ASEP, q-TASEP and integrable many body systems

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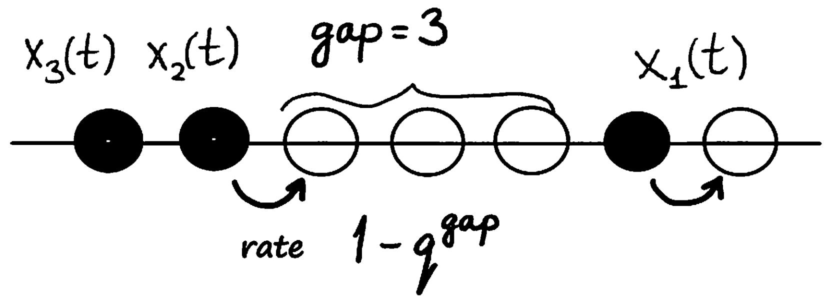
Basic reason that all these models turned out to be accessible is the existence of a large family of observables whose averages are explicit.

<u>Example 1: q-TASEP</u> [Borodin-Corwin, 2011]

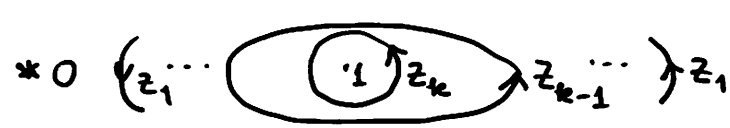


$0 < q < 1$

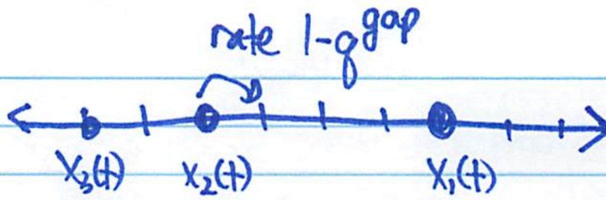
<u>*Theorem [B-C'11], [B-C-Sasamoto'12]*</u> *For q-TASEP with step initial data* $\{x_n(0)=-n\}_{n\geq 1}$

$$\mathbb{E}\left[q^{(x_{N_1}(t)+N_1)+\ldots+(x_{N_k}(t)+N_k)}\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint \cdots \oint \prod_{A<B} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{e^{(q-1)t z_j}}{(1-z_j)^{N_j}} \frac{dz_j}{z_j}$$

$(N_1 \geq N_2 \geq \ldots \geq N_k)$



Part 1: q-TASEP

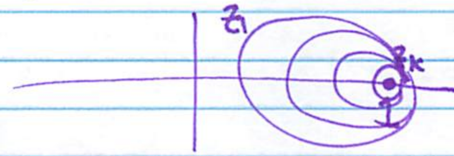


$0<q<1$

Theorem [Borodin-C '11, Borodin-C-Sasamoto] For step initial data $\{x_n(0)=-n\}_{n\geq 1}$

$$h(t;\vec{n}) = \mathbb{E}\left[\prod_{i=1}^{k} q^{x_{n_i}(t)+n_i}\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint\cdots\oint \prod_{1\leq A<B\leq k} \frac{z_A-z_B}{z_A-qz_B} \prod_{j=1}^{k} \frac{e^{(q-1)tz_j}}{(1-z_j)^{n_j}} \frac{dz_j}{z_j}$$

$n_1\geq n_2\geq \cdots \geq n_k$



"Nested contour integral"

(Quantum) many body system approach (for q-TASEP)

1. Find observables ~~with closed~~ whose expectations satisfy closed "true evolution equations".

2. Rewrite true evolution equation in "integrable form" as k one body free evolution eqn with $k-1$ two-body boundary cond.

3. Solve free system with b/c via "nested contour integral ansatz" (a version of Bethe ansatz)

Generally, not (always) clear how to find systems which are amenable to this form of solvability (Macdonald processes and algebraic Bethe ansatz provide more structural approaches)

Step 1 True evolution eq^n.

k=1:

$$dq^{X_n(t)+n} = \left(q^{X_n(t)+1+n} - q^{X_n(t)+n}\right)\left(1-q^{X_{n-1}(t)-X_n(t)-1}\right)dt + dM$$

$$= (1-q)\nabla q^{X_n(t)+n}\, dt + dM$$

$\nabla f(n) = f(n-1) - f(n)$

so

(unique)

$$\boxed{\begin{cases} \frac{d}{dt}\mathbb{E}\left[q^{X_n(t)+n}\right] = (1-q)\nabla \mathbb{E}\left[q^{X_n(t)+n}\right] \\ \mathbb{E}\left[q^{X_0(t)}\right] \equiv 0 \end{cases}}$$

K=2: Assume $n_1 \geq n_2$

$\sum_{i=1}^{2}(1-q)\nabla_i$

For $n_1 > n_2$:

$$\boxed{\frac{d}{dt}\mathbb{E}\left[q^{X_{n_1}(t)+n_1} q^{X_{n_2}(t)+n_2}\right] = (1-q)(\nabla_1+\nabla_2)\mathbb{E}\left[\prod_{i=1}^{2} q^{X_{n_i}(t)+n_i}\right]}$$

For $n_1 = n_2 (=n)$

$$dq^{2(X_n(t)+n)} = \left(q^{2(X_n(t)+1+n)} - q^{2(X_n(t)+n)}\right)\left(1-q^{X_{n-1}(t)-X_n(t)-1}\right)dt + dM$$

so

$$\boxed{\frac{d}{dt}\mathbb{E}\left[\prod_{i=1}^{2} q^{X_{n_i}(t)+n_i}\right] = (1-q^2)\nabla_2 \mathbb{E}\left[\prod_{i=1}^{2} q^{X_{n_i}(t)+n_i}\right]}$$

∇ in n_2

K≥3: For each cluster in $\vec{n} = (n_1, \ldots n_k)$, ~~the~~ apply

$$\frac{d}{dt}\mathbb{E}[\ ] = \sum_{\text{clusters}}\left(1-q^{|\text{cluster}|}\right)\nabla_{\text{top label cluster}}\mathbb{E}[\ ].$$

(duality to a TAZRP

)

Step 2:
The true evolution equation is not constant coefficient or separable.

- Bethe's idea (1931): Rewrite in terms of solution of K particle free evolution equation subject to K-1 two body boundary conditions. Advanced Reflection Principle !!

Usually not possible and one has to ~~consider~~ impose multiparticle boundary conditions. BUT, if possible $\rightsquigarrow$ "Integrable"!

- Idea in motion for $K=2$.

Consider $u: \mathbb{R}_{\geq 0} \times \mathbb{Z}^2_{\geq 0} \to \mathbb{R}$ solving

$$\frac{d}{dt} u(t;\vec{n}) = \sum_{i=1}^{2} (1-q) \nabla_i u(t;\vec{n})$$

can violate $\vec{n}$ order

For $n_1 > n_2$ this matches true evolution equation.

But for $n_1 = n_2$ differs by

$$(*) \quad \sum_{i=1}^{2} (1-q) \nabla_i u(t;\vec{n}) - (1-q^2) \nabla_2 u(t;\vec{n})$$

If u such that $(*) \equiv 0$ then restricted to $\{n_1 \geq n_2\}$ u solves true evolution equation.

$$(*) = 0 \longleftrightarrow (\nabla_1 - q\nabla_2) u|_{n_1 = n_2} \equiv 0$$

For $k \geq 3$ a priori may have boundary conditions for all compositions of k
($n_1 = n_2$, $n_2 = n_3$, also $n_1 = n_2 = n_3$)

Amazingly: Only $k-1$ boundary conditions from $n_i = n_{i+1}$ must be imposed, all others follow from them!

- Proposition [Borodin-C-Sasamoto '12] "Free evol^{n} eq^{n} with $k-1$ boundary cond."

If $u : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}^{k} \to \mathbb{R}$ solves

(1) $\forall\, \vec{n} \in \mathbb{Z}_{\geq 0}^{k}$, $t \in \mathbb{R}_{\geq 0}$

$$\frac{d}{dt} u(t;\vec{n}) = \sum_{i=1}^{k} (1-q) \nabla_i u(t;\vec{n})$$

(2) $\forall\, \vec{n} \in \mathbb{Z}_{\geq 0}^{k}$ s.t. $n_i = n_{i+1}$ $(1 \leq i \leq k-1)$

$$(\nabla_i - q \nabla_{i+1})\, u(t;\vec{n}) \equiv 0$$

(3) $\forall\, \vec{n} \in \mathbb{Z}_{\geq 0}^{k}$ with $n_k = 0$, $u(t;\vec{n}) \equiv 0$

(4) $\forall\, \vec{n} \in \mathbb{Z}_{\geq 0}^{k}$: $\underbrace{n_1 \geq n_2 \geq \cdots \geq n_k}_{W_{\geq 0}^{k}}$, $u(0;\vec{n}) = h(0;\vec{n})$

Then $\forall\, \vec{n} \in W_{\geq 0}^{k}$,

$$h(t;\vec{n}) = u(t;\vec{n}).$$

"Restriction solves true evolution equation".

Step 3:

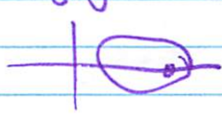
Check for step initial data, integral solves free system...

($X_i(0) = -i$ means $q^{X_i(0)+i} \equiv 1$ so $h(0;\vec{n}) \equiv 1$)

- Multiplicative term $\prod_{j=1}^{k} \frac{e^{(q-1)tz_j}}{(1-z_j)^{n_j}}$ solves (1)

- If $n_i = n_{i+1}$ (eg $i=1, i+1=2$) apply $(\nabla_i - q\nabla_{i+1})$ to integrand of integral. Introduces $(z_i - qz_{i+1})$ factor, cancels term in denominator, allows to deform z_i, z_{i+1} contours together

$$\iint G(z_i, z_{i+1})(z_i - z_{i+1}) = 0 \qquad \text{hence (2)}$$



↑ symmetric
contains other
integrals

- (3), (4) via residue calculus.

Hence integral satisfies (1),(2),(3),(4) $\Rightarrow$ Theorem $\blacksquare$.

Note: Inserting other symmetric f^n of $z_1, \dots z_k$ (right now $\prod \frac{1}{z_j}$) will relate to other initial data of true evol^n eqn.

The inverse map remains a challenge!

Could be solved if we can diagonalize the Hamiltonian.

A good way to check the solution, but how to produce it?

- Macdonald processes [B-C]: Integrable properties of Macdonald sym. polynomials led to q-TASEP (Pieri rules), the observables (eig$^{=}$ values of D_1) and integral formulas (eig values + Cauchy identity).

  Many body system $\longleftrightarrow$ Commutation relation involving D_1.

  Only useful for certain class of initial data.

- Coordinate Bethe ansatz [B-C-Petrov-Sasamoto '13]: Diagonalize free evol$^{\underline{n}}$ eq$^{\underline{n}}$ ~~gives~~ via proving a new Plancherel theorem (gen. of H.O.)

  Gives approach for general initial data.

- Algebraic Bethe ansatz [Sasamoto-Wadati '98]: Produces from "q-Boson" the time evol$^{\underline{n}}$ eqns and some indications of their integrability (without any reference to q-TASEP), ~~about~~ but not really studied or exploited yet.

One application of q-TASEP formulas.

Note: Expectation of observables completely determine distribution of $\{X_n(t)\}_{n\geq 1}$

- One point marginal via q-Laplace transform of $q^{X_n(t)+n}$:

$$\mathbb{E}^{step}\left[e_q(\zeta q^{X_n(t)+n})\right] = \mathbb{E}^{step}\left[\sum_{k=0}^{\infty} q^{k(X_n(t)+n)} \frac{\zeta^k}{(1-q)\cdots(1-q^k)}\right]$$

Hahn '49

Justified for $|\zeta|$ small

$$= \sum_{k=0}^{\infty} \mathbb{E}^{step}\left[q^{k(X_n(t)+n)}\right] \frac{\zeta^k}{(1-q)\cdots(1-q^k)}$$

Deforming nested contours together (and keeping track of ~~exact~~ residue expansion)

Theorem [B-C '11]: For step initial data q-TASEP

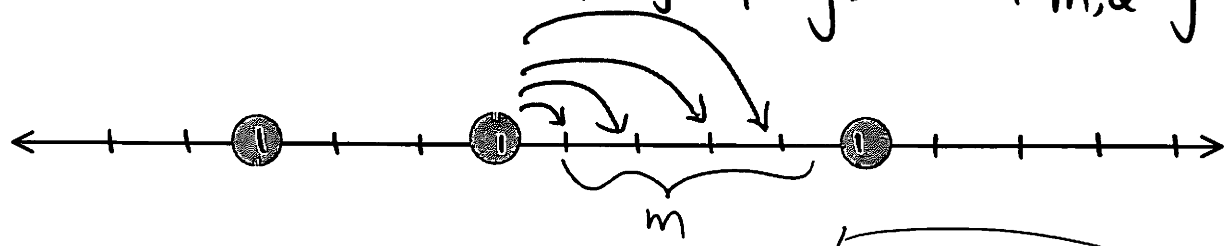
$$\mathbb{E}^{step}\left[e_q(\zeta q^{X_n(t)+n})\right] = \det(I + K_{\zeta,n,t})$$

explicit kernel.

This result is suitable for asymptotics such as necessary to prove $t^{1/3}$ TW GUE asymptotics (i.e. KPZ universality), as well as one-point statistics for things like KPZ equation.

(Parallel) Geometric discrete time q-TASEP [Borodin-C '13]:

$$\mathbb{P}(\text{jump}=j) = P_{m,\alpha}(j)$$



$q \in (0,1)$, $\alpha \in (0,1)$

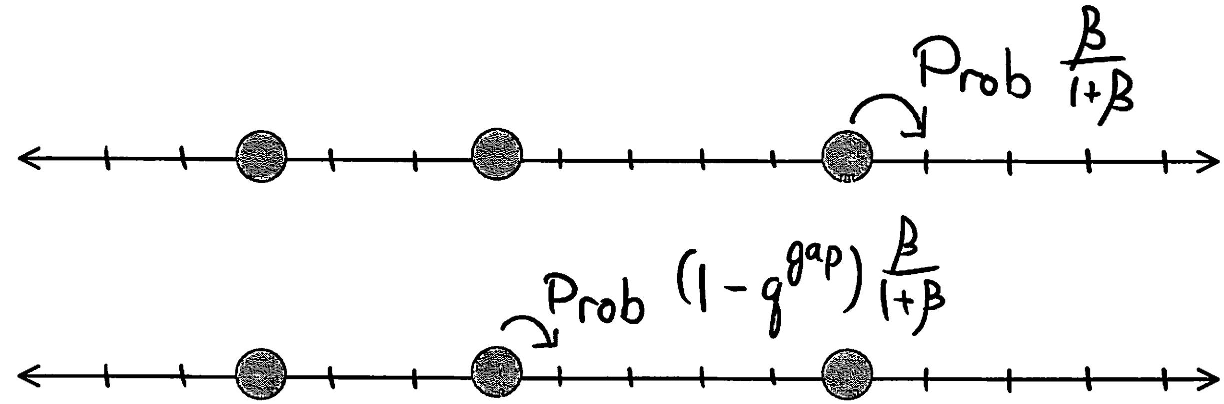
$$P_{m,\alpha}(j) = \alpha^j (\alpha;q)_{m-j} \frac{(q;q)_m}{(q;q)_{m-j}(q;q)_j} \mathbb{1}_{0\le j\le m} \qquad \left[(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j) \right]$$

At q=0 -> parallel geometric TASEP with blocking

[Warren-Windridge '09]

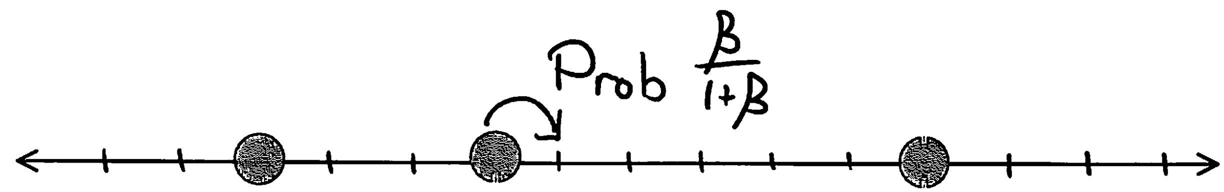
(Sequential) Bernoulli discrete time q-TASEP [Borodin-C '13]:

1)



$q \in (0,1)$, $\beta \in (0,\infty)$

2)



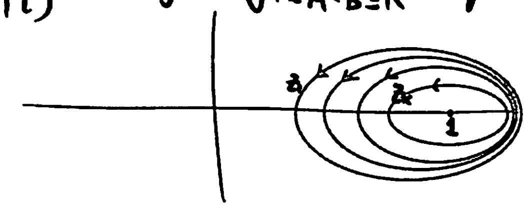
3) ...

At $q=0$ -> sequential Bernoulli TASEP [Borodin-Ferrari '08]

q-TASEP joint moments satisfy various many body systems

<u>Theorem [Borodin-C '13]</u>: For $n_1 \geq n_2 \geq \cdots \geq n_k > 0$

$$\mathbb{E}^{\text{step}}\left[\prod_{j=1}^{k} q^{x_{n_j}(t)+n_j}\right] = \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k} \oint \cdots \oint \prod_{1\leq A<B\leq k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{1}{(1-z_j)^{n_j}} \frac{f(q z_j)}{f(z_j)} \frac{dz_j}{z_j}$$



$$f(z) = \begin{cases} e^{tz}, & \text{Poissonian continuous } q\text{-TASEP} \\ \left(\frac{1}{(\alpha z; q)_\infty}\right)^t, & \text{Geometric discrete } q\text{-TASEP} \\ (1+\beta z)^t, & \text{Bernoulli discrete } q\text{-TASEP} \end{cases}$$

$(a;q)_n = (1-a)(1-qa)\cdots(1-q^{n-1}a)$

$$\mathbb{E}\left[\frac{1}{(\zeta q^{X_n(t)+n};q)_\infty}\right] = \mathbb{E}\left[\sum_{k=0}^{\infty} \frac{\zeta^k}{(1-q)\cdots(1-q^k)} q^{k(X_n(t)+n)}\right]$$

$\boxed{\text{justified for small } \zeta}$

$$= \sum_{k=0}^{\infty} \frac{\zeta^k}{(1-q)\cdots(1-q^k)} \mathbb{E}\left[q^{k(X_n(t)+n)}\right]$$

LHS: "q-Laplace transform of $q^{X_n(t)+n}$" identifies dist.

RHS: we have explicit formulas.

$\boxed{\text{Slide}}$

Theorem (Borodin-Corwin '11): q-TASEP step initial data, $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$

$$\mathbb{E}\left[\frac{1}{(\zeta q^{X_n(t)+n};q)_\infty}\right] = \det(I + K_{\zeta,n,t})_{L^2(C_1)}$$

$$K_{\zeta,n,t}(w,w') = \frac{1}{2\pi i}\int_{i\mathbb{R}+1/2} \frac{\pi}{\sin(-\pi s)}(-\zeta)^s \frac{g(w)}{g(q^s w)} \frac{ds}{q^s w - w'}$$

$$g(w) = \frac{e^{-tw}}{(w;q)_\infty^n}$$

q-Laplace trans

$$e_q(x) := \frac{1}{((1-q)x;q)_\infty}, \quad (a;q)_n = (1-a)(1-qa)\cdots(1-q^{n-1}a)$$

$$E_q(x) := (-(1-q)x;q)_\infty$$

HW As $q \to 1$, $e_q(x) = e^x$

HW

Prop: $f \in \ell^1(\mathbb{Z}_{\geq 0})$, for $z \in \mathbb{C} / \{q^{-m}\}_{m \geq 0}$ define

$$\hat{f}^q(z) := \sum_{n=0}^\infty \frac{f(n)}{(zq^n;q)_\infty} \quad e_q \qquad z \rightsquigarrow (1-q)z$$

E_q

Then $$f(n) = -q^n \frac{1}{2\pi i} \int (q^{n+1}z;q)_\infty \, \hat{f}^q(z)\,dz$$

contains $\{z = q^{-m} : 0 \leq m \leq n\}$.

- linearity, scaling, shift, trans. under q-der. / int.

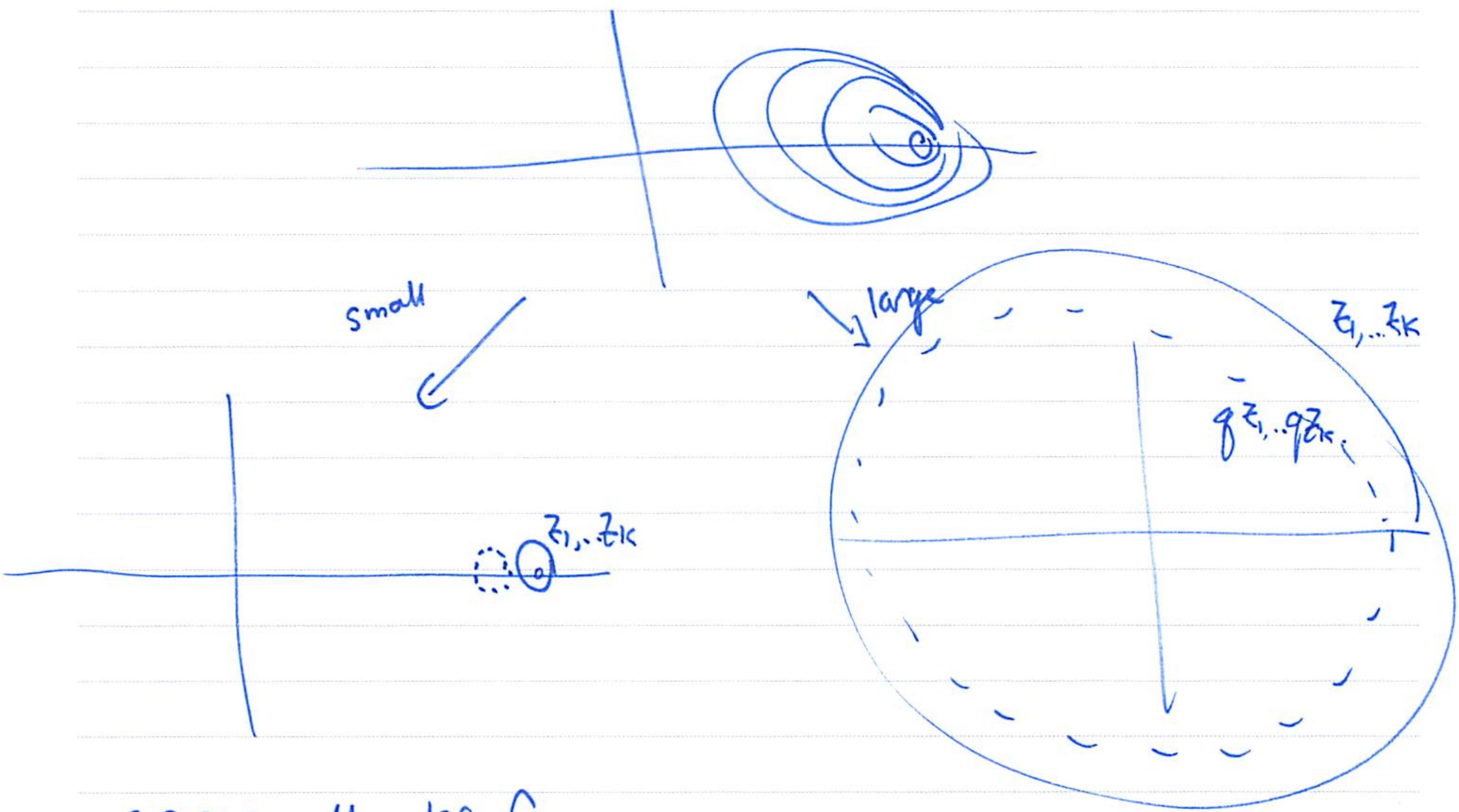
q-prod / convolution. , useful in solving q-diff. equations.

Nested contours to
Fredholm determinants.

Let $\mu_k := \dfrac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \int \cdots \int \prod_{1 \le A < B \le k} \dfrac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \dfrac{1}{(1-z_j)^{N_j}} \dfrac{f(q z_j)}{f(z_j)} \dfrac{dz_j}{z_j}$

$\mu_k(N_1, \ldots N_k)$

Contours become cumbersome as k grows:

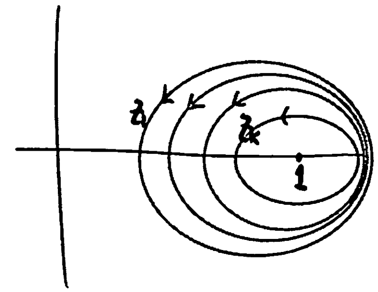


crosses all poles from
$\prod_{A<B} \dfrac{z_A - z_B}{z_A - q z_B}$ term

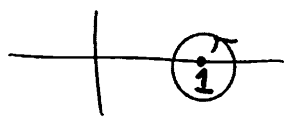
crosses pole at $z_j = 0$.

<u>Proposition [Borodin-C '11]</u>: For "nice" $f(x)$

$$\mathcal{M}_k(\vec{n}) := \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k} \oint \cdots \oint \prod_{1\le A<B\le k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{1}{(1-z_j)^{n_j}} \frac{f(qz_j)}{f(z_j)} \frac{dz_j}{z_j}$$



$$= \sum_{\substack{\lambda \vdash k \\ \lambda = 1^{m_1} 2^{m_2} \cdots}} \frac{(1-q)^k}{m_1! m_2! \cdots} \cdot \frac{1}{(2\pi i)^{\ell(\lambda)}} \oint \cdots \oint \det\left[\frac{1}{q^{\lambda_i} - w_j}\right]_{i,j=1}^{\ell(\lambda)} E_{\vec{n}}(w_1, qw_1, \ldots, q^{\lambda_1 - 1} w_1, w_2, \ldots, q^{\lambda_2 - 1} w_2, \ldots, w_{\ell(\lambda)}, \ldots, q^{\lambda_{\ell(\lambda)}-1} w_{\ell(\lambda)})\, dw$$



where

$$E_{\vec{n}}(z_1, \ldots, z_k) = \prod_{j=1}^{k} \frac{f(qz_j)}{f(z_j)} \cdot \sum_{\sigma \in S_k} \prod_{k \ge B > A \ge 1} \frac{z_{\sigma(A)} - q z_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}} \prod_{j=1}^{k} \frac{1}{(1 - z_{\sigma(j)})^{n_j}}$$

Slide

(need much less)

To prove

Prop: Assume $f_\Pi(z)$ holomorphic $(f_\Pi(z)=e^{zt})$ then

$$\mu_f(q^{N_1\ldots N_k}) = \sum_{\substack{\lambda\vdash k\\ \lambda=1^{m_1}2^{m_2}\ldots}} \frac{(1-q)^k}{m_1!m_2!\ldots} \frac{1}{(2\pi i)^{\ell(\lambda)}} \oint dw_1 \cdots \oint dw_\ell \cdot \det\left[\frac{1}{w_i q^{\lambda_i} - w_j}\right]_{i,j=1}^{\ell}$$

$$\times E(w_1, qw_1, \ldots, q^{\lambda_1-1}w_1, w_2, \ldots, q^{\lambda_2-1}w_2, \ldots w_{\ell(\lambda)}, \ldots q^{\lambda_{\ell(\lambda)}-1}w_{\ell(\lambda)})$$

$$E(z_1,\ldots z_k) = \prod_{j=1}^{k} \frac{f_\Pi(qz_j)}{f_\Pi(z_j)} \sum_{\sigma\in S_k} \prod_{A>B} \frac{z_{\sigma(A)} - qz_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}} \prod_{j=1}^{k} \frac{1}{(1-z_{\sigma(j)})^{N_j}}$$

~~Rk~~ E important object - eigenfunction for many body system, and the integration is related to completeness.

If $N_j \equiv N$, E simplifies

$$E(z_1,\ldots,z_k) = \prod_{j=1}^{k} \frac{f_\Pi(qz_j)}{f_\Pi(z_j)} \frac{1}{(1-z_j)^N} \cdot \underbrace{\sum_{\sigma\in S_k} \prod_{A>B} \frac{z_{\sigma(A)} - qz_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}}}_{C_k}$$

Lem: $C_k = \dfrac{(1-q)(1-q^2)\cdots(1-q^k)}{(1-q)^k} := k_q!$

Lem: $k_q! \to k!$ as $q\to 1$.

So $\mu_k(N) = (1-q)\cdots(1-q^k) \cdot \sum_{\lambda \vdash k} \frac{1}{m_1! m_2! \cdots} \frac{1}{(2\pi i)^{\ell(\lambda)}} \int \cdots \int \det\left[\frac{1}{w_i q^{\lambda_i} - w_j}\right]$

$$\prod_{j=1}^{\ell(\lambda)} \frac{f(q^{\lambda_j} w_j)}{f(w_j)} \cdot \left(\frac{1}{(w_j;q)_{\lambda_j}}\right)^N$$

$$\frac{\mu_k(N) \cdot \zeta^k}{(1-q)\cdots(1-q^k)} = \cancel{(1-q)\cdots(1-q^k)} \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \sum_{\lambda_1=1}^{\infty} \cdots \sum_{\lambda_\ell=1}^{\infty} \mathbb{1}\{\lambda_1+\cdots+\lambda_\ell = k\} \oint \frac{dw_1}{2\pi i} \cdots \int \frac{dw_\ell}{2\pi i}$$

$$\det\left[\tilde{K}_{\zeta,n,t}(\lambda_i, w_i ; \lambda_j, w_j)\right]_{i,j=1}^{\ell}$$

$$\tilde{K}_{\zeta,n,t}(\lambda, w ; \lambda', w') = \zeta^{\lambda} \frac{f(q^{\lambda} w)}{f(w)} \left(\frac{1}{(w;q)_\lambda}\right)^N \cdot \frac{1}{w q^{\lambda} - w'}$$

$$G(\zeta) = \sum_{k=0}^{\infty} \frac{\mu_k(N) \zeta^k}{(1-q)\cdots(1-q^k)} = 1 + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \sum_{\lambda_1=1}^{\infty} \cdots \sum_{\lambda_\ell=1}^{\infty} \int_{C_1} \cdots \int \det[\tilde{K}_{\zeta,n,t}]$$

$$= \det(I + \tilde{K}_{\zeta,n,t})_{L^2(\mathbb{Z}_{>0} \times C_1)}$$

$$= 1 + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \int \cdots \int \det(K_{\zeta,n,t}(w, w'))$$

$$=: \det(I + K)_{L^2(C_1)}$$

so with $K_\zeta(w,w') = \sum_{\lambda=1}^{\infty} \zeta^\lambda \cdot g_{w,w'}(q^\lambda)$

$$g_{w,w'}(q^\lambda) = \frac{\Pi(q^\lambda w)\,(q^\lambda w;q)_\infty^N}{\Pi(w)\,(w;q)_\infty^N} \cdot \frac{1}{wq^\lambda - w'}$$

This is not yet ready for $q \nearrow 1$ asymptotics since the series for K has termwise limit which is divergent as $q \to 1$.

Instead, should sum the series and then take $q \nearrow 1$.

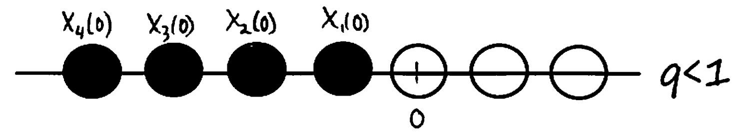
Identity "Mellin-Barnes representation"

$$\sum_{\lambda=1}^{\infty} g(q^\lambda)\,\zeta^\lambda = \frac{1}{2\pi i} \int_{\substack{\text{contour}\\ \text{contains}\\ 1,2,\ldots}} \frac{\pi}{\sin(-\pi s)} (-\zeta)^s\, g(q^s)\, ds$$

Valid under some conditions on g and contour.

Taking contour away from $1,2,\ldots$  this is now ammenable to asymptotic analysis.

Theorem (Borodin-C '11): q-TASEP step initial data



$$\mathbb{E}\left[\frac{1}{(\zeta q^{x_n(t)};q)_\infty}\right] = \det\left(I + K_\zeta^{q\text{-TASEP}}\right)$$

where

$$K_\zeta(w,w') = \frac{1}{2\pi i}\int \frac{\pi}{\sin(\pi s)}(-\zeta)^s \frac{g(w)}{g(q^s w)} \frac{1}{q^s w - w'}\, ds$$

$$g(w) = \frac{e^{-tw}}{(w;q)_\infty^n}, \quad (a;q) = (1-a)(1-qa)(1-q^2a)\cdots$$

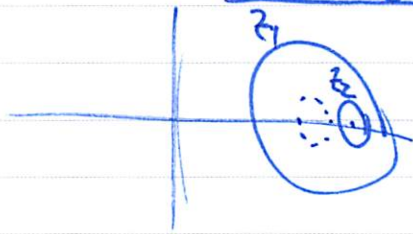
q-Laplace transform [Hahn '49] *identifies* $x_n(t)$ *distribution.*

Good for asymptotics (Patrik's talk)!

Proof of nested contour expansion result.

See yellow notes pg 23 - 25

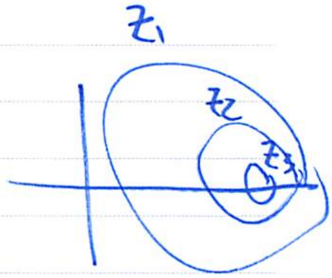
Example $k=2$: $\dfrac{z_1-z_2}{z_1-qz_2}$



pole $z_1 = qz_2$ so pick residue at $z_1 = qz_2$

and remaining integral.

Example $k=3$: $\dfrac{z_1-z_2}{z_1-qz_2}\cdot\dfrac{z_1-z_3}{z_1-qz_3}\cdot\dfrac{z_2-z_3}{z_2-qz_3}$



Shrink z_2, cross pole at $z_2 = qz_3$.

Residue term and integral term

$$\frac{z_1-qz_3}{z_1-q^2z_3}\cdot\frac{z_1-z_3}{\cancel{z_1-qz_3}}\cdot\cancel{qz_3-z_1}$$

Apparent pole at $z_1 = qz_3$ not present, only pole at $z_1 = q^2 z_3$.

We decompose into (geometric) strings of residues ~~at geometric~~

Generally shrink $z_{k-1}, \dots z_1$ and express integral ~~over~~ as sum over integral in which some variables are along C_1 ($+\theta$) and others have taken residues on geo. Strings. ~~Var~~

Index strings by partitions $\lambda \vdash k$ ~~and~~ example, $\ell(\lambda)$, $\lambda = 1^{m_1} 2^{m_2} \dots$

$$z_{i_1} = q z_{i_2} = q^2 z_{i_3} = \dots = q^{\lambda_1 - 1} z_{i_{\lambda_1}} \quad i_1 < i_2 < \dots < i_{\lambda_1}$$

$$z_{j_1} = \quad \dots \quad = q^{\lambda_2 - 1} z_{j_{\lambda_2}} \quad j_1 < j_2 < \dots < j_{\lambda_2}$$

$$\vdots$$

Rather complex... rewrite in more canonical way:

$$(z_{i_1}, \dots, z_{i_{\lambda_1}}) \xmapsto{\text{Change var.}} (y_{\lambda_1}, \dots, y_1)$$

$$(z_{j_1}, \dots, z_{j_{\lambda_2}}) \longmapsto (y_{\lambda_1+\lambda_2}, \dots y_{\lambda_1+1})$$

$$\vdots$$

If λ has some elements of multiplicity we can interchange levels. Lets count them all, but divide each by multiplicity.

Let $\sigma \in S_k$: $(z_1, \dots, z_k) = (y_{\sigma(1)}, \dots, y_{\sigma(k)})$

— Not all σ arise —

Call $y_1 = w_1$, $y_{\lambda_1+1} = w_2, \dots$

$$\mu_k(\vec{n}) = \sum_{\lambda \vdash k} \frac{1}{m_1! m_2! \cdots} \frac{1}{(2\pi i)^\ell} \int \frac{dw_1}{\cancel{2\pi i}} \cdots \int dw_\ell \; \cancel{\mathrm{Res}}$$

$$\mathrm{Res} \sum_{\sigma \in S_k} \prod_{A<B} \frac{y_{\sigma(A)} - y_{\sigma(B)}}{y_{\sigma(A)} - q y_{\sigma(B)}} \prod_{j=1}^{k} \frac{1}{(1-y_{\sigma(j)})^{n_j}} \frac{f(q y_{\sigma(j)})}{f(y_{\sigma(j)})} \frac{1}{y_{\sigma(j)}}$$

$y_{\lambda_1} = q y_{\lambda_1 - 1} = \cdots q^{\lambda_1 - 1} y_1$, $y_1 \leftarrow w_1$

$y_{\lambda_1+\lambda_2} = \cdots = q^{\lambda_2 - 1} y_{\lambda_1+1}$, $y_{\lambda_1+1} \leftarrow w_2$

initially not all of S_k, but other terms have zero residue.

$$\prod_{A<B} \frac{y_{\sigma(A)} - y_{\sigma(B)}}{y_{\sigma(A)} - q y_{\sigma(B)}} = \prod_{A \neq B} \frac{y_A - y_B}{y_A - q y_B} \cdot \prod_{A>B} \frac{y_{\sigma(A)} - q y_{\sigma(B)}}{y_{\sigma(A)} - y_{\sigma(B)}}$$

Can see — has residue — has no residue.

$$\mathrm{Res}\Big(\prod_{A \neq B} \Big) \rightsquigarrow \det\left(\frac{1}{w_i q^{\lambda_i} - w_j} \right)$$

Rest gives $E(w_1, \dots, q^{\lambda_1 - 1} w_1, \dots)$ $\square$

q-TASEP satisfies:

$$\begin{cases} d\, q^{x_n(t)+n} = (1-q)\nabla q^{x_n(t)+n}\, dt + q^{x_n(t)+n}\, d\underset{\substack{\uparrow \\ \text{Martingale}}}{M_n}(t) \\ q^{x_n(0)+n} \equiv 1 \text{ (step)}, \quad q^{x_0(t)+0} \equiv 0 \quad (x_0 = \infty) \end{cases}$$

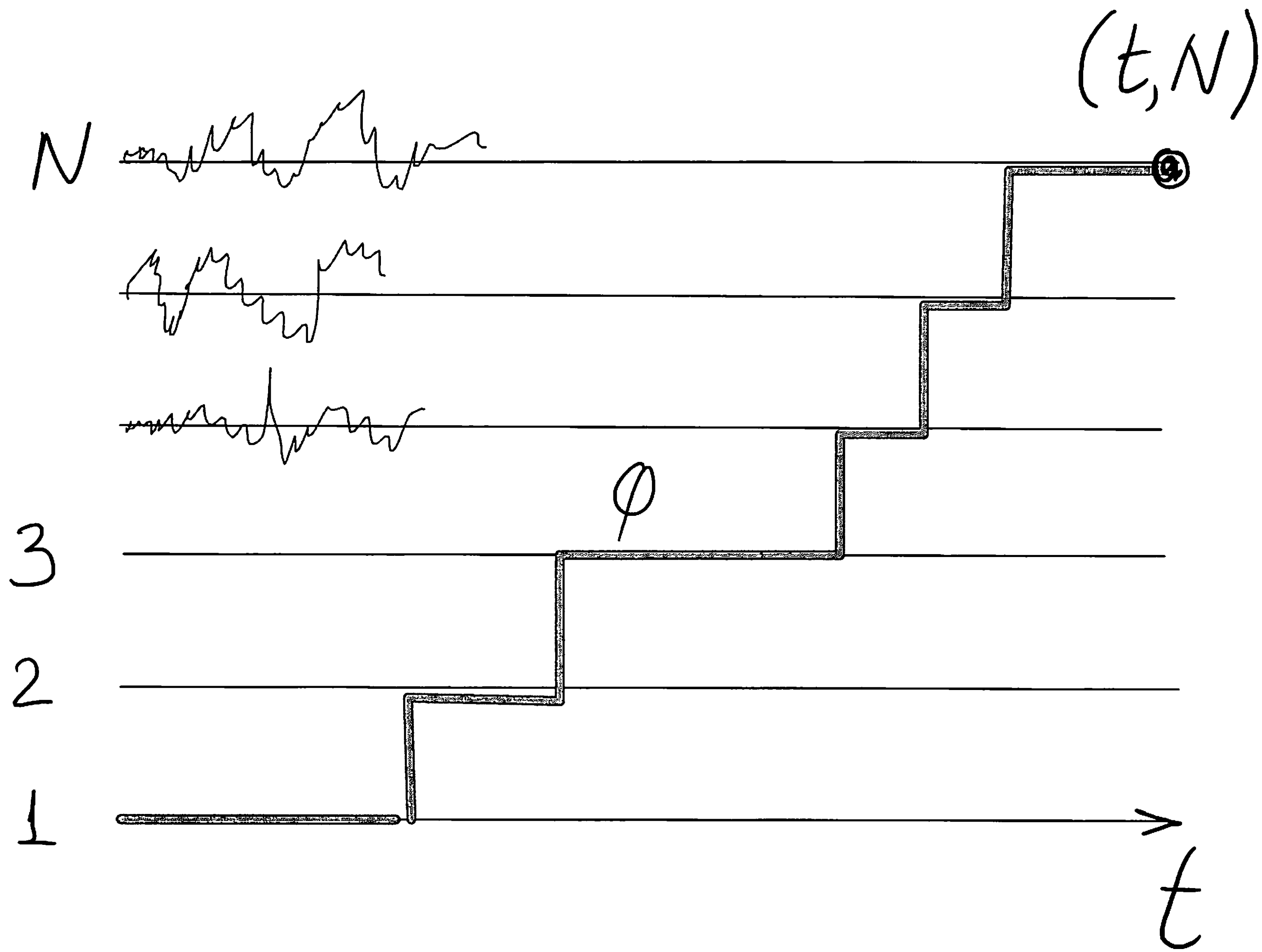
<u>Theorem [Borodin-C '11]</u>: For q-TASEP with step init. cond.

scale $q = e^{-\varepsilon}$, $t = \varepsilon^{-2}\tau$, $X_n(t) = \varepsilon^{-2}\tau - (n-1)\varepsilon^{-1}\log\varepsilon^{-1} - \varepsilon^{-1}F_\varepsilon(\tau, n)$

and call $Z_\varepsilon(\tau, n) = \exp\left\{-\frac{3\tau}{2} + F_\varepsilon(\tau, n)\right\}$. Then as $\varepsilon \searrow 0$,

$Z_\varepsilon(\cdot,\cdot) \Rightarrow Z(\cdot,\cdot)$ where Z solves the <u>semi-discrete SHE</u>:

$$\begin{cases} dZ(\tau, n) = \nabla Z(\tau, n)\, d\tau + Z(\tau, n)\, dB_n(\tau) \quad \leftarrow \text{ind. BM's} \\ Z(0, n) = \mathbb{1}_{n=0}, \quad Z(\tau, 0) \equiv 0 \end{cases}$$



(t,N)
N
φ
3
2
1
t

Vignette 1: Polymer replica method limit

(From duality) we know q-TASEP dynamics imply

$$d\, q^{x_n(t)+n} = (1-q)\nabla q^{x_n(t)+n}\, dt + q^{x_n(t)+n}\, d\underset{\substack{\uparrow \\ \text{martingale}}}{M_n(t)}$$

$$q^{x_0(t)+0} \equiv 0 \qquad \text{zero b/c } (x_0 = \infty)$$

$$q^{x_n(0)+n} \equiv 1 \qquad \text{step initial data}$$

Def^n: $Z : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ solves the semidiscrete stochastic heat equation (SHE) with $Z_0(n)$ initial data if:

$$dZ(\tau;n) = \nabla Z(\tau;n)\, d\tau + Z(\tau;n)\, d\underset{\substack{\uparrow \\ \text{ind BM's}}}{B_n(\tau)}$$

$$Z(\tau;0) \equiv 0$$

$$Z(0;n) = Z_0(n)$$

Theorem [Borodin-C]: For q-TASEP with step initial condition set

$$q = e^{-\varepsilon},\quad t = \varepsilon^{-2}\tau,\quad x_n(t) = \varepsilon^{-2}\tau - (n-1)\varepsilon^{-1}\log\varepsilon^{-1} - \varepsilon^{-1} f_\varepsilon(\tau;n)$$

and call $Z_\varepsilon(\tau;n) = e^{\frac{-3\tau}{2}} e^{f_\varepsilon(\tau;n)}$. Then, as a space-time process, $Z_\varepsilon(\tau;n)$ converges weakly to $Z(\tau;n)$ with $Z_0(n) = \mathbb{1}_{n=1}$

Sketch of stochastic analysis proof:

- Initial data: $Z_\varepsilon(0;n) = \varepsilon^{n-1} e^{\varepsilon n} \longrightarrow \mathbb{1}_{n=1}$

- Dynamics: $dF_\varepsilon(\tau;n) = F_\varepsilon(\tau;n) - F_\varepsilon(\tau - d\tau;n)$

$$= \varepsilon^{-1} d\tau - \varepsilon\left[X_n(\varepsilon^{-2}\tau) - X_n(\varepsilon^{-2}\tau - \varepsilon^{-2}d\tau)\right]$$

q-TASEP jump rate in rescaled time variables is

$$1 - q^{X_{n-1}(t) - X_n(t) - 1} = 1 - \varepsilon e^{F_\varepsilon(\tau;n-1) - F_\varepsilon(\tau;n)} + O(\varepsilon^2)$$

so in time $\varepsilon^{-2} d\tau$ (by convergence of poisson pt. proc. to BM)

$$\varepsilon\left[X_n(\varepsilon^{-2}\tau) - X_n(\varepsilon^{-2}\tau - \varepsilon^{-2}d\tau)\right] \approx \varepsilon^{-1} - e^{F_\varepsilon(\tau;n-1) - F_\varepsilon(\tau;n)} d\tau - \underset{B_n(\tau) - B_n(\tau - d\tau)}{\overset{\shortparallel}{dB_n(\tau)}}$$

Thus we see $dF_\varepsilon(\tau;n) \approx e^{F_\varepsilon(\tau;n-1) - F_\varepsilon(\tau;n)} d\tau + dB_n(\tau) + O(1)$

Exponentiating and applying Itô's Lemma gives

$$d e^{F_\varepsilon(\tau;n)} = \left(\frac{1}{2} e^{F_\varepsilon(\tau;n)} + e^{F_\varepsilon(\tau;n-1)}\right) d\tau + e^{F_\varepsilon(\tau;n)} dB_n + O(1)$$

or in terms of Z_ε as in statement of theorem

$$dZ_\varepsilon(\tau;n) = \nabla Z_\varepsilon(\tau;n)\, d\tau + Z_\varepsilon(\tau;n)\, dB_n + O(1)$$

and as $\varepsilon \to 0$ we recover the claimed formula $\square$

start building a picture

q-TASEP $\quad q^{x_n(t)+n}$

$\downarrow$

semi discrete SHE $\quad Z(\tau;n)$

Implications:

- Weak convergence as $q \nearrow 1$ implies q-Laplace trans. of $q^{x_n(t)+n} \rightsquigarrow$ Laplace trans. of $Z(\tau;n)$

hence we rigorously prove formula:

$$\boxed{\mathbb{E}^{\mathbb{1}_{n=1}}\left[e^{-\varsigma Z(\tau;n)}\right] = \det\left(I + \tilde{K}_{\varsigma,\tau,n}\right)}$$

which identifies distribution of ~~...~~ $Z(\tau;n)$.

~~Remark: Could~~ **Corollary** [Borodin-C, Borodin-C-Ferrari]

set $F(\tau;n) = \frac{3\tau}{2} + \log Z(\tau;n)$ $\left(= \lim_{\varepsilon\to 0} F_\varepsilon(\tau;n)\right)$ then for all $\kappa > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{F(\kappa n;n) - n\bar{f}_\kappa}{n^{1/3}} \le r\right) = F_{GUE}\left(\left(\frac{\bar{g}_\kappa}{2}\right)^{-1/3} r\right)$$

$\bar{f}_\kappa = \inf_{t>0}\left(\kappa t - \Psi(t)\right) \quad$ arg inf $\bar{t}_\kappa \quad \Psi(t) = (\log\Gamma)'(t)$

$\bar{g}_\kappa = -\Psi''(\bar{t}_\kappa)$.

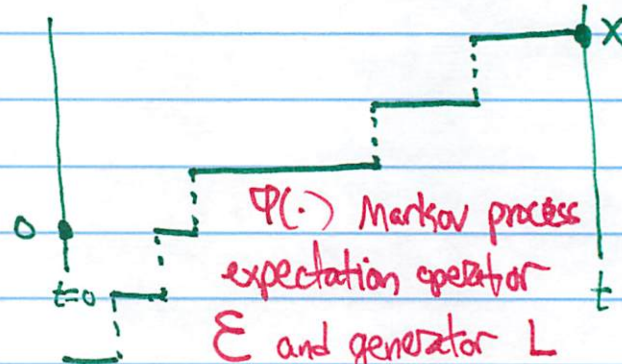
- $\bar{f}_\kappa$ conj O'Connell-Yor, proved O'Connell-Moriarty
- Seppäläinen-Valko proved upper bound of $O(t^{2/3})$ on variance of $F(\kappa n,n)$
- Similar result should hold for q-TASEP directly

Limit of q-TASEP duality, many body systems and moment formulas
(either take limit or use replica method)

Aside of Feynman-Kacs representation:

- Consider homogeneous Markov process generator L and deterministic potential V
- Goal: solve $\frac{d}{dt} Z(t,x) = (LZ)(t,x) + V(t,x)Z(t,x) \;;\; Z(0,x) = Z_0(x)$
- Probabilistic interpretation

  for $L = \nabla$



$\varphi(\cdot)$ Markov process expectation operator $\mathcal{E}$ and generator L run backwards in time from t to 0.

- $p(t,x) = \mathcal{E}^{\varphi(t)=x}\left[\mathbb{1}_{\varphi(0)=0}\right]$
- For potential $V \equiv 0$ by superposition/linearity of expectation

$$Z(t,x) = \mathcal{E}^{\varphi(t)=x}\left[Z_0(\varphi(0))\right]$$

- When V is turned on, Duhamel's principle shows

$$Z(t,x) = \int_{-\infty}^{\infty} p(t,x-y)Z_0(y)\,dy + \int_0^t ds \int_{-\infty}^{\infty} dy\, p(t-s,y-x)Z(s,y)V(s,y)$$

- Apply identity multiple times yields series which sums to

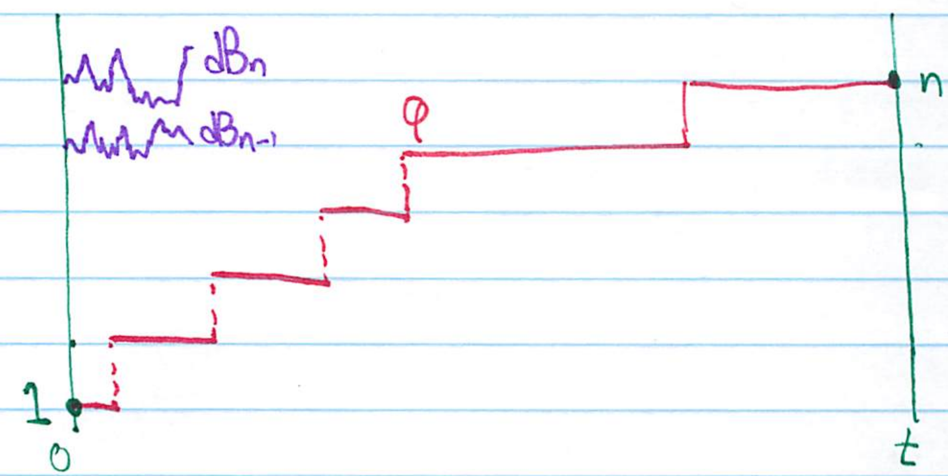
Feynman-Kacs representation $\rightarrow$ $$Z(t,x) = \mathcal{E}^{\varphi(t)=x}\left[\exp\left\{\int_0^t V(s,\varphi(s))\,ds\right\}\cdot Z_0(\varphi(0))\right]$$

This V is random, care is needed in defining stochastic integrals.

Generally leads to correction in exponential which goes by $\overset{\text{name}}{\wedge}$ Wick or Girsanov correction.

- For $Z(\tau;n)$, $V(\tau;n) = dB_n(\tau)$, $L = \nabla$ so

$$Z(\tau;n) = \mathcal{E}^{\varphi(\tau)=n}\left[\exp\left\{\int_0^{\tau}\left(dB_{\varphi(s)}(s) - \frac{ds}{2}\right)\right\} Z_0(\varphi(0))\right]$$



O'Connell-Yor polymer partition function

Using ~~the~~ these path integrals we can show (like last week)

that $\bar{Z} = \mathbb{E}\left[\prod_{i=1}^{k} Z(\tau;n_i)\right]$ solves delta Bose gas

$$\frac{d}{dt}\bar{Z}(\tau,\vec{n}) = H\bar{Z}(\tau;\vec{n}), \quad H = \sum_{i=1}^{k}\nabla_i + \sum_{i<j}\mathbb{1}_{n_i=n_j}$$

- This semi-discrete delta Bose gas is integrable and equivalent to solving free evol^{n} eq^{n} with k-1 two body boundary cond. of form

$$\boxed{(\nabla_i - \nabla_{i+1} - 1)\, u\big|_{n_i = n_{i+1}} = 0.}$$

- Either as limit of q-TASEP moments or directly from above system we get nested contour integral formulas for $Z(t;\vec{n})$.

Could we have found Laplace transform from these moments?

- No! They grow like $\mathbb{E}[Z(\tau,n)^k] \approx e^{ck^2}$.

So RHS of below "equality" is divergent

$$\mathbb{E}\left[e^{-\varsigma Z(\tau;n)}\right] \underset{\text{illegal interchange}}{\text{"="}} \sum \frac{(-\varsigma)^k}{k!}\, \mathbb{E}\left[Z(\tau;n)^k\right]$$

Could proceed formally (shadowing q version) and, after "summing" the divergent series would get right answer.

Thus, polymer replica method is shadow of rigorous approach explained earlier.

There is a further limit to continuum SHE / polymer and we get rigorous proof of Laplace transform formula there (earlier proof by [Amir-Corwin-Quastel]).

add

q-TASEP $\quad q^{x_n(t)+n}$

$\downarrow$

semi discrete SHE $\quad Z(\tau; n)$

$\downarrow$

add to picture

continuum SHE $\quad Z(t; x)$

Slide : Theorem to prove

Two versions of ASEP

- Occupation process $\eta \in \{0,1\}^{\mathbb{Z}}$, $\cancel{(L^{occ} f)(\eta) = \sum_{y\in\mathbb{Z}} p\eta_y(1-\eta_{y+1})}$

$$(L^{occ} f)(\eta) = \sum_{y\in\mathbb{Z}} \left(p\eta_y(1-\eta_{y+1}) + q(1-\eta_y)\eta_{y+1}\right)\left[f(\eta^{y,y+1}) - f(\eta)\right]$$

$$N_y(\eta) = \sum_{x\geq y} \eta_x$$

- k particle process. $\{y_1 < \dots < y_k\}$.

$$\left(L^{k,part} g\right)(\vec{y}) = \sum_{\text{clusters}} p\left(g(\vec{y}_i^-) - g(\vec{y})\right) + q\left(g(\vec{y}_j^+) - g(\vec{y})\right)$$

$\cancel{y_i \quad y_{i+1} \quad \dots \quad y_j}$

$i \qquad j$

Def^n: $\eta(\cdot) \in X$, $y(\cdot) \in Y$ ind^t markov dual wrt $H : X \times Y \to \mathbb{R}$

if $\forall \eta, y, t$, $\mathbb{E}^\eta H(\eta(t), y) = \mathbb{E}^y H(\eta, y(t))$.

Theorem: Duality for

$$H(\eta, y) = \prod_{i=1}^{k} \tilde{Q}_{y_i}(\eta) = \prod_{i=1}^{k} \tau^{N_{y_i-1}(\eta)} \eta_{y_i}$$

also $G(\eta, y) = \prod_{i=1}^{k} \tau^{N_{y_i}(\eta)}$.

Remarks

- $p = q$ H duality $\rightsquigarrow$ SSEP cor. F^{n}.
- H duality $\rightsquigarrow$ Schutz '97 ($\exists$ generalizations)
- G duality at $k=1$ is Gartner's transform.

Proof directly from showing $L^{occ} H(\eta, y) = L^{k,part} H(\eta, y)$.

Proof of formula from earlier much like q-TASEP.

1. True evol^{n} eq^{n}. (uniqueness more subtle than q-TASEP)
   Duality implies $h(t;\vec{y}) := \mathbb{E}^{\eta}[H(\eta(t),\vec{y})]$.
   solves $\frac{d}{dt} h(t;\vec{y}) = L^{k,part} h(t;\vec{y})$.

2. Can rewrite as free evolution eq^{n} with $k=1$ ~~free~~ boundary cond.

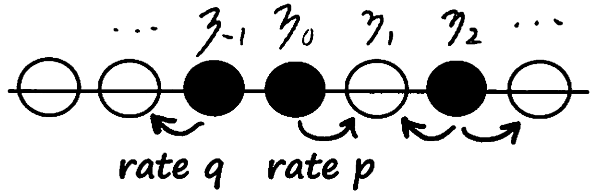
3. Check directly.

One hopes to the compute asymptotics. Suitable combinations of these observables gives formula for $\mathbb{E}[\tau^{kN_y(t)}]$ and this leads to det. formula.

– Slides –

Basic reason that all these models turned out to be accessible is the existence of a large family of observables whose averages are explicit.

<u>Example 2: ASEP</u>



$0<p<q<1, \quad p+q=1$

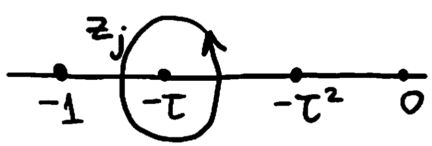
Set $\tau = p/q < 1$, $\quad N_y(t) = \sum_{x \ge y} \eta_x(t)$, $\quad \tilde{Q}_y = \dfrac{\tau^{N_y} - \tau^{N_{y-1}}}{\tau - 1}$.

<u>*Theorem [B-C-Sasamoto, 2012]*</u> *For ASEP with step initial data* $\{x_n(0) = -n\}_{n\ge 1}$

$$\mathbb{E}\left[\tilde{Q}_{y_1}(t)\cdots\tilde{Q}_{y_k}(t)\right] = \frac{\tau^{k(k-1)/2}}{(2\pi i)^k}\cdot\oint\cdots\oint\prod_{A<B}\frac{z_A - z_B}{z_A - \tau z_B}$$

$(y_1 < y_2 < \ldots < y_k)$

$$* \prod_{j=1}^{k} e^{-\frac{z_j (p-q)^2 t}{(1+z_j)(p+q z_j)}} \left(\frac{1+z_j/\tau}{1+z_j}\right)^{y_j+1} \frac{dz_j}{\tau + z_j}$$



Suitable combinations of $\mathbb{E}^{step}\left[\prod_{i=1}^{k} \tau^{N_{x_i - 1}(\eta(t))} \eta_{x_i}(t)\right]$ yields $\mathbb{E}^{step}\left[\tau^{n N_x(\eta(t))}\right]$

<u>Theorem [Borodin-C-Sasamoto '12]</u>: For step initial condition ASEP with $a_i \equiv 1$ and $p<q$ (hence $\tau = p/q < 1$, $\gamma = q-p > 0$)

$$\mathbb{E}\left[\frac{1}{(\zeta \tau^{N_x(\eta(t))};\tau)_\infty}\right] = \det(I + K_\zeta)_{L^2}$$

Mellin Barnes type

$$\mathbb{E}\left[\frac{1}{(\zeta \tau^{N_x(\eta(t))};\tau)_\infty}\right] = \frac{1}{(\zeta;\tau)_\infty}\det(I - \zeta \tilde{K})_{L^2}$$

Cauchy type

$$K_\zeta(w,w') = \frac{1}{2\pi i}\int \frac{\pi \, ds}{\sin(-\pi s)} (-\zeta)^s \frac{g(w)}{g(\tau^s w)} \frac{1}{w' - \tau^s w}$$

$$g(w) = e^{\gamma t \frac{\tau}{\tau + w}} \left(\frac{\tau}{\tau + w}\right)^x$$

$$\tilde{K}(w,w') = \frac{e^{\varepsilon'(w)t}}{\tau w - w'}$$

$$\varepsilon'(w) = -\frac{1}{q}\frac{w\gamma^2}{(1+w)(\tau + w)}$$

Corollary [Tracy-Widom '09, Borodin-C-Sasamoto '12]:

$$\lim_{t\to\infty} \mathbb{P}^{step}\left(\frac{N_0(\eta(t/\gamma)) - t/4}{t^{1/3}} \geq -r\right) = F_{GUE}(2^{4/3} r)$$

Recovering the celebrate Tracy-Widom / Johansson result.

Remarks:

- Mellin Barnes Fredhold det. new and easy for asymptotics
- Inversion of Cauchy Fredholm det. equivalent to initial det. in [Tracy-Widom '09]
- Completely parallel to q-TASEP formulas

Coordinate approach of [Tracy-Widom '08-'09]:

- Study k particle ASEP and use coordinate Bethe ansatz (cf. [Schutz '97] for k=2) to compute Green's functions.
- Manipulate formulas to extra one-point marginal.
- Approach step initial condition by taking k to infinity and observe an integral transform of Cauchy type Fredholm det.
- Functional analysis to rework for asymptotic analysis.

Using k-particle Green's functions can write solution of duality ODEs as k! k-fold contour integrals [Imamura-Sasamoto '11]. Equivalence to nested formula is non-trivial.