtrations $E = E_1 \supsetneq \cdots \supsetneq E_n \supsetneq 0$ and weight sequences $\underline{w}_1 \leq \cdots \leq \underline{w}_n \in \mathbb{Z}^n$, cover $\mathscr{D}eg(\mathcal{M}, [E])$.*

Proof. We will show that for any rational cone $\mathbb{R}^n_{\geq 0} \to |\mathbf{DF}(\mathcal{M}, E)_\bullet|$, corresponding to a non-degenerate filtration $f : \Theta_K^n \to \mathcal{M}$, there is a decomposition of $\mathbb{R}^n_{\geq 0}$ into rational polyhedral sub-cones such that the restriction of f to along any map $\phi : \Theta_K^p \to \Theta_K^n$ for which $\phi(\mathbb{R}^p_{\geq 0}) \subset \mathbb{R}^n_{\geq 0}$ is contained in one of the subcones, the restriction of $f \circ \phi$ factors through one of our canonical maps $\sigma(E_\bullet, \underline{w}_\bullet) : \Theta_K^N \to \mathcal{M}$. In the language of Section 3, if $F_\bullet \subset \mathbf{DF}(\mathcal{M}, E)_\bullet$ denotes the smallest sub-fan containing cones of the form $\sigma(E_\bullet, \underline{w}_\bullet)$, then we will show that the inclusion $F_\bullet \subset \mathbf{DF}(\mathcal{M}, E)_\bullet$ is a bounded inclusion. Because $F_1 = \mathbf{DF}(\mathcal{M}, E)_1$ and because $\mathbf{DF}(\mathcal{M}, E)_\bullet$ is quasi-separated, it follows from Corollary 3.19.1 that $\mathbb{P}(F_\bullet) \to \mathscr{D}eg(\mathcal{M}, E)$ is a homeomorphism.

Consider a presentation of E of the form (35). Note that the objects E_i need not be totally ordered by inclusion. However, if the weights $\underline{w}_i$ are totally ordered with respect to the partial order on $\mathbb{Z}^n$, then we can re-express the object $E = \sum \mathcal{O}_{X_K}[t_1, \ldots, t_n] \cdot t^{-\underline{w}_i} \otimes E'_i$, where $E'_i = \sum_{\underline{w}_j \geq \underline{w}_i} E_j$. If $E'_i = E'_j$ for some $i \neq j$, then we only keep one term in the sum with E_i corresponding to the largest value of $\underline{w}_j$ for which $E_j = E_i$. The result, after reindexing, is an equality $E = \sum \mathcal{O}_{X_K}[t_1, \ldots, t_n] \cdot t^{-\underline{w}_i} \otimes E'_i$ where $E'_1 \supsetneq E'_2 \supsetneq \cdots \supsetneq E'_N \supsetneq 0$ and $\underline{w}_i \geq \underline{w}_{i+1}$ (and not equal) for all i. Let A be the $N \times n$ matrix whose i^{th} row vector is $\underline{w}_i$ for $i = 1, \ldots, N$. Then A classifies a map $\Theta_K^n \to \Theta_K^N$ such that the given $\mathbb{Z}^n$-weighted filtration $f : \Theta_K^n \to \mathcal{M}$ is the composition of this map with the standard map $\sigma(E'_\bullet, \underline{w}_\bullet) : \Theta_K^N \to \mathcal{M}$. Thus the filtration corresponding to E lies in $F_\bullet$.

Now for a general filtration $f : \Theta_K^n \to \mathcal{M}$, consider the subdivision of the positive cone $(\mathbb{R}_{\geq 0})^n$ into rational polyhedral cones, where codimension 1 walls are the hyperplanes $\{r \in \mathbb{R}^n_{\geq 0} | r \cdot (\underline{w}_i - \underline{w}_j) = 0\}$ for some i and j appearing. Let $A : \mathbb{R}^p \to \mathbb{R}^n$ be a linear map with integer coefficients mapping $\mathbb{R}^p_{\geq 0}$ to a single cone in this decomposition of $\mathbb{R}^n_{\geq 0}$. Then by definition $r \cdot A^\tau \underline{w}_i - r \cdot A^\tau \underline{w}_j$ has the same sign (or vanishes) for all $r \in \mathbb{R}^p_{\geq 0}$. In particular the vectors $A^\tau \underline{w}_i$ are totally ordered with respect to the partial order on $\mathbb{Z}^p$ discussed above. It follows from the previous paragraph that the restriction to $f \circ \phi$ factors through a map of the form $\sigma(E'_\bullet, \underline{w}'_\bullet)$. $\square$

Remark 5.16. The canonical filtration $\sigma(E_\bullet, w_\bullet)$ defines a cone $\mathbb{R}^n_{\geq 0} \to |\mathbf{DF}(\mathcal{M}, E)_\bullet|$. It is natural to regard vectors $a \in \mathbb{R}^n_{\geq 0}$ as assigning the real weights $a \cdot w_1 \leq a \cdot w_2 \leq \cdots \leq a \cdot w_n$ to the filtration $E_1 \supset E_2 \supset \cdots \supset E_n$. We therefore interpret Proposition 5.15 as identifying $|\mathbf{DF}(\mathcal{M}, E)_\bullet|$ with the space of finite real weighted filtrations of E, and identifying $\mathscr{D}eg(\mathcal{M}, E)$ with the space of finite real weighted filtrations of E up to positive rescaling of weights. See Section 5.4 for further discussion.

5.1.2. *Θ-reductivity.* For the usual t-structure on $\mathrm{D}^b(X)$ we have seen that $\mathcal{M}$ is Θ-reductive. Indeed this is equivalent to the classical fact that the flag schemes of a flat family of coherent sheaves are proper over the base of the family. The following proposition generalizes this observation, and even holds when $\mathcal{M}$ is not an algebraic stack.

Proposition 5.17. *Let X be a projective k-scheme, and consider a noetherian t-structure on $\mathrm{D}^b(X)$. Then $\mathrm{ev}_1 : \mathrm{Filt}(\mathcal{M}) \to \mathcal{M}$ satisfies the valuative criterion for properness for any discrete valuation ring which is essentially finite type over B. If $\mathcal{M}$ is an algebraic stack with quasi-compact flag spaces, then it is Θ-reductive.*

Lemma 5.18. *Let R be a valuation ring over k, and consider a t-structure on $\mathrm{D}^b(X)$ with noetherian heart. Then an object $N \in \mathrm{D}^b(X_R)$ is R-flat if and only if $\mathrm{Cone}(N \to N \otimes_R K) \in \mathrm{D}_{qc}(X_R)^\heartsuit$.*

Proof. $R \to K$ is a flat resolution of K/R, so $\mathrm{Cone}(N \to N \otimes_R K) \simeq N \otimes_R (K/R)$, and the "only if" direction is essentially the definition of flatness.

Conversely, by part (3) of Lemma 5.5 and the fact that all finitely generated ideals are principal, N is R-flat if and only if $\forall x \in R, N \otimes_R (R/(x)) \in \mathrm{D}_{qc}(X_{R/(x)})^\heartsuit$, or equivalently that $N \xrightarrow{\times x} N$ is injective in $\mathrm{D}_{qc}(X_R)^\heartsuit$. The commutative diagram in $\mathrm{D}_{qc}(X_R)^\heartsuit$

$$\begin{tikzcd} N \arrow[r] \arrow[d, "\times x"] & N \otimes_R K \arrow[d, "\times x"] \\ N \arrow[r] & N \otimes_R K \end{tikzcd}$$

combined with the fact that $\times x$ is an isomorphism on $N \otimes_R K$ implies that $N \xrightarrow{\times x} N$ is injective if $N \to N \otimes_R K$ is. □

Proof of Proposition 5.17. We must show that given a t-flat $N \in \mathrm{D}^b(X_R)^\heartsuit$ (note that every complex in $\mathrm{D}^b(X_R)$ is relatively perfect because R has finite projective dimension) and a descending weighted filtration of $N \otimes_R K$, then there is a unique descending weighted filtration of N inducing the given one on $N \otimes_R K$ and such that all of the subquotients of N are R-flat as well. The subquotients will lie in $\mathrm{D}^b(X_R)$ by Proposition 5.6.

Because any filtration can be built from a sequence of one step filtrations, one can reduce to showing the following: if $M' \subset N \otimes_R K$ is a subobject in $\mathrm{D}_{qc}(X_R)^\heartsuit$, then there is a unique extension to a map of short exact sequences in $\mathrm{D}_{qc}(X_R)^\heartsuit$

$$\begin{tikzcd} 0 \arrow[r] & N' \arrow[r] \arrow[d, dashed] & N \arrow[r] \arrow[d] & N'' \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & M' \arrow[r] & N \otimes_R K \arrow[r] & M'' \arrow[r] & 0 \end{tikzcd}$$

which identifies $M' \simeq N' \otimes_R K$ and $M'' \simeq N'' \otimes_R K$ and such that both dotted arrows are injective in $\mathrm{D}_{qc}(X_R)^\heartsuit$. To construct this diagram, we let N' be the intersection (i.e. fiber product in $\mathrm{D}_{qc}(X_R)^\heartsuit$) of N and M' in $N \otimes_R K$, and we let N'' be the image of the composition $N' \to N \otimes_R K \to M''$. The injectivity of the dotted arrows and the exactness of the top sequence follow immediately from this construction.

What remains is to show that the canonical map $N' \otimes_R K \to M'$ is an isomorphism (and thus so is $N'' \otimes_R K \to M''$). The functor $(-) \otimes_R K$ is t-extact, so the homomorphism $N' \otimes_R K \to N \otimes_R K \simeq M$ is injective. This homomorphism factors through M', so $N' \otimes_R K \to M'$ is injective. For surjectivity, it suffices to show that every map $P \to M'$ from $P \in \mathrm{D}^b(X_R)^\heartsuit$ factors through $N' \otimes_R K$: Indeed this would imply that $\mathrm{Hom}(P, N' \otimes_R K) \to \mathrm{Hom}(P, M')$ is an isomorphism for all $P \in \mathrm{D}^b(X_R)^\heartsuit$ and thus for any $P \in \mathrm{D}_{qc}(X_R)^\heartsuit$, because any object of $\mathrm{D}_{qc}(X_R)^\heartsuit$ is a filtered colimit of objects in $\mathrm{D}^b(X_R)^\heartsuit$.[20]

So consider a map $P \to M'$ with $P \in \mathrm{D}^b(X_R)^\heartsuit$. Because $N \otimes_R K = \mathrm{colim}(N \to N \to N \to \cdots)$ is a filtered colimit of uniformly bounded homological amplitude, we may find a dotted arrow fitting into the diagram

$$\begin{tikzcd} & & & N \arrow[d] \\ P \arrow[r, "\times \pi^m"'] \arrow[rrru, dashed] & P \arrow[r] & M' \arrow[r] & N \otimes_R K \end{tikzcd}$$

By definition of N' as a fiber product the dotted arrow factors through N', which shows that our initial map $P \to M'$ factors through $N' \otimes_R K$ after multiplication by π^m for some m. Because multiplication by π^m is an isomorphism on both $N' \otimes_R K$ and M', it follows that the original map $P \to M'$ factors through $N' \otimes_R K$.

If $\mathcal{M}$ is an algebraic stack with quasi-compact flag spaces, then $\mathrm{ev}_1 : \mathrm{Filt}(\mathcal{M}) \to \mathcal{M}$ is representable locally on $\mathcal{M}$ by a disjoint union of algebraic spaces which are finite type over k. It follows that the valuative criterion for discrete valuation rings which are essentially finite type over k implies that each connected component is proper (See the discussion in the proof of Theorem 2.7). Hence ev_1 satisfies the valuative criterion for properness with respect to arbitrary valuation rings. $\square$

[20]To show this, write P as a filtered colimit of perfect complexes, then use the fact that the truncation functors commute with filtered colimits.

5.2. **Slope semistability as Θ-stability.** We let $\mathcal{N}(X)$ denote the numerical K-group of coherent sheaves on X, which we define as the image of the map $K_0(\mathrm{D}^b(X)) \to \mathrm{Hom}(K^0(\mathrm{Perf}(X)), \mathbb{Z})$ induced by the Euler pairing. $\mathcal{N}(X)$ is the quotient of $K_0(\mathrm{D}^b(X))$ by the subgroup of classes $[F]$ for which $\chi(X, E \otimes F) = 0, \forall E \in \mathrm{Perf}(X)$. Likewise we define $\mathcal{N}^{perf}(X)$ to be the quotient of $K^0(\mathrm{Perf}(X))$ by the subgroup of classes $[E]$ for which $\chi(E \otimes F) = 0, \forall F \in \mathrm{D}^b(X)$. By construction $\mathcal{N}(X)$ and $\mathcal{N}^{perf}(X)$ are torsion-free, and χ descends to a bilinear pairing $\chi : \mathcal{N}^{perf}(X) \otimes \mathcal{N}(X) \to \mathbb{Z}$. When $k = \mathbb{C}$, then Riemann-Roch shows that $\mathcal{N}^{perf}(X)$ and $\mathcal{N}(X)$ are finitely generated, and hence χ is a perfect pairing after tensoring with $\mathbb{Q}$.

Lemma 5.19. *Let T be a connected k scheme of finite type, and let $E \in \mathrm{D}^b(X_T)$ be relatively perfect. For any finite type point $t \in T$, consider the class*

$$v := \frac{1}{\deg(\kappa(t)/k)}[E_{\kappa(t)}] \in \mathcal{N}(X)_{\mathbb{Q}},$$

where we regard $E_{\kappa(t)}$ as a complex on X via pushforward along the map $X_{\kappa(t)} \to X$. Then v is independent of the choice of t, and we may write $\mathcal{D}^b_{pug}(X)$ (respectively $\mathcal{M}$) as a disjoint union of open and closed substacks $\mathcal{D}^b_{pug}(X)_v$ (respectively $\mathcal{M}_v$) parameterizing families of a fixed class $v \in \mathcal{N}(X)$.

Proof. If $p : X_T \to T$ denotes the projection, then the semicontinuity theorem implies that $p_*((\mathcal{O}_T \boxtimes F) \otimes E) \in \mathrm{Perf}(T)$ for any $F \in \mathrm{Perf}(X)$. Therefore the Euler characteristic

$$\chi(X_{\kappa(t)}, E_{\kappa(t)} \otimes (\kappa(t) \otimes_k F)) = \deg(\kappa(t)/k)\chi(X, E_{\kappa(t)} \otimes_k F)$$

does not depend on the finite type point $t \in T$, and the claim follows. This splits $\mathcal{D}^b_{pug}(X)$ into open and closed substacks because $\mathcal{D}^b_{pug}(X)$ is algebraic and locally of finite type over k, and it splits $\mathcal{M}$ into open and closed substacks (even when $\mathcal{M}$ is not algebraic), because $\mathcal{M}$ is a substack of $\mathcal{D}^b_{pug}(X)$. □

Definition 5.20. A *slope-stability condition* on X consists of the following data: a t-structure on $\mathrm{D}^b(X)$, an abelian subcategory $\mathcal{A} \subset \mathrm{D}^b(X)^\heartsuit$ which is numerical in the sense that there is a finitely generated subgroup $\Lambda \subset \mathcal{N}(X)$ such that $\mathcal{A} = \{E \in \mathrm{D}^b(X)^\heartsuit | [E] \in \Lambda\}$, and a group homomorphism $Z : \Lambda \to \mathbb{C}$, known as the central charge, such that $Z(\mathcal{A}) \subset \mathbb{H} \cup \mathbb{R}_{\leq 0}$.

Note that for any short exact sequence in $\mathrm{D}^b(X)^\heartsuit$, if any two of the objects lie in the category $\mathcal{A}$ then so does the third. The previous lemma allows us to regard $\mathcal{M}_v$ for $v \in \Lambda$ as the moduli of objects in $\mathcal{A}$, and in particular filtrations in $\mathcal{M}_v$ correspond under Lemma 5.12 to weighted filtrations in the abelian category $\mathrm{D}^b(X)^\heartsuit$ whose associated graded object lies in $\mathcal{A}$.

Example 5.21. The simplest example is where X is a smooth curve, and we let $\mathcal{A}$ be the category of coherent sheaves on X. The central charge is $Z(E) = -\deg(E) + i\,\mathrm{rank}(E)$.

We can generalize this fundamental example in two directions.

Example 5.22. We can let $(Z, \mathcal{A})$ be an arbitrary numerical Bridgeland stability condition on a smooth projective variety X. In addition to the Harder-Narasimhan property, which we discuss below, a Bridgeland stability condition must have $Z(E) \neq 0$ for nonzero $E \in \mathcal{A}$. We will not need this nondegeneracy hypothesis until Section 5.3.

Example 5.23. We can fix a projective variety X with ample invertible sheaf L, and choose an integer $0 < d \leq \dim X$. We consider the usual t-structure on $\mathrm{D}^b(X)$ and let $\Lambda \subset \mathcal{N}(X)$ be the subgroup of classes whose Hilbert polynomial has degree $\leq d$. Then $\mathcal{A} = \mathrm{Coh}(X)_{\leq d}$ is the category of coherent sheaves whose support has dimension $\leq d$. This is a full abelian subcategory of $\mathrm{Coh}(X)$. The assignment

$$E \mapsto p_E(n) := \chi(E \otimes L^n) = \sum \frac{p_{E,k}}{k!} n^k$$

defines a group homomorphism $K_0(X) \to \mathbb{Z}[n]$. We define our central charge to be $Z(E) := i p_{E,d} - p_{E,d-1}$. This has the property that $Z(E) \subset \mathbb{H} \cup \mathbb{R}_{\leq 0}$ for any $E \in \mathcal{A}$: if $\dim(\mathrm{supp}(E)) < d-1$ then $Z(E) = 0$, if $\dim(\mathrm{supp}\, E) = d-1$ then $p_{E,d} = 0$ and $p_{E,d-1} > 0$, and if $\dim(\mathrm{supp}\, E) = d$ then $p_{E,d} > 0$.

For a nonzero $E \in \mathcal{A}$, the phase $\phi(E) \in (0,1]$ is the unique number such that $Z(E) = |Z(E)| e^{2\pi i \phi(E)}$, and by convention $\phi = 1$ if $Z(E) = 0$. The following is a common notion of semi-stability in abelian categories:

Definition 5.24. An object $E \in \mathcal{A}$ is *slope semistable* if there are no subobjects $F \subset E$ in the abelian category $\mathcal{A}$ such that $\phi(F) > \phi(E)$. We say that $(Z, \mathcal{A})$ has the *Harder-Narasimhan* property if every object E admits a filtration by subobjects $E = E_1 \supset \cdots \supset E_p \supset E_{p+1} = 0$ in $\mathcal{A}$ with each subquotient $\mathrm{gr}_j E_\bullet$ semistable and $\phi(\mathrm{gr}_j E_\bullet)$ increasing in j.

Our goal in this subsection is to construct a numerical invariant on $\mathcal{M}_v$ such that Θ-stability and the HN problem correspond to the notions of Definition 5.24.

It follows from the fact that Λ is finitely generated that the Euler characteristic induces a perfect pairing $\Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} (\mathcal{N}^{perf}(X)/\Lambda^\perp) \otimes \mathbb{Q} \to \mathbb{Q}$, where $\Lambda^\perp := \{v \in \mathcal{N}(X)^{perf} | \chi(v \cdot \lambda) = 0, \forall \lambda \in \Lambda\}$. Therefore we can find a unique element $\omega_Z \in (\mathcal{N}^{perf}(X)/\Lambda^\perp) \otimes \mathbb{C}$ such that

$$Z(x) = \chi(\omega_Z \otimes x), \forall x \in \Lambda \otimes \mathbb{C}$$

We will regard ω_Z as a class $\omega_Z \in K^0(\mathrm{Perf}(X)) \otimes \mathbb{C}$ by choosing a lift under the surjective map $K_0(\mathrm{Perf}(X)) \otimes \mathbb{C} \twoheadrightarrow (\mathcal{N}^{perf}(X)/\Lambda^\perp) \otimes \mathbb{C}$. Let $\mathcal{E}$ denote the universal object in the derived category of $\mathcal{M} \times X$, and consider the diagram

$$\mathcal{M}_v \xleftarrow{p_1} \mathcal{M}_v \times X \xrightarrow{p_2} X$$

We define the cohomology classes

$$\begin{aligned} l &:= |Z(v)|^2 \operatorname{ch}_1\left((p_1)_*\left(\mathcal{E} \otimes p_2^* \Im\left(\tfrac{-\omega_Z}{Z(v)}\right)\right)\right) \\ b &:= 2 \operatorname{ch}_2\left((p_1)_*\left(\mathcal{E} \otimes p_2^* \Im(\omega_Z)\right)\right) \end{aligned} \tag{36}$$

Where $\Im : \mathcal{N}(X) \otimes \mathbb{C} \to \mathcal{N}(X) \otimes \mathbb{R}$ denotes the imaginary part. Note that the cohomology classes (36) are real, but if $Z(\mathcal{A}) \subset \mathbb{Q} + i\mathbb{Q}$, then we may assume $\omega_Z \in K^0(\mathrm{Perf}(X)) \otimes (\mathbb{Q} + i\mathbb{Q})$, and the cohomology classes l and b are rational.

Notation 5.25. We define the *rank-degree-weight* sequence associated to a $\mathbb{Z}$-weighted filtration $\cdots \supset E_w \supset E_{w+1} \supset \cdots$ as the order sequence of triples

$$\alpha = \left\{(r_j, d_j, w_j) = (\Im Z(\operatorname{gr}_j E_\bullet), -\Re Z(\operatorname{gr}_j E_\bullet), w_j)\right\}_{j=1,\ldots,p} \tag{37}$$

where $w_j \in \mathbb{Z}$ are the finite set of integers w for which $\operatorname{gr}_w(E_\bullet) \neq 0$. Because $Z(E)$ only depends on the numerical class of E, α is locally constant on $\mathrm{Filt}(\mathcal{M}_v)$. We sometimes use the notation $E_j := E_{w_j}$, and regard the data of a $\mathbb{Z}$-weighted filtration as a finite filtration $E = E_0 \supsetneq E_1 \supsetneq \cdots \supsetneq E_p \supsetneq E_{p+1} = 0$ along with the strictly increasing weights $w_0 < w_1 < \cdots < w_p$ in $\mathbb{Z}$. It is natural to allow the weights w_i to lie in $\mathbb{Q}$ or even $\mathbb{R}$, and we refer to such data as a *descending weighted filtration*.

Lemma 5.26. *Let k'/k be a field extension and let $f : \Theta_{k'} \to \mathcal{M}_v$ correspond to a descending weighted filtration $E_\bullet$. Let $\{(r_j, d_j, w_j)\}$ be the rank-degree-weight sequence associated to $E_\bullet$ as in (37) and let $Z(v) = -D + iR$. Then we have*

$$\tfrac{1}{q} f^* l = \sum_{j=1}^{p} w_j (R d_j - D r_j), \quad \textit{and} \quad \tfrac{1}{q^2} f^* b = \sum_{j=1}^{p} w_j^2 r_j \tag{38}$$

Proof. If $\pi : \Theta_{k'} \times X \to \Theta$ and $\mathcal{E}_f$ is the object classified by f, then in K-theory, we have $\mathcal{E}_f \simeq \sum_j u^{-w_j} [\operatorname{gr}_j E_\bullet]$ under the decomposition $K_0(X \times \Theta_{k'}) \simeq K_0(X_{k'}) \otimes \mathbb{Z}[u^{\pm}]$. We can thus compute

$$\begin{aligned} f^* l &= |Z(v)|^2 \operatorname{ch}_1\left(\pi_*(\mathcal{E}_f \otimes p_2^* \Im(\frac{-\omega_Z}{Z(v)}))\right) \\ &= |Z(v)|^2 \operatorname{ch}_1\left(\sum u^{-w_j} \Im\left(\frac{-\chi(X, \omega_Z \otimes \operatorname{gr}_j E_\bullet)}{Z(v)}\right)\right) \\ &= \sum w_j q \Im\left(\chi(X, \omega_Z \otimes \operatorname{gr}_j E_\bullet) \cdot \overline{Z(v)}\right) \end{aligned}$$

By the defining property of ω_Z, we have $\chi(\omega_Z \otimes \operatorname{gr}_j E_\bullet) = (-d_j + i r_j)$, and the claim follows. The computation for $f^* b$ is almost identical, so we omit it. $\square$

We will need the following notion for a slope-stability condition $(Z, \mathcal{A})$:

Definition 5.27. We refer to $E \in \mathcal{A}$ as *torsion* if $Z(E) \in \mathbb{R}_{\leq 0}$, and we call an object *torsion-free* if it has no torsion subobjects (in particular a torsion-free object has $\Im Z(E) > 0$). We define $\mathcal{T} \subset \mathcal{A}$ (resp. $\mathcal{F} \subset \mathcal{A}$) to be the full subcategory consisting of torsion objects (resp. torsion-free objects). Note that torsion objects are automatically semistable.

Example 5.28. Continuing Example 5.23, let $\mathcal{A}$ is the category of sheaves whose support has dimension $\leq d$. The torsion subcategory $\mathcal{T}$ consists of sheaves whose support has dimension $\leq d-1$, and the $\mathcal{F}$ consists of sheaves which are pure of dimension d.

We shall consider the numerical invariant $\mu : \mathcal{U} \subset \mathscr{C}omp(\mathcal{M}_v) \to \mathbb{R}$ associated to the cohomology classes (36). Observe from (38) that for any filtration $f : \Theta_{k'} \to \mathcal{M}_v$, $f^*(b) = 0$ if and only if $r_i = 0$ for any i for which $w_i \neq 0$. It follows that b is not positive definite on $\mathcal{M}_v$, and μ is not stictly quasi-concave, if $\mathcal{A}$ has torsion objects. The rational points of the subset $\mathcal{U}_E \subset \mathscr{D}eg(\mathcal{M}_v, E)$ on which μ is defined corresponds to the set of all $\mathbb{Z}$-weighted filtrations which are *not* of the form

$$0 = 0 = \cdots = E_1 \subset E_0 \subset E_{-1} \subset E_{-2} \subset \cdots \subset E$$

with $\mathrm{gr}_i(E_\bullet)$ torsion for all $i < 0$. The main result of this subsection states that the HN problem for this numerical invariant on $\mathcal{M}_v$ is equivalent to the Harder-Narasimhan property for $(Z, \mathcal{A})$.

Theorem 5.29. *Let $(Z, \mathcal{A})$ be a slope-semistability condition on a projective k-scheme X. Let μ be the numerical invariant on $\mathcal{M}_v$ associated to the cohomology classes (36) as in Definition 4.9, and let M^μ the corresponding stability function of Definition 4.1. Then $E \in \mathcal{F}$ is slope semistable if and only if $M^\mu([E]) \leq 0$, and the following are equivalent:*

(1) The pair $(Z, \mathcal{A})$ has the Harder-Narasimhan property; and
(2) Every object in $\mathcal{A}$ has a maximal torsion subobject, and for every unstable $E \in \mathcal{F}$, the function $\mu : \mathcal{U}_E \subset \mathscr{D}eg(\mathcal{M}_v, E) \to \mathbb{R}$ obtains a maximum.

If $\mathcal{A}$ is Noetherian, this is equivalent to the condition:

(3) For any $E \in \mathcal{A}$, the set $\{\phi(F) | F \subset E\}$ has a maximal element.

Furthermore, if (2) holds then the maximum of μ occurs at the point in $\mathscr{D}eg(\mathcal{M}_v, E)$ corresponding to the Harder-Narsimhan filtration $E = E_1 \supset E_2 \supset \cdots \supset E_n$ with weights proportional to the slopes (defined below) of the associated graded pieces E_i/E_{i+1}, and this is the unique maximizer corresponding to a filtration whose associated graded pieces are torsion-free. If Z is such that $Z(E) = 0 \Rightarrow E = 0$ for all $E \in \mathcal{A}$, then this is the unique maximizer of μ.

We shall prove Theorem 5.29 at the end of this subsection, after collecting several intermediate results. The proof bears a strong formal resemblance to the analysis of semistability for filtered isocrystals in p-adic Hodge theory as presented in [DOR].

As in the context of Bridgeland stability conditions, we let $\mathcal{P}(\phi) \subset \mathcal{A}$ be the full subcategory of semistable objects of phase ϕ. In addition, for any subset $I \subset (0,1]$, we let $\mathcal{P}(I)$ be the full subcategory generated under extensions by semistable objects whose phase lies in I.

Lemma 5.30. *For any* $\epsilon \in (0,1]$, $\mathrm{Hom}(E,F) = 0$ *for all* $E \in \mathcal{P}([\epsilon,1])$ *and* $F \in \mathcal{P}((0,\epsilon))$.

Proof. It suffices to show that $\mathrm{Hom}(E_1,E_2) = 0$ for semistable objects with $\phi(E_1) > \phi(E_2)$. Letting $f : E_1 \to E_2$, if $\mathrm{im}(f)$ is nonzero, then the semistability of E_1 and E_2 implies that $\phi(E_1) \leq \phi(\mathrm{im}(f)) \leq \phi(E_2)$. Hence if this inequality is violated we must have $f = 0$. $\square$

If $(Z,\mathcal{A})$ has the Harder-Narasimhan property, we let $E^{\phi \geq \epsilon} \subset E$ denote the largest sub-object in the Harder-Narasimhan filtration whose associated graded pieces have phase $\geq \epsilon$.

Corollary 5.30.1. *If* $(Z,\mathcal{A})$ *has the Harder-Narasimhan property, then the Harder-Narsimhan filtration is functorial in the sense that for any* $\epsilon \in (0,1]$, *any homomorphism* $E \to F$ *maps* $E^{\phi\geq\epsilon}$ *to* $F^{\phi\geq\epsilon}$.

Proof. The map $E^{\phi\geq\epsilon} \to F^{\phi<\epsilon} := F/F^{\phi\geq\epsilon}$ must vanish by the previous lemma. $\square$

For any ordered sequence of complex numbers $(z_1,\ldots,z_p)$ with $z_j \in \mathbb{H} \cup \mathbb{R}_{\leq 0}$, we associate the convex polyhedron

$$\mathrm{Pol}(\{z_j\}) = \left\{ \sum \lambda_j z_j + c \,\middle|\, 0 \leq \lambda_1 \leq \cdots \leq \lambda_p \leq 1 \text{ and } c \in \mathbb{R}_{\geq 0} \right\} \subset \mathbb{H} \cup \mathbb{R}_{\geq 0}$$

This is the convex hull of the points $z_p, z_p + z_{p-1}, \ldots, z_p + \cdots + z_1$ plus an arbitrary shift by $c \in \mathbb{R}_{\geq 0}$. For any filtration $E_\bullet$ in $\mathcal{A}$, we let $\mathrm{Pol}(E_\bullet) = \mathrm{Pol}(\{Z(\mathrm{gr}_j E_\bullet)\})$, and for any object $E \in \mathcal{A}$ we let $\mathrm{Pol}^{\mathrm{HN}}(E) := \mathrm{Pol}(E_\bullet)$, where $E_\bullet$ is the HN filtration of E.

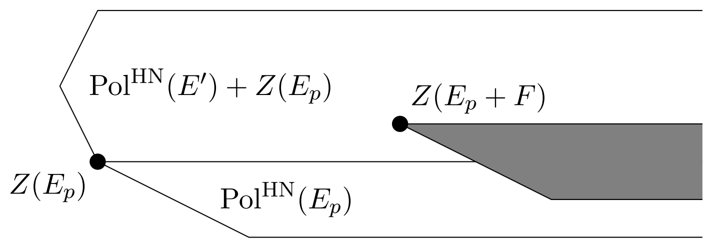
Lemma 5.31. *If* $(Z,\mathcal{A})$ *has the Harder-Narasimhan property, then* $\mathrm{Pol}(E_\bullet) \subset \mathrm{Pol}^{\mathrm{HN}}(E)$ *for any weighted descending filtration* $E_\bullet$ *of* E.

Proof. It suffices to show that for any E and any subobject $F \subset E$, one has $Z(F) \in \mathrm{Pol}^{\mathrm{HN}}(E)$, and we prove this by induction on the length of the HN filtration of E. The base case is when E is semistable, in which case the statement of the lemma is precisely the definition of semistability, combined with the fact that $\Im Z(F) \leq \Im Z(E)$ for any subobject.

Let $E_p \subset E$ be the first subobject in the HN filtration, so by hypothesis E_p is semistable, say with phase ϕ_p. Corollary 5.30.1 guarantees that any subobject of E must have phase $\leq \phi_p$. Let $E' = E/E_p$ and let F' be the image of $F \to E'$. Then can pullback the defining sequence for E' to F':

$$\begin{tikzcd}
0 \arrow[r] & E_p \arrow[r] \arrow[d, equal] & E_p + F \arrow[r] \arrow[d, hook] & F' \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & E_p \arrow[r] & E \arrow[r] & E' \arrow[r] & 0
\end{tikzcd}$$

The length of the HN filtration of E' is one less than E, and $\mathrm{Pol}^{\mathrm{HN}}(E)$ decomposes as a union of $\mathrm{Pol}^{\mathrm{HN}}(E_p)$ and a shift of $\mathrm{Pol}^{\mathrm{HN}}(E')$ by $Z(E_p)$. Thus by the inductive hypothesis $Z(E_p + F) = Z(F') + Z(E_p)$ lies in $Z(E_p) + \mathrm{Pol}^{\mathrm{HN}}(E')$, which we illustrate graphically:



We also have the short exact sequence $0 \to E_p \cap F \to E_p \oplus F \to E_p + F \to 0$, which implies that

$$\begin{aligned} Z(F) &= Z(E_p + F) + Z(E_p \cap F) - Z(E_p) \\ &= Z(E_p + F) - Z(E_p / E_p \cap F) \end{aligned}$$

We have $\Im Z(E_p/E_p \cap F) \leq \Im Z(E_p)$, and the phase of $Z(E_p/E_p \cap F)$ is $\geq \phi_p$ because E_p is semistable. It follows that $Z(F)$ must lie in the shaded region above, which is a translate of $\mathrm{Pol}^{\mathrm{HN}}(E_p)$, and hence $Z(F) \in \mathrm{Pol}^{\mathrm{HN}}(E)$. Even though we have drawn the diagram assuming that $\Im Z(E_p) > 0$, the argument works without modification in the case where E_p is torsion as well. □

Let $\alpha = \{(r_j, d_j, w_j)\}$ be a rank-degree-weight sequence. For $j = 1, \ldots, p$, we define the j^{th} phase $\phi_j \in (0, 1]$ by the property that $ir_j - d_j \in \mathbb{R}_{>0} e^{i\phi_j}$. Note that by Lemma 5.26, the numerical invariant can be expressed formally in terms of the rank-degree-weight sequence $\mu = \mu(\alpha)$. Furthermore it extends continuously to rank-degree-weight sequences where the w_j are arbitrary real numbers rather than integers.

Lemma 5.32 (Insertion and deletion)**.** *Let α be a rank-degree-weight sequence such that $r_k + r_{k+1} > 0$ for some k, and let α' be the rank-degree-weight sequence obtained from α by discarding the $(k+1)^{st}$ element and relabelling $r'_k = r_k + r_{k+1}$ and $d'_k = d_k + d_{k+1}$, and*

$$w'_j := \begin{cases} w_j, & \text{if } j < k \\ \frac{w_j r_j + w_{j+1} r_{j+1}}{r_j + r_{j+1}}, & \text{if } j = k \\ w_{j+1}, & \text{if } j > k \end{cases}.$$

Then we have:

- *If $\mu(\alpha) \geq 0$ and $\phi_k \geq \phi_{k+1}$, then $\mu(\alpha') \geq \mu(\alpha)$ with equality if and only if $\phi_k = \phi_{k+1}$ and $\mu(\alpha) = 0$.*
- *If $\mu(\alpha') \geq 0$ and $\phi_k < \phi_{k+1}$, then $\mu(\alpha) > \mu(\alpha')$.*

Proof. We denote $\mu = L/\sqrt{B}$. Substituting α' for α, the numerator and denominator change by

$$\begin{aligned}\Delta L &= w'_k(d_k + d_{k+1}) - w_k d_k - w_{k+1} d_{k+1} \\ &= \frac{w_{k+1} - w_k}{r_k + r_{k+1}}(d_k r_{k+1} - r_k d_{k+1}) \\ \Delta B &= (w'_k)^2(r_{k+1} + r_k) - w_k^2 r_k - w_{k+1}^2 r_{k+1} \\ &= -\frac{r_k r_{k+1}}{r_k + r_{k+1}}(w_k - w_{k+1})^2\end{aligned}$$

Note that $\Delta B \leq 0$. Also the sign of $d_k r_{k+1} - d_{k+1} r_k$, and hence the sign of ΔL, is the same as the sign of $\phi_k - \phi_{k+1}$, and $\Delta L \neq 0$ if $\phi_k \neq \phi_{k+1}$. The claim follows from these observations. □

In terms of descending weighted filtrations, the modification $\alpha \mapsto \alpha'$ in Lemma 5.32 corresponds to deleting the $(k+1)^{st}$ subobject from the filtration $E = E_1 \supset \cdots \supset E_p \supset E_{p+1} = 0$ and adjusting the weights appropriately. We also observe that if $\phi_k \geq \phi_{k+1}$ then this does not change $\operatorname{Pol}(E_\bullet)$. Hence we have the following:

Corollary 5.32.1. *For any descending weighted filtration $(E_\bullet, w_\bullet)$, there is a sequence of deletions resulting in a descending weighted filtration $(E'_\bullet, w'_\bullet)$ which is convex, in the sense that $\phi'_1 < \cdots < \phi'_{p'}$, such that $\operatorname{Pol}(E_\bullet) = \operatorname{Pol}(E'_\bullet)$ and $\mu(E'_\bullet) \geq \mu(E_\bullet)$, with strict inequality if $E_\bullet$ was not convex to begin with.*

Given a sequence $(z_1, \ldots, z_p)$ in $\mathbb{H} \cup \mathbb{R}_{\leq 0}$, we let $R = \sum r_j$ and define a continuous piecewise linear function $h_{\{z_j\}} : [0, R] \to \mathbb{R}$,

$$h_{\{z_j\}}(r) := \sup\{x \,|ir - x \in \operatorname{Pol}(\{z_j\})\}.$$

Note that this only depends on $\operatorname{Pol}(\{z_j\})$.

Lemma 5.33. *Let $ir_j - d_j \in \mathbb{H}$ be sequence of points such that $\phi_1 < \cdots < \phi_p < 1$. Let $\nu_j := d_j/r_j$, $D := \sum d_j$, $R := \sum r_j$, and $\nu := D/R$, and let $h(x) := h_{\{ir_j - d_j\}}(x)$. Then μ is maximized by assigning the weights $w_j \propto \nu_j - \nu$. The maximum is*

$$\frac{\mu}{R} = \sqrt{\left(\sum \nu_j^2 r_j\right) - \nu^2 R} = \sqrt{\int_0^R (h'(x))^2 dx - \nu^2 R}. \tag{39}$$

Proof. We can think of the numbers $\mathrm{ev}_1, \cdots, r_p$ as defining an inner product $\vec{a} \cdot \vec{b} = \sum a_j b_j r_j$. Then given a choice of weights $\vec{w} = (w_1, \cdots, w_p)$, the numerical invariant can be expressed as

$$\mu = \frac{R}{|\vec{w}|}\vec{w} \cdot (\vec{\nu} - \nu\vec{1})$$

where $\vec{\nu} = (\nu_1, \cdots, \nu_p)$ and $\vec{1} = (1, \ldots, 1)$. From linear algebra we know that this quantity is maximized when $\vec{w} \propto \vec{\nu} - \nu\vec{1}$, and the maximum value is $R|\vec{\nu} - \nu\vec{1}|$. In the case when $\nu_1 < \cdots < \nu_p$ the assignment $\vec{w} \propto \vec{\nu} - \nu\vec{1}$

satisfies the constraints $w_1 < w_2 < \cdots < w_p$. The integral formula simply expresses $\nu_j^2 r_j$ as the integral of $h'(x)$ on an interval of length r_j along which it is constant of value ν_j. □

Lemma 5.34. *The expression* (39) *is strictly monotone increasing with respect to inclusion of polyhedra.*

Proof. Let $h_1, h_2 : [0, R] \to \mathbb{R}$ be two continuous piecewise linear functions with $h_i'(x)$ decreasing and with $h_1(x) \leq h_2(x)$ with equality at the endpoints of the interval. We must show that $\int_0^R (h_1'(x))^2 < \int_0^R (h_2'(x))^2$. First by suitable approximation with respect to a Sobolev norm it suffices to prove this when h_i are smooth functions with $h'' < 0$. Then we can use integration by parts

$$\int_0^R (h_2')^2 - (h_1')^2 dx = (h_2' + h_1')(h_2 - h_1)|_0^R - \int_0^R (h_2 - h_1)(h_2'' + h_1'')dx$$

The first term vanishes because $h_1 = h_2$ at the endpoints, and the second term is strictly positive unless $h_1 = h_2$. □

Proof of Theorem Theorem 5.29. First we show that $E \in \mathcal{F}$ is unstable if and only if $M^\mu([E]) > 0$. If $F \subset E$ with $\phi(F) > \phi(E)$, then consider the two step filtration $\operatorname{gr}_2 E_\bullet = F$ and $\operatorname{gr}_1 E_\bullet = E/F$ and $w_2 > w_1$ arbitrary. Because E is torsion-free we know $Z(F) \neq 0$, so by Lemma 5.26

$$\begin{aligned}\frac{1}{q} f^* l &= w_1 \Im\left((Z(E) - Z(F))\overline{Z(E)}\right) + w_2 \Im\left(Z(F)\overline{Z(E)}\right) \\ &= (w_2 - w_1) \Im\left(Z(F)\overline{Z(E)}\right) > 0,\end{aligned}$$

and hence $M^\mu(E) > 0$. Conversely, if $f : \Theta \to \mathcal{M}_v$ corresponds to a weighted filtration $E_\bullet$ of E such that $\mu(f) > 0$, then Corollary 5.32.1 provides a convex filtration $E'_\bullet$ with $\mu(E'_\bullet) \geq \mu(E_\bullet) > 0$. In particular $E'_\bullet$ is a non-trivial filtration, and so the first subobject $E'_{p'} \subset E$ destabilizes E.

Proof that (1) $\Rightarrow$ (2)*:*

If $(Z, \mathcal{A})$ has the Harder-Narasimhan property, then $E^{\phi \geq 1} \subset E$ is the maximal torsion subsheaf of any $E \in \mathcal{A}$. Now consider $E \in \mathcal{F}$. By Corollary 5.32.1, it thus suffices to maximize μ over the set of convex filtrations, and because E is torsion-free any convex filtration will have $\phi < 1$ for all graded pieces. For a fixed convex filtration, Lemma 5.33 computes the weights which maximize μ for a fixed filtration, and this expression is strictly monotone increasing with respect to inclusion of polyhedra by Lemma 5.34. By Lemma 5.31 we have $\operatorname{Pol}(E_\bullet) \subset \operatorname{Pol}^{\mathrm{HN}}(E)$ for any filtration, so if $\mu(E_\bullet) > 0$ for some weighted descending filtration of E, then $\mu(E_\bullet)$ is strictly less than the value of μ obtained by assigning weights to the HN filtration as in Lemma 5.33.

Proof that (2) $\Rightarrow$ (1)*:*

First if any E admits a maximal torsion subobject $T \subset E$, then T is semistable, and it suffices to show that the torsion-free quotient E/T admits an HN filtration with phases < 1. Assume $E \in \mathcal{F}$ is unstable and choose a point in $\mathscr{D}eg(\mathcal{M}_v, E)$ which maximizes μ, corresponding to a descending filtration $E_1 \supset \cdots \supset E_n$ with real weights $w_1 \leq \cdots \leq w_n$ (see Remark 5.16). Corollary 5.32.1 implies that $\phi_1 < \cdots < \phi_p < 1$, where $\phi_j := \phi(E_j/E_{j+1})$.

Consider a subobject $F \subset E_j/E_{j+1}$. We refine the filtration $E_\bullet$ to a new filtration

$$E'_\bullet = (E_n \subset \cdots \subset E_{j+1} \subset \tilde{F} \subset E_j \subset \cdots \subset E_1),$$

where $\tilde{F}$ is the preimage of F under the map $E_j \to E_j/E_{j+1}$. One automatically has $\mathrm{Pol}(E_\bullet) \subset \mathrm{Pol}(E'_\bullet)$, and the inclusion is strict if $Z(F) \neq 0$ and $\phi(F) > \phi(E_j/E_{j+1})$. Because $\mu(E_\bullet)$ is maximal, Corollary 5.32.1 and Lemma 5.34 imply that $\mathrm{Pol}(E_\bullet) = \mathrm{Pol}(E'_\bullet)$, so it follows that either $Z(F) = 0$ or $\phi(F) \leq \phi_j$.

This shows that each object E_j/E_{j+1} is either torsion-free, in which case it is semistable, or the maximal torsion subobject $F \subset E_j/E_{j+1}$ has $Z(F) = 0$. By a simple inductive procedure, re-defining E_{j+1} to be the preimage of the maximal torsion subobject in E_j/E_{j+1}, we can construct a new filtration of the same length $E'_1 \supset \cdots \supset E'_n$ with $Z(E'_j) = Z(E_j)$ for all j and E'_j/E'_{j+1} torsion-free. It follows that $E'_\bullet$ maximizes μ, so the analysis of the previous paragraph implies that E'_j/E'_{j+1} is semistable of increasing phase, hence $E'_\bullet$ is a Harder-Narasimhan filtration for E. Note that this analysis shows that *any* maximizer of μ for which E_j/E_{j+1} is torsion-free is actually the Harder-Narasimhan filtration with weights given by Lemma 5.33.

Proof that (3) $\Leftrightarrow$ (1) *when* $\mathcal{A}$ *is noetherian:*

The proof is well-known in closely related contexts, see for instance [B2, Proposition 2.4], but for completeness we summarize it here: For any E, let ϕ be the maximal phase of a subobject. Any subobject $F \subset E$ of phase ϕ must be semistable, and for any two subobjects $F, F' \subset E$ of phase ϕ the image $F + F' = \mathrm{im}(F \oplus F' \to E)$ has phase ϕ as well, or else the kernel would be a subobject of $F \oplus F'$ with phase $> \phi$. Because $\mathcal{A}$ is noetherian it follows that there is a maximal subobject of phase ϕ. Using this fact, one can build an ascending sequence $0 \subsetneq E_0 \subsetneq E_1 \subset \cdots \subset E$ such that $\phi(\mathrm{gr}_i(E_\bullet))$ is increasing in i, and E_i is the preimage of the maximal subobject of minimal phase in E/E_{i-1}. Again because $\mathcal{A}$ is noetherian this sequence must stabilize, and it is a HN filtration by construction.

□

5.3. **Θ-stratification of the moduli of torsion-free objects.** Given a slope-stability condition on a projective k-scheme X, we define a moduli functor $\mathcal{M}_v^{\mathcal{F}}$ parameterizing families of torsion-free objects of class $v \in \mathcal{N}(X)$. We then show that the numerical invariant of the previous section defines a Θ-stratification of $\mathcal{M}_v^{\mathcal{F}}$ under suitable hypotheses.

5.3.1. *The moduli stack of torsion-free objects.* For any finite extension k'/k, we can define a slope-stability condition $(\mathcal{A}_{k'}, Z_{k'})$ on $\mathrm{D}^b(X_{k'})$ where $\mathcal{A}_{k'}$ is the heart of the induced t-structure, hence noetherian by Proposition 5.6, and $Z_{k'}(E) = Z(p_*(E))$, where $p_* : \mathrm{D}^b(X_{k'}) \to \mathrm{D}^b(X)$ is the pushforward. For the corresponding torsion theory $(\mathcal{T}_{k'}, \mathcal{F}_{k'})$ on $\mathcal{A}_{k'}$, $E \in \mathcal{T}_{k'}$ if and only if $p_*(E) \in \mathcal{T}$ by definition. Using the fact that for any object $E \in \mathcal{A}_{k'}$ the canonical map $k' \otimes_k p_*(E) \to E$ is surjective one can show that $E \in \mathcal{F}_{k'}$ if and only if $p_*(E) \in \mathcal{F}$ as well.

We define our moduli functor for families of torsion-free objects using the "cheap" trick of defining families only over finitely generated k-algebras, and then using algebraicity to implicitly extend this to all k-algebras.

Definition 5.35. Let $(Z, \mathcal{A})$ be a slope-stability condition for which $\mathcal{A}$ is noetherian and which satisfies the generic flatness condition. For any finite type k-algebra R, define the groupoid of *flat families of torsion-free objects* of class $v \in \mathcal{N}(X)$ in $\mathcal{A}$ as

$$\mathcal{M}_v^{\mathcal{F}}(R) := \left\{ E \in \mathcal{M}_v(R) | E_p \in \mathcal{F}_{\kappa(p)} \subset \mathcal{A}_{\kappa(p)}, \forall \text{ closed points } p \in \operatorname{Spec}(R) \right\}.$$

This condition is local for the smooth topology on affine schemes over $\mathcal{M}_v$, and because $\mathcal{M}_v$ is algebraic and locally of finite presentation over k it therefore defines a substack of $\mathcal{M}_v$.

We shall introduce a condition under which the stack $\mathcal{M}_v^{\mathcal{F}}$ is algebraic. Recall that a subset $\Sigma \subset |\mathfrak{X}|$, where $\mathfrak{X}$ is an algebraic stack locally of finite type over k, is said to be bounded if there is a finite type k-scheme T and a morphism $T \to \mathfrak{X}$ such that Σ lies in the image of $|T| \to |\mathfrak{X}|$. This is equivalent to requiring Σ to lie in some quasi-compact open substack of $\mathfrak{X}$.

Definition 5.36 (Boundedness of quotients)**.** We say that $(Z, \mathcal{A})$ satisfies the *Boundedness of quotients* condition if for any $E \in \mathcal{A}$ and any $\phi \in (0, 1)$, the set of points of $\mathcal{M}$ which parameterize a torsion-free object $E' \in \mathcal{F}_{k'}$ over a finite extension k'/k of phase $\leq \phi$ which admits a surjection $E \otimes_k k' \to E'$ is bounded.

This condition is satisfied in Example 5.23, as is shown for instance in [HL4, Lemma 1.7.9].

Lemma 5.37. *If $(Z, \mathcal{A})$ satisfies the Boundedness of quotients property, then for any finite type scheme T, $E \in \mathcal{M}(T)$, and $\phi \in (0, 1)$, the set of finite type points of $\mathcal{M}$ parameterizing objects $E' \in \mathcal{F}_{\kappa(t)}$ of phase $\leq \phi$ which arise as quotients $E_t \twoheadrightarrow E'$ for some finite type point $t \in T$ is bounded.*

Proof. As in the proof of [T4, Proposition 3.17], one may use the fact that for an affine test scheme T, every $\mathcal{E} \in \mathcal{M}(T)$ admits a surjection of the form $\mathcal{O}_T \otimes_k E \twoheadrightarrow \mathcal{E}$ to reduce the boundedness claim for $\mathcal{E}$ to showing the boundedness of the set of quotients of the single object $E \in \mathcal{A}$ of phase $< \phi$. □

We will not use this below, but we note the following immediate corollary of the previous lemma and Proposition 5.17:

Corollary 5.37.1. *If $(Z, \mathcal{A})$ is a slope-stability condition for which $\mathcal{A}$ is noetherian and which satisfies the generic flatness and Boundedness of quotients conditions, then $\mathcal{M}$ has quasi-compact flag spaces and is Θ-reductive. More precisely, for any family $\xi : T \to \mathcal{M}$ over a finite type k-scheme T, the open and closed subspace of $\mathrm{Flag}(\xi)$ parameterizing filtrations with a fixed rank-degree-weight sequence is proper over T.*

In the context of Bridgeland stability, we have an alternative formulation of the Boundedness of quotients condition.

Lemma 5.38 (Lemma 3.15 of [T4])**.** *Let $(Z, \mathcal{A})$ be an algebraic Bridgeland stability condition on $\mathrm{D}^b(X)$. Assume that the functor $\mathcal{M}_v^{ss}$ is bounded for every class $v \in \mathcal{N}(X)$. Then $(Z, \mathcal{A})$ satisfies the boundedness of quotients property.*

In particular in [T4, Section 4], Toda shows that the conditions of the lemma hold for the class of Bridgeland stability conditions on $K3$ surfaces over $\mathbb{C}$ constructed in [B+2]. We have the following

Lemma 5.39. *Let $(Z, \mathcal{A})$ be a slope-stability condition for which $\mathcal{A}$ is noetherian and which satisfies the generic flatness condition. If $(Z, \mathcal{A})$ satisfies the Boundedness of quotients condition, then $\mathcal{M}_v^{\mathcal{F}} \subset \mathcal{M}_v$ is an open substack and is thus algebraic.*

Proof. Let $T \to \mathcal{M}_v$ classify a flat family of objects $E_t \in \mathcal{A}$ of class v over a k-scheme T of finite type. The locus of t for which $E_t \in \mathcal{F}$ is the complement of the image under ev_1 of a certain substack $\mathfrak{Y} \subset \mathrm{Filt}(\mathcal{M}_v)$, defined as the union of the connected components classifying weighted descending filtrations of the form $\cdots = 0 \subset F_1 \subset F_0 = \cdots$ where F_0 is of class v and F_1 is of torsion class. The Boundedness of quotients condition implies that $\mathfrak{Y} \times_{\mathrm{ev}_1, \mathcal{M}_v} T$ is quasi-compact, hence Proposition 5.17 implies that the image of $\mathfrak{Y}$ in T is closed and that its complement is open. $\square$

We will now apply Theorem 4.38 to construct a Θ-stratification of $\mathcal{M}_v^{\mathcal{F}}$.

Proposition 5.40. *Let X be a projective scheme over a field k, and let $(Z, \mathcal{A})$ be a Bridgeland stability condition on $\mathrm{D}^b(X)$ with $Z(\mathcal{A}) \subset \mathbb{Q} + i\mathbb{Q}$ for which $\mathcal{A}$ is noetherian and which satisfies the generic flatness and Boundedness of quotients conditions. Then the numerical invariant associated to the classes of Equation (36) defines a Θ-stratification of the stack $\mathcal{M}_v^{\mathcal{F}}$ of torsion-free objects in $\mathcal{A}$. In particular $\mathcal{M}_v^{ss} \subset \mathcal{M}_v^{\mathcal{F}} \subset \mathcal{M}_v$ are open immersions for all $v \in \mathcal{N}(X)$ with $\Im(Z(v)) > 0$.*

Proof of Proposition 5.40. We verify the hypotheses of Theorem 4.38 for the numerical invariant μ associated to the classes of Equation (36). Note that by Lemma 5.26 the class $b \in H^4(\mathcal{M}_v^{\mathcal{F}})$ is positive definite, so we are in the situation where $\mathcal{U} = \mathscr{C}omp(\mathcal{M}_v^{\mathcal{F}})$. The existence and uniqueness of HN filtrations for μ is Theorem 5.29.

We know from Theorem 5.29 that for any k-point μ is maximized by a filtration whose rank-degree sequence is convex, and applying the theorem after a finite extension k'/k implies that the same is true for any k'-point. If $\xi : T \to \mathcal{M}_v^{\mathcal{F}}$ is a family over a finite type k-scheme, then if we restrict to points $f \in \mathrm{Flag}(\xi)$ parameterizing filtrations whose rank-degree sequence is convex, the boundedness condition for $(Z, \mathcal{A})$ implies that the family of objects in $\mathcal{A}$ arising as $\mathrm{gr}(f)$ for f in this set is bounded (see Lemma 5.37), hence lies in a quasi-compact substack of $\mathcal{M}_v^{\mathcal{F}}$. This verifies (B2).[21]

All that remains is to verify the HN-specialization property (2). Consider a family $E \in \mathcal{M}_v^{\mathcal{F}}(R)$ over a DVR R with fraction field K and residue field κ and let $(E_K)_\bullet$ be an HN filtration in $\mathcal{M}_v^{\mathcal{F}}$ of E_K. Note that $(E_K)_\bullet$ is also an HN filtration in $\mathcal{M}_v$ by Theorem 5.29. The stack $\mathcal{M}_v$ is Θ-reductive by Proposition 5.17, so the filtration extends uniquely to a filtration in $\mathcal{M}_v$ over R. The filtration over the special fiber lies in $\mathcal{U}_{E_\kappa} \subset \mathscr{D}eg(\mathcal{M}_v, E_\kappa)$ and has the same value of μ as the filtration of E_K. This implies that $M^\mu(E_K) \leq M^\mu(E_\kappa)$. If equality holds, then the filtration we constructed over $\mathrm{Spec}(R)$ is an HN filtration at the special point, and it follows from Theorem 5.29 that the filtration of E lies in $\mathcal{M}_v^{\mathcal{F}}$.

Upgrading from a weak Θ-stratification to a Θ-stratification:

In order to show that we have a Θ-stratification, it suffices, by Lemma 2.5, to show that for any finite extension k'/k and any point $E_\bullet \in \mathrm{Filt}(\mathcal{M}_v^{\mathcal{F}})(k')$ corresponding to a *split* Harder-Narasimhan filtration (with its canonical weights), the map of vector spaces $\mathrm{Lie}(\mathrm{Aut}_{\mathrm{Filt}(\mathcal{M}_v)}(E_\bullet)) \to \mathrm{Lie}(\mathrm{Aut}_{\mathcal{M}_v}(E))$, which is simply the restriction map

$$\mathrm{Hom}_{X \times \Theta_{k'}}(E_\bullet, E_\bullet) \to \mathrm{Hom}_{X'_k}(E, E),$$

is surjective (see [B1] for a version of this argument). Let us write $E = F_1 \oplus \cdots \oplus F_n$, where F_i is semistable and the phase $\phi(F_i)$ is increasing in i. Then the HN filtration $E_\bullet$ of E has $E_i = \bigoplus_{j \geq i}$ with $\mathbb{Z}$-weights $w_1 < \cdots < w_n$ which are positively proportional to the slopes $\nu_i = d_i/r_i$ of F_i. The $\mathbb{Z}$-weighted filtration $E_\bullet$ corresponds to the complex

$$\bigoplus_i F_i \otimes_{k'} k'[t] \cdot t^{-w_i} \subset E \otimes_{k'} k'[t^\pm] \in \mathrm{D}^b(X \times \Theta_{k'}).$$

Then $\mathrm{Hom}_{X \times \Theta_{k'}}(E_\bullet, E_\bullet) \simeq \bigoplus_{i \leq j} \mathrm{Hom}_{X_{k'}}(F_i, F_j)$, and the surjectivity of the map above is equivalent to the claim that $\mathrm{Hom}_{X_{k'}}(F_i, F_j) = 0$ for $i > j$. This follows from Lemma 5.30.

□

[21]Another way of establishing local finiteness would be to verify (B1+). Indeed the set of rational simplices of $\mathscr{C}omp(\mathcal{M}_v^{\mathcal{F}})$ corresponding to convex rank-degree sequences will satisfy this condition.

5.4. **Loose ends.** Our goal with Proposition 5.40 is to illustrate how one can apply the machinery of numerical invariants in Section 4 to moduli problems which are beyond the scope of geometric invariant theory: the stack $\mathcal{M}_v^{\mathcal{F}}$ is not finite-type, and it is not known to be a local quotient stack. We note, however that in the context of slope-semistability there are many ways to extend Proposition 5.40 to construct Θ-stratifications which do not fit neatly into the framework of numerical invariants as discussed in Section 4:

(1) One can extend the Θ-stratification of Proposition 5.40 to a Θ-stratification of $\mathcal{M}_v$, where the HN filtration of every unstable point corresponds to some choice of weights for the Harder-Narasimhan filtration in the sense of slope semistability. For every Harder-Narasimhan polytope whose last subobject is torsion, simply choose a weight for this maximal torsion subobject which is greater than the weights appearing in the HN filtration for the torsion-free quotient. Using Theorem 2.7 one can show that this defines a Θ-stratification of $\mathcal{M}_v$, but the weights have not been chosen canonically by a numerical invariant.

(2) If the conditions of Proposition 5.40 are satisfied for enough stability conditions in a given connected component of $\mathrm{Stab}(X)$, we expect that the methods of [T4, PT] can be used to show that the conclusion of Proposition 5.40 holds for *every* stability condition in that connected component, i.e. $\mathcal{M}_{\phi,v}^{\mathrm{ss}} \subset \mathcal{M}_{\phi,v}$ is open for every class v and phase ϕ, and $\mathcal{M}_{v,\phi}$ admits a Θ-stratification according to the Harder-Narasimhan type of an unstable object.

    However, if $Z(A)$ does not lie in $\mathbb{Q}+i\mathbb{Q}$, the the cohomology classes (36) are not rational, and the unique maximizer of the numerical invariant μ in Theorem 5.29 need not lie at a rational point. The simplest way to deal with this is to perturb the canonical assignment of real weight of Lemma 5.33 for each HN polytope so that they are rational, and the resulting rational point of $\mathscr{D}eg(\mathcal{M}_v, E)$ will correspond to a filtration $f : \Theta_k \to \mathcal{M}$ up to ramified covers. This again has the drawback of non-canonical weights.

(3) The more classical theory Harder-Narasimhan filtrations of coherent sheaves in the context of Gieseker stability is also not determined by a numerical invariant as studied in Section 4. As above, one can certainly construct a Θ-stratification of $\mathcal{M}_v$ such that HN filtrations correspond to some choice of weights for the Harder-Narasimhan filtration of a coherent sheaf – this essentially follows from the work of Hoskins [HK] and Zamora [Z] – but the weights are again non-canonical.

Even though $\mathcal{M}_v$ is Θ-reductive, the obstacle to applying Theorem 4.39 to these extensions of Proposition 5.40 is the lack of a positive definite class in $H^4(\mathcal{M}_v;\mathbb{Q})$ with which to define a numerical invariant. We expect that one can incorporate these examples into a more general notion of numerical invariant taking values in a suitable totally ordered field rather than $\mathbb{R}$. We discussed polynomial-valued numerical invariants in Section 4.6 arising from $l \in H^2(\mathcal{M};\mathbb{Q}[n])$ and $b \in H^4(\mathcal{M};\mathbb{Q})$, but here we need to allow b to have coefficients in a larger group as well in order for b to be positive definite.

For example, the natural way to incorporate Gieseker stability would be to consider a central charge valued in polynomials $Z : K_0(\mathrm{D}^b(X)) \to \mathbb{C}[n]$ as in [B+1], leading to cohomology classes on $\mathcal{M} = \underline{\mathrm{Coh}}(X)$ with values in $\mathbb{Q}[n]$. The natural generalization of the class in (36) is

$$b = 2\,\mathrm{ch}_2((p_1)_*(\mathcal{E} \boxtimes L^n)) \in H^4(\mathcal{M};\mathbb{Q}[n]),$$

where $p_1 : \underline{\mathrm{Coh}}(X) \times X \to \underline{\mathrm{Coh}}(X)$ is the projection and $\mathcal{E}$ is the universal coherent sheaf on the product $\underline{\mathrm{Coh}}(X) \times X$. This class is positive definite in the sense that for any filtration $f : \Theta_k \to \mathcal{M}$, $f^*(b) \in \mathbb{Q}[n]$ will be a polynomial which takes positive values for $n \gg 0$, because the higher cohomology of $R\Gamma(X, E \otimes L^n)$ vanishes for $n \gg 0$. If one uses this class, however, then the numerical invariant $\mu(f) = f^*(l)/\sqrt{f^*(b)}$ would take values in the fraction field F of the ring of germs of analytic functions in one real variable n in a neighborhood of $n = \infty$.

If one allows numerical invariants with values in F, however, then the canonical weights obtained by maximizing the numerical invariant will tend to take values in F as well. In order to properly develop this theory, one should discuss geometric descriptions of filtrations with weights in F, and generalize the construction of $\mathscr{D}eg(\mathfrak{X}, x)$ so that points correspond to F-weighted filtrations. We shall postpone a full discussion of this theory until future work.

Remark 5.41. In a more speculative generalization of Proposition 5.40, one might ask if one can associate a numerical invariant, perhaps not even one arising from Definition 4.9, to a Bridgeland stability condition in such a way that one obtains a Θ-stratification of the full (derived) stack of relatively perfect complexes $\mathcal{D}^b_p(X)$ on X, in the sense that any $E \in \mathrm{D}^b(X)$ extends canonically to a map $\Theta \to \mathrm{D}^b(X)$ classifying the canonical "filtration" of E whose subquotients are semistable objects of increasing phase. The phases appearing in this filtration will not lie in the interval $(0,1]$ if $E \notin \mathrm{D}^b(X)^\heartsuit$. As the stack $\mathrm{D}^b_p(X)$ is not a 1-stack, the algebraicity of the mapping stack $\mathrm{Filt}(\mathcal{D}^b_p(X))$ is not on entirely rigorous footing (although see [HP] for a sketch of one possible approach), and so a treatment of this is beyond the scope of this paper. Still the degeneration space $\mathscr{D}eg(\mathcal{D}^b_p(X), [E])$ is well defined, and one could ask to what extent the stratification of $\mathcal{D}^b_p(X)$ is controlled by a continuous function induced by Z.

Appendix A. Proof of Theorem 1.37

The piece of the proof of Theorem 1.36 which reduces the general case of X/G to the case of pt/G applies verbatim. It therefore suffices to show that if G is a smooth algebraic k group and $T \subset G$ a split maximal torus, then the canonical P_ψ bundles E_ψ on Θ_k for any homomorpism $\psi : \mathbb{G}_m^n \to T$ induce equivalences

$$\bigsqcup_{\psi \in \mathrm{Hom}(\mathbb{G}_m^n, T)/W} \mathrm{pt}/P_\psi \xrightarrow{\simeq} \mathrm{Filt}^n(\mathrm{pt}/G) \quad \text{and}$$
$$\bigsqcup_{\psi \in \mathrm{Hom}(\mathbb{G}_m^n, T)/W} \mathrm{pt}/L_\psi \xrightarrow{\simeq} \mathrm{Grad}^n(\mathrm{pt}/G).$$

We will prove this in the case where $n = 1$. The case for general n follows from a similar argument. Alternatively, using the fact that $T \subset P_\psi$ will be a split maximal torus for all $\psi \in \mathrm{Hom}(\mathbb{G}_m^n, T)$, one can prove the general statement inductively using the fact that

$$\underline{\mathrm{Map}}(\Theta^n, \mathrm{pt}/G) \simeq \underline{\mathrm{Map}}(\Theta, \underline{\mathrm{Map}}(\Theta, \ldots, \underline{\mathrm{Map}}(\Theta, \mathrm{pt}/G))) \ldots),$$

and likewise for $\underline{\mathrm{Map}}(\mathrm{pt}/\mathbb{G}_m^n, \mathrm{pt}/G)$.

Proposition A.1. *Let k be a perfect field, let S be a connected finite type k-scheme, and let E be a G-bundle over $\Theta_S := \Theta \times S$. Let $s \in S(k)$, thought of as the point $(0, s) \in \mathbb{A}^1_S$, and assume that $\mathrm{Aut}(E_s) \simeq G$. Let $\lambda : \mathbb{G}_m \to G$ be a 1PS conjugate to the one parameter subgroup $\mathbb{G}_m \to \mathrm{Aut}(E_s)$. Then*

(1) There is a unique reduction of structure group $E' \subset E$ to a P_λ-torsor such that $\mathbb{G}_m \to \mathrm{Aut}(E'_s) \simeq P_\lambda$ is conjugate in P_λ to λ, and
(2) the restriction of E' to $\{1\} \times S$ is canonically isomorphic to the sheaf on the étale site of S mapping $T/S \mapsto \mathrm{Iso}((E_\lambda)_{\Theta_T}, E|_{\Theta_T})$.

Proof. $(E_\lambda)_{\Theta_S} = E_\lambda \times S/\mathbb{G}_m$ is a G-bundle over Θ_S, and $\underline{\mathrm{Iso}}((E_\lambda)_{\Theta_S}, E)$ is a sheaf over Θ_S representable by a (relative) scheme over Θ_S. In fact, if we define a twisted action of $\mathbb{G}_m$ on E given by $t \star e := t \cdot e \cdot \lambda(t)^{-1}$, then

$$\underline{\mathrm{Iso}}((E_\lambda)_{\Theta_S}, E) \simeq E/\mathbb{G}_m \text{ w.r.t. the } \star\text{-action} \tag{40}$$

as sheaves over Θ_S.[22]

The twisted $\mathbb{G}_m$ action on E is compatible with base change. Let $T \to S$ be an S-scheme. From the isomorphism of sheaves (40), there is a natural bijection between the set of isomorphisms $(E_\lambda)_{\mathbb{A}^1_T} \xrightarrow{\simeq} E|_{\mathbb{A}^1_T}$ as $\mathbb{G}_m$-equivariant

[22]To see this, note that a map $T \to \Theta_S$ corresponds to a $\mathbb{G}_m$-bundle $P \to T$ along with a $\mathbb{G}_m$ equivariant map $f : P \to \mathbb{A}^1 \times S$. Then the restrictions $(E_\lambda \times S)_T$ and $E|_T$ correspond (via descent for G-bundles) to the $\mathbb{G}_m$-equivariant bundles $f^{-1}(E_\lambda \times S)$ and $f^{-1}E$ over P. Forgetting the $\mathbb{G}_m$-equivariant structure, the G-bundle $E_\lambda \times S$ is trivial, so an isomorphism $f^{-1}(E_\lambda \times S) \to f^{-1}E$ as G-bundles corresponds to a section of $f^{-1}E$, or equivalently to a lifting

$$\begin{tikzcd} & E \arrow[d] \\ P \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & \mathbb{A}^1 \times S \end{tikzcd}$$

to a map $\tilde{f} : P \to E \to \mathbb{A}^1 \times S$. The isomorphism of G-bundles defined by the lifting $\tilde{f}$ descends to an isomorphism of $\mathbb{G}_m$-equivariant G-bundles $f^{-1}(E_\lambda \times S) \to f^{-1}E$ if and only if the lift $\tilde{f}$ is equivariant with respect to the twisted $\mathbb{G}_m$ action on E.

G-bundles and the set of $\mathbb{G}_m$-equivariant sections of $E|_{\mathbb{A}^1_T} \to \mathbb{A}^1_T$ with respect to the twisted $\mathbb{G}_m$ action.

The morphism $E|_{\mathbb{A}^1_T} \to \mathbb{A}^1_T$ is separated, so a twisted equivariant section is uniquely determined by its restriction to $\mathbb{G}_m \times T$, and by equivariance this is uniquely determined by its restriction to $\{1\} \times T$. Thus we can identify $\mathbb{G}_m$-equivariant sections with the set of maps $T \to E$ such that $\lim_{t\to 0} t \star e$ exists and $T \to E \to \mathbb{A}^1_S$ factors as the given morphism $T \to \{1\} \times S \to \mathbb{A}^1_S$.

If we define the subsheaf of E over $\mathbb{A}^1_S$

$$E'(T) := \left\{ e \in E(T) | \mathbb{G}_m \times T \xrightarrow{t\star e(x)} E \text{ extends to } \mathbb{A}^1 \times T \right\} \subset E(T),$$

then we have shown that $E'|_{\{1\}\times S}(T) \simeq \mathrm{Iso}((E_\lambda)_{\Theta_T}, E|_{\Theta_T})$. Next we show in several steps that the subsheaf $E' \subset E$ over $\mathbb{A}^1_S$ is a torsor for the subgroup $P_\lambda \subset G$, so E' is a reduction of structure group to P_λ.

Step 1: *E' is representable:* The functor E' is exactly the functor of Corollary 1.32.1, so Proposition 1.31 implies that E' is representable by a disjoint union of $\mathbb{G}_m$ equivariant locally closed subschemes of E.

Step 2: *$P_\lambda \subset G$ acts simply transitively on $E' \subset E$:* Because E is a G-bundle over $\mathbb{A}^1_S$, right multiplication $(e, g) \mapsto (e, e \cdot g)$ defines an isomorphism $E \times G \to E \times_{\mathbb{A}^1_S} E$. The latter has a $\mathbb{G}_m$ action, which we can transfer to $E \times G$ using this isomorphism.

For $g \in G(T)$, $e \in E(T)$, and $t \in \mathbb{G}_m(T)$ we have $t \star (e \cdot g) = (t \star e) \cdot (\lambda(t) g \lambda(t)^{-1})$. This implies that the $\mathbb{G}_m$ action on $E \times G$ corresponding to the diagonal action on $E \times_{\mathbb{A}^1_S} E$ is given by

$$t \cdot (e, g) = (t \star e, \lambda(t) g \lambda(t)^{-1})$$

The subfunctor of $E \times G$ corresponding to $E' \times_{\mathbb{A}^1_S} E' \subset E \times_{\mathbb{A}^1_S} E$ consists of those points for which $\lim_{t\to 0} t \cdot (e, g)$ exists. This is exactly the subfunctor represented by $E' \times P_\lambda \subset E \times G$. We have thus shown that E' is equivariant for the action of P_λ, and $E' \times P_\lambda \to E' \times_{\mathbb{A}^1_S} E'$ is an isomorphism of sheaves.

Step 3: *$p : E' \to \mathbb{A}^1_S$ is smooth:* Proposition 1.34 implies that E' and $E^{\mathbb{G}_m} \subset E'$ are both smooth over S. The restriction of the tangent bundle $T_{E/S}|_{E^{\mathbb{G}_m}}$ is an equivariant locally free sheaf on a scheme with trivial $\mathbb{G}_m$ action, hence it splits into a direct sum of vector bundles of fixed weight with respect to $\mathbb{G}_m$. The tangent sheaf $T_{E'/S}|_{E^{\mathbb{G}_m}} \subset T_{E/S}|_{E^{\mathbb{G}_m}}$ is precisely the subsheaf with weight ≥ 0. By hypothesis $T_{E/S} \to p^* T_{\mathbb{A}^1_S/S}$ is surjective, and $p^* T_{\mathbb{A}^1_S/S}|_{E^{\mathbb{G}_m}}$ is concentrated in nonnegative weights, therefore the map

$$T_{E'/S}|_{E^{\mathbb{G}_m}} = (T_{E/S}|_{E^{\mathbb{G}_m}})_{\geq 0} \to p^* T_{\mathbb{A}^1_S/S}|_{E^{\mathbb{G}_m}} = (p^* T_{\mathbb{A}^1_S/S}|_{E^{\mathbb{G}_m}}))_{\geq 0}$$

is surjective as well.

Thus we have shown that $T_{E'/S} \to p^*T_{\mathbb{A}^1_S/S}$ is surjective when restricted to $E^{\mathbb{G}_m} \subset E'$, and by Nakayama's Lemma it is also surjective in a Zariski neighborhood of $E^{\mathbb{G}_m}$. On the other hand, the only equivariant open subscheme of E' containing $E^{\mathbb{G}_m}$ is E' itself. It follows that $T_{E'/S} \to p^*T_{\mathbb{A}^1_S/S}$ is surjective, and therefore that the morphism p is smooth.

Step 4: $p : E' \to \mathbb{A}^1_S$ *admits sections étale locally:* We consider the $\mathbb{G}_m$ equivariant G-bundle $E|_{\{0\}\times S}$. After étale base change we can assume that $E|_{\{0\}\times S}$ admits a non-equivariant section, hence the $\mathbb{G}_m$-equivariant structure is given by a homomorphism $(\mathbb{G}_m)_{S'} \to G_{S'}$. Lemma A.3 implies that after further étale base change this homomorphism is conjugate to a constant homomorphism. Thus $E|_{\{0\}\times S'}$ is isomorphic to the trivial equivariant $\mathbb{G}_m$-bundle $(E_\lambda)_{\mathbb{A}^1_S} = \mathbb{A}^1_S \times G \to \mathbb{A}^1_S$ with $\mathbb{G}_m$ acting by left multiplication by $\lambda(t)$.

It follows that $E|_{\{0\}\times S'}$ admits an invariant section with respect to the twisted $\mathbb{G}_m$ action. In other words $(E_{\mathbb{A}^1_{S'}})^{\mathbb{G}_m} \to \{0\} \times S'$ admits a section, and $E^{\mathbb{G}_m} \subset E'$, so we have shown that $E' \to \mathbb{A}^1_{S'}$ admits a section over $\{0\} \times S'$. On the other hand, because $p : E' \to \mathbb{A}^1_{S'}$ is smooth and $\mathbb{G}_m$-equivariant, the locus over which p admits an étale local section is open and $\mathbb{G}_m$-equivariant. It follows that p admits an étale local section over every point of $\mathbb{A}^1_{S'}$.

□

Remark A.2. In fact we have shown something slightly stronger than the existence of étale local sections of $E' \to \mathbb{A}^1_S$ in Step 4. We have shown that there is an étale map $S' \to S$ such that $E'|_{S'} \to \mathbb{A}^1_{S'}$ admits a $\mathbb{G}_m$-equivariant section.

The following fact, that families of one parameter subgroups of G are étale locally constant up to conjugation, was the key input to the proof of Proposition A.1.

Lemma A.3. *Let S be a connected scheme of finite type over a perfect field k, let G be a smooth affine k-group, and let $\phi : (\mathbb{G}_m)_S \to G_S$ be a homomorphism of group schemes over S. Let $\lambda : \mathbb{G}_m \to G$ be a 1PS conjugate to ϕ_s for some $s \in S(k)$. Then the subsheaf*

$$F(T) = \left\{ g \in G(T) | \phi_T = g \cdot (\mathrm{id}_T, \lambda) \cdot g^{-1} : (\mathbb{G}_m)_T \to G_T \right\} \subset G_S(T)$$

is an L_λ-torsor. In particular ϕ is étale-locally conjugate to a constant homomorphism.

Proof. Verifying that $F \times L_\lambda \to F \times_S F$ given by $(g, l) \mapsto (g, gl)$ is an isomorphism of sheaves is straightforward. The more important question is whether $F(T) \neq \emptyset$ étale locally.

As in the proof of Proposition A.1 we introduce a twisted $\mathbb{G}_m$ action on $G \times S$ by $t \star (g, s) = \phi_s(t) \cdot g \cdot \lambda(t)^{-1}$. Then $G \times S \to S$ is $\mathbb{G}_m$ invariant, and the functor $F(T)$ is represented by the map of schemes $(G \times S)^{\mathbb{G}_m} \to S$. By Proposition 1.34, $(G \times S)^{\mathbb{G}_m} \to S$ is smooth, and in particular it admits a section after étale base change in a neighborhood of a point $s \in S(k)$ for which $(G \times S)^{\mathbb{G}_m}_s = (G \times \{s\})^{\mathbb{G}_m} \neq \emptyset$. By construction this set is nonempty precisely when ϕ_s is conjugate to λ, so by hypothesis it is nonempty for some $s \in S(k)$.

By the same reasoning, for every finite separable extension k'/k, every k'-point has an étale neighborhood on which ϕ is conjugate to a constant homomorphism determined by some one parameter subgroup defined over k'. Because k is perfect and S is locally finite type, we have a cover of S by Zariski opens over each of which ϕ is etale-locally conjugate to a constant homomorphism determined by a one parameter subgroup defined over some finite separable extension of k. Because S is connected, all of these 1PS's are conjugate to λ, possibly after further finite separable field extensions. Thus $(G \times S)^{\mathbb{G}_m} \to S$ admits a global section after étale base change. □

Because T is split, the set of one parameter subgroups up to conjugacy by W is unaffected by passing to the algebraic closure, so it suffices to assume that $k = \bar{k}$. Proposition A.1 implies that the map $\bigsqcup_{\mathrm{Hom}(\mathbb{G}_m,T)/W} \mathrm{pt}/P_\lambda \to \mathrm{Filt}(\mathrm{pt}/G)$ is an equivalence over the sub-site of k-schemes of finite type. The functor $\mathrm{Filt}(\mathrm{pt}/G)$ is limit preserving by the formal observation

$$\mathrm{Map}(\varprojlim_i T_i, \underline{\mathrm{Map}}(\Theta, \mathrm{pt}/G)) \simeq \varprojlim_i \mathrm{Map}(T_i \times \Theta, \mathrm{pt}/G)$$

where the last equality holds because pt/G is an algebraic stack locally of finite presentation. The stack $\bigsqcup_I \mathrm{pt}/P_\lambda$ is locally of finite presentation and thus limit preserving as well. Every affine scheme over k can be written as a limit of finite type k schemes, so the isomorphism for finite type k-schemes implies the isomorphism for all k-schemes.

To show that $\bigsqcup_{\lambda \in \mathrm{Hom}(\mathbb{G}_m,T)/W} \mathrm{pt}/L_\lambda \to \mathrm{Grad}(\mathrm{pt}/G)$ is an equivalence, note that for S of finite type over $k = \bar{k}$, the proof of Proposition A.1 carries over unchanged for G-bundles over $(\mathrm{pt}/\mathbb{G}_m) \times S$, showing that étale locally in S they are isomorphic to $S \times G$ with $\mathbb{G}_m$ acting by left multiplication by $\lambda(t)$ on G. In fact, we had to essentially prove this when we considered the $\mathbb{G}_m$-equivariant bundle $E|_{\{0\} \times S}$ in Step 4 of that proof. The amplification of the statement from finite type $\bar{k}$-schemes to all $\bar{k}$-schemes, and from $\bar{k}$-schemes to k-schemes, is identical to the argument for $\mathrm{Filt}(\mathrm{pt}/G)$ above. This completes the proof of Theorem 1.37.

Remark A.4. The argument above does not a priori use the fact that $\mathrm{Filt}(\mathrm{pt}/G)$ is algebraic. It thus provides an alternative proof for the algebraicity of $\mathrm{Filt}(X/G)$ when G is a split algebraic k-group and X is a G-quasi-projective scheme which is independent of the general results of [HP].

Appendix B. On the slice theorem of Alper-Hall-Rydh

In the body of the paper, we need a strengthened version of the Alper-Hall-Rydh slice theorem [AHR1, Theorem 1.2]. In a follow up to their original paper, Alper, Hall, and Rydh are preparing a second paper which strengthens and extends several of their results. Among other things, they establish the following:

Theorem B.1 ([AHR2])**.** *Let $\mathcal{X}$ be a quasi-compact quasi-separated algebraic stack with affine stabilizers which is finitely presented over a quasi-separated algebraic space S which is defined over a field. Given a locally closed immersion $x : \mathrm{Spec}(k)/G \hookrightarrow \mathcal{X}$, with G linearly reductive, there is an étale map $U/G' \to \mathcal{X}$ extending x, where U is affine and G' is a linearly reductive embeddable group scheme over $U//G'$.*

We will also need a "non-local" variant of this theorem, which we formulate here, and which we will prove jointly with Alper, Hall, and Rydh in a follow up paper [AHHLR]. Consider a Noetherian affine base scheme S and a linearly reductive smooth affine group scheme G over S such that BG has the resolution property (every coherent sheaf is a quotient of a locally free sheaf). Examples include:

- G is a linearly reductive group scheme over a noetherian Dedekind domain $S = \mathrm{Spec}(R)$, or
- $G = \mathbb{G}_m^r$ over any Noetherian affine base scheme S.

Theorem B.2. *Let $\mathfrak{X}$ be a quasi-compact quasi-separated algebraic stack which is finitely presented over S and has affine stabilizer groups. Let $\mathfrak{Y} \subset \mathfrak{X}$ be a locally closed substack, and let $Y/G \to \mathfrak{Y}$ be a representable surjective smooth (respectively étale) map with Y affine over S. There is a G-scheme U which is affine over S and a smooth (resp. étale) map $U/G \to \mathfrak{X}$ such that the map $(U/G) \times_{\mathfrak{X}} \mathfrak{Y} \to \mathfrak{Y}$ can be factored as a representable surjective étale map $(U/G) \times_{\mathfrak{X}} \mathfrak{Y} \to Y/G$ followed by the original map $Y/G \to \mathfrak{Y}$.*

In other words, given a smooth (resp. étale) cover of the locally closed substack $\mathcal{Y}$, after passing to a further étale cover one can extend this to a smooth (rep. étale) map to $\mathcal{X}$.

References

[A] Jarod Alper, *Good moduli spaces for artin stacks [bons espaces de modules pour les champs dartin]*, Annales de linstitut fourier, 2013, pp. 2349–2402.

[AB] Michael Francis Atiyah and Raoul Bott, *The Yang-Mills equations over Riemann surfaces*, Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences **308** (1983), no. 1505, 523–615.

[AHHLR] Jarod Alper, Jack Hall, Daniel Halpern-Leistner, and David Rydh, *A non-local version of the slice theorem for stacks* (2018).

[AHLH] Jarod Alper, Daniel Halpern-Leistner, and Jochen Heinloth, *On the existence of good moduli spaces* (In preparation).

[AHR1] Jarod Alper, Jack Hall, and David Rydh, *A luna étale slice theorem for algebraic stacks*, arXiv preprint arXiv:1504.06467 (2015).

[AHR2] ______, *Private communication, will appear in the sequel to "a luna étale slice theorem for algebraic stacks"* (2018).

[AP] Dan Abramovich and Alexander Polishchuk, *Sheaves of t-structures and valuative criteria for stable complexes*, Journal fur die reine und angewandte Mathematik (Crelles Journal) **2006** (2006), no. 590, 89–130.

[B1] Kai A Behrend, *Semi-stability of reductive group schemes over curves*, Mathematische Annalen **301** (1995), no. 1, 281–305.

[B2] Tom Bridgeland, *Stability conditions on triangulated categories*, Annals of Mathematics (2007), 317–345.

[BHL] Bhargav Bhatt and Daniel Halpern-Leistner, *Tannaka duality revisited*, Advances in Mathematics **316** (2017), 576–612.

[B^+1] Arend Bayer et al., *Polynomial bridgeland stability conditions and the large volume limit*, Geometry & Topology **13** (2009), no. 4, 2389–2425.

[B^+2] Tom Bridgeland et al., *Stability conditions on $k3$ surfaces*, Duke Mathematical Journal **141** (2008), no. 2, 241–291.

[C] Brian Conrad, *Reductive group schemes*, Autour des schémas en groupes. Vol. I, 2014, pp. 93–444. MR3362641

[DG] Vladimir Drinfeld and Dennis Gaitsgory, *On a theorem of braden*, Transformation Groups **19** (2014), no. 2, 313–358.

[DH] Igor V. Dolgachev and Yi Hu, *Variation of geometric invariant theory quotients*, Inst. Hautes Études Sci. Publ. Math. **87** (1998), 5–56. With an appendix by Nicolas Ressayre. MR1659282 (2000b:14060)

[DOR] Jean-François Dat, Sascha Orlik, and Michael Rapoport, *Period domains over finite and p-adic fields*, Cambridge university press, 2010.

[E] Ryszard Engelking, *General topology, sigma series in pure mathematics, vol. 6*, Heldermann Verlag, Berlin, 1989.

[EG] Dan Edidin and William Graham, *Equivariant intersection theory (with an appendix by angelo vistoli: The chow ring of m2)*, Inventiones mathematicae **131** (1998), no. 3, 595–634.

[F1] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry. MR1234037 (94g:14028)

[F2] ______, *Introduction to toric varieties*, Princeton University Press, 1993.

[FS] Maksym Fedorchuk and David Ishii Smyth, *Alternate compactifications of moduli spaces of curves*, arXiv preprint arXiv:1012.0329 (2010).

[G] R Gonzales, *Equivariant operational chow rings of t-linear schemes*, Doc. Math **20** (2015), 401–432.

[H1] Jochen Heinloth, *Bounds for behrend s conjecture on the canonical reduction*, International Mathematics Research Notices **2008** (2008), rnn045.

[H2] Wim H. Hesselink, *Concentration under actions of algebraic groups*, Paul Dubreil and Marie-Paule Malliavin Algebra Seminar, 33rd Year (Paris, 1980), 1981, pp. 55–89. MR633514 (82m:14029)

[HK] Victoria Hoskins and Frances Kirwan, *Quotients of unstable subvarieties and moduli spaces of sheaves of fixed harder–narasimhan type*, Proceedings of the London Mathematical Society **105** (2012), no. 4, 852–890.

[HL1] Daniel Halpern-Leistner, *The derived category of a git quotient*, Journal of the American Mathematical Society **28** (2015), no. 3, 871–912.

[HL2] ______, *Theta-stratifications, theta-reductive stacks, and applications*, arXiv preprint arXiv:1608.04797 (2016).

[HL3] ______, *θ-stratifications, derived categories, and k3 surfaces* (In preparation).

[HL4] Megumi Harada and Gregory D. Landweber, *Surjectivity for hamiltonian g-spaces in k-theory*, Trans. Amer. Math. Soc. **359** (2007), no. 12, 6001–6025 (electronic). MR2336314 (2008e:53165)

[HL5] Daniel Huybrechts and Manfred Lehn, *The geometry of moduli spaces of sheaves*, Springer, 2010.

[HP] D. Halpern-Leistner and A. Preygel, *Mapping stacks and categorical notions of properness*, ArXiv e-prints (February 2014), available at 1402.3204.

[HR] Jack Hall and David Rydh, *Coherent tannaka duality and algebraicity of hom-stacks*, arXiv preprint arXiv:1405.7680 (2014).

[K1] George R. Kempf, *Instability in invariant theory*, Ann. of Math. (2) **108** (1978), no. 2, 299–316. MR506989 (80c:20057)

[K2] Frances Clare Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes, vol. 31, Princeton University Press, 1984. MR766741 (86i:58050)

[K3] Andrew Kresch, *Cycle groups for artin stacks*, Inventiones mathematicae **138** (1999), no. 3, 495–536.

[L1] Max Lieblich, *Moduli of complexes on a proper morphism*, arXiv preprint math (2005).

[L2] Jacob Lurie, *Dag viii: Quasi-coherent sheaves and tannaka duality theorems*, Preprint available from authors website as http://math. harvard. edu/˜lurie/papers/DAG-VIII. pdf (2012).

[L3] ———, *Dag xiv: Representability theorems*, Preprint available from authors website as http://math. harvard. edu/˜lurie/papers/DAG-XIV. pdf (2012).

[L4] ———, *Higher algebra*, 2012.

[LMB] Gérard Laumon and Laurent Moret-Bailly, *Champs algébriques*, Vol. 39, Springer, 2018.

[MFK] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, Third, Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR1304906 (95m:14012)

[M] T. Mochizuki, *Donaldson type invariants for algebraic surfaces: Transition of moduli stacks*, Lecture Notes in Mathematics, Springer Berlin Heidelberg, 2009.

[N1] Nitin Nitsure, *Schematic harder–narasimhan stratification*, International Journal of Mathematics **22** (2011), no. 10, 1365–1373.

[N2] Behrang Noohi, *Homotopy types of topological stacks*, Adv. Math. **230** (2012), no. 4-6, 2014–2047. MR2927363

[P1] Sam Payne, *Equivariant Chow cohomology of toric varieties*, Math. Res. Lett. **13** (2006), no. 1, 29–41. MR2199564 (2007f:14052)

[P2] ———, *Moduli of toric vector bundles*, Compos. Math. **144** (2008), no. 5, 1199–1213. MR2457524 (2009h:14071)

[P3] A Polishchuk, *Constant families of t-structures on derived categories of coherent sheaves*, Mosc. Math. J **7** (2007), no. 1, 109–134.

[PT] Dulip Piyaratne and Yukinobu Toda, *Moduli of bridgeland semistable objects on 3-folds and donaldson–thomas invariants*, Journal für die reine und angewandte Mathematik (Crelles Journal) (2015).

[S1] Stephen S. Shatz, *The decomposition and specialization of algebraic families of vector bundles*, Compositio Math. **35** (1977), no. 2, 163–187. MR0498573 (58 #16668)

[S2] The Stacks Project Authors, *Stacks Project*, 2018.

[T1] Constantin Teleman, *The quantization conjecture revisited*, Ann. of Math. (2) **152** (2000), no. 1, 1–43. MR1792291 (2002d:14073)

[T2] Michael Thaddeus, *Geometric invariant theory and flips*, J. Amer. Math. Soc. **9** (1996), no. 3, 691–723. MR1333296 (96m:14017)

[T3] Robert W Thomason, *Algebraic k-theory and étale cohomology*, Annales scientifiques de l'école normale supérieure, 1985, pp. 437–552.

[T4] Yukinobu Toda, *Moduli stacks and invariants of semistable objects on k3 surfaces*, Advances in Mathematics **217** (2008), no. 6, 2736–2781.

[T5] Burt Totaro, *The resolution property for schemes and stacks*, Journal für die reine und angewandte Mathematik (Crelles Journal) **2004** (2004), no. 577, 1–22.

[TW] Constantin Teleman and Christopher T Woodward, *The index formula for the moduli of G-bundles on a curve*, Annals of mathematics (2009), 495–527.

[V] Angelo Vistoli, *Grothendieck topologies, fibered categories and descent theory*, Fundamental algebraic geometry, 2005, pp. 1–104. MR2223406

[Z] Alfonso Zamora, *GIT characterizations of Harder–Narasimhan filtrations*, arXiv preprint arXiv:1407.4223 (2014).