KPZ equation and universality class

June 3, 2013, Rennes FR

(Ivan Corwin)

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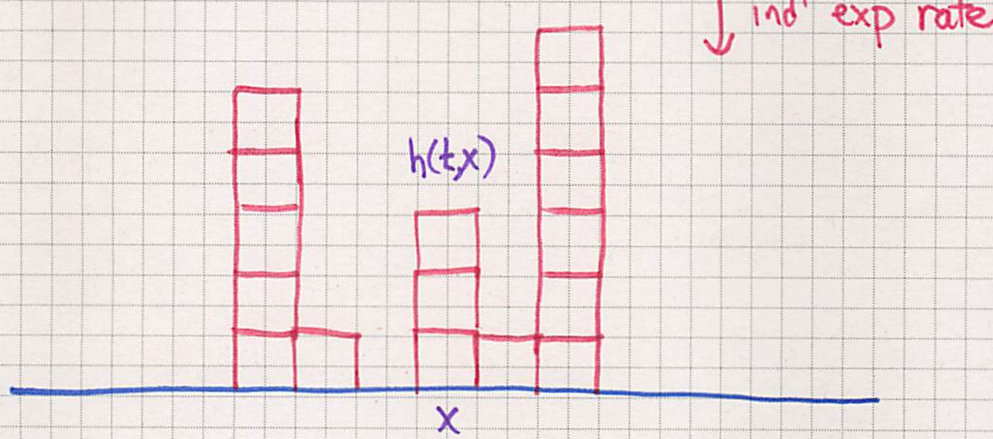
- Trivial example: Random deposition:

Video



- Limit shape and fluctuations:

$W_{x,i}$: waiting time block i in column x to fall – iid exp.

$$\{h(t,x) < n\} = \{\sum_{i=1}^{n} W_{x,i} > t\}$$

Shape LLN in each column

$$\lim_{t\to\infty} \frac{h(t,x)}{t} = 1$$

Fluctuations CLT in each column ⇝ ind^t $N(x)$ No spatial correlation

$$\frac{h(t,x) - t}{\sqrt{t}} \xrightarrow{(d)} N(x)$$

- Gaussian Universality class: $(1/2, 0)$

Under $t^{1/2}$ fluct. scaling and t^0 spatial scaling
Converge to Gaussian.

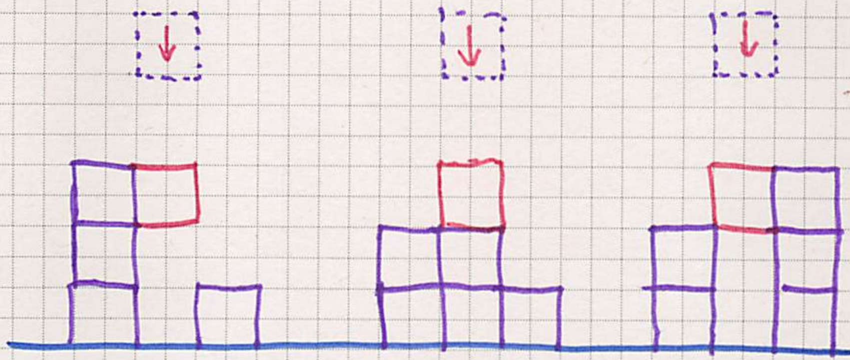
$$\varepsilon^{1/2}\{h(\varepsilon^{-1}t, \varepsilon^{0}x) - \bar{h}\} \Longrightarrow \mathfrak{h}(t,x) \text{ ind}^t \text{ BM's}$$

Varying distribution of $W_{x,i}$ does not change result.

- Ballistic deposition: Sticky blocks

indt exp rate

Video



Sticky blocks creates spatial correlation

- If started flat, it is known that there is an asymptotic growth rate, but exact value is not known.

- No ~~rigorous~~ exact results on fluctuations either.

video: hailstones on windshield

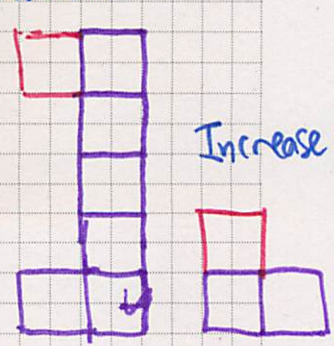
video: coffee stain

What is the Universality class?

- Key properties:
  1. Local growth mechanism.
  2. Smoothing (fill big valleys quickly).
  3. Lateral growth / Slope dependent growth rate. ← Most important
  4. Independent space-time noise.

Increase 4

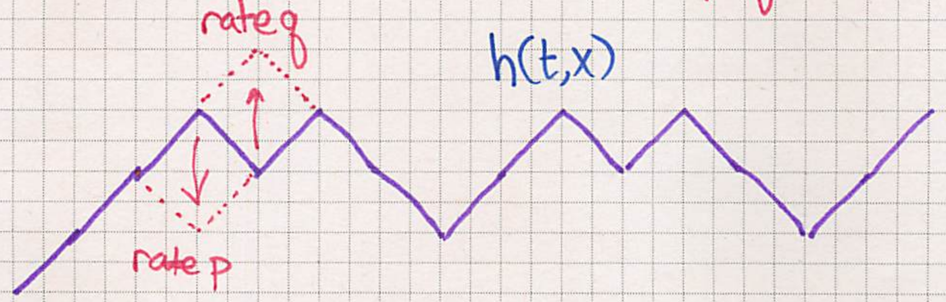
Increase 1



Many physical/probabalistic systems share these features.

- ASEP: An exactly solvable model

$p+q=1$

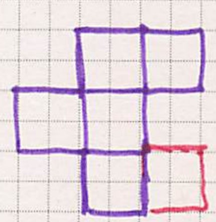


Video: TASEP ($q=1, p=0$) from flat and wedge.

Limit shape is known as is (now) fluctuations

We will return soon to this model, as it is one of few such examples which can be solved.

- Eden growth (pure lateral growth)



Growth of box at each boundary point at rate 1

Video: Takeuchi & Sano droplet and flat

Other examples include slow burning fronts

Growth rate and fluctuations not known.

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- Kardar-Parisi-Zhang's idea '86

1. All growth models with four (previously noted) properties should demonstrate same scaling behavior (i.e. same universality class)

KPZ universality class

2. Proposed a continuum model in this class

KPZ equation

$$\underbrace{\frac{\partial}{\partial t} h}_{\text{local rule}} = \underbrace{\nu \frac{\partial^2}{\partial x^2} h}_{\text{smoothing}} + \underbrace{\frac{\lambda}{2}\left(\frac{\partial}{\partial x} h\right)^2}_{\text{lateral growth}} + \sqrt{D} \cdot \underbrace{\xi}_{\text{space-time white noise}}$$

Standard scaling $\nu = 1/2, \lambda = D = 1$

3. Using non-rigorous dynamical renormalization group analysis of $\frac{\partial}{\partial x} h$ by Forster-Nelson-Stephen '77 predicted that $h(t,x)$ should show large time fluctuations of $O(t^{1/3})$ with spatial correlations of $O(t^{2/3})$.

Sparked a lot of interest with the physics community and over late 80's to early 90's many numerical simulations, experiments and scaling conjectures (theories) were developed, along with the breadth of the universality class

* This work and ensuing decade could be called the zeroth KPZ revolution *

- Related equations

As an SPDE, KPZ equation is not obviously well-posed
(see Martin's talks, Peter's talks and Massimiliano's lecture)
Formally (standard) KPZ equation transforms as $Z(t,x) = e^{h(t,x)}$ to

- Stochastic Heat Equation (SHE) with multiplicative noise

$$\frac{\partial}{\partial t} Z = \frac{1}{2} \frac{\partial^2}{\partial x^2} Z + \xi \cdot Z$$

which is well-posed (later in my talk).

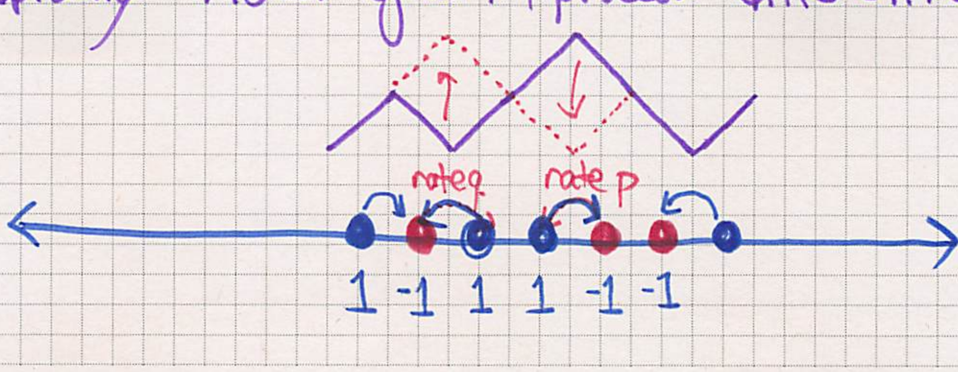
Via Feynmac-Kac representation this relates to the study of
Directed polymers in disordered media

- Stochastic Burgers Equation formally $u(t,x) = \frac{\partial}{\partial x} h(t,x)$

$$\frac{\partial}{\partial t} u = \frac{1}{2} \frac{\partial^2}{\partial x^2} u + \frac{1}{2} \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial x} \xi$$

This may be interpreted as a continuum interacting particle system

Microscopically ASEP growth process differentiated becomes



- KPZ scaling "theory" (prediction/conjecture from physics)

For any nearest-neighbor interacting particle system with at most one particle per site (i.e. ± 1 slope growth process) : $\{\eta_x(t)\}_{x\in\mathbb{Z}} = \eta(t)$

Assumption: The spatially ergodic and time stationary measures of $\eta(t)$ are precisely labeled by average density

$$\rho = \lim_{a\to\infty} \frac{1}{2a+1} \sum_{|j|\le a} \eta_j$$

with $|\rho| \le 1$ (call measure μ_ρ)

- **Average steady state current**:

$$j(\rho) := \mu_\rho\big(c_{0,1}(\eta) \cdot (\eta_0 - \eta_1)\big)$$

$c_{0,1}(\eta)$ jump rate across $0-1$ bond

"limit speed at slope ρ of inter-face"

- **Integrated covariance**: ~~of slope field~~

$$A(\rho) := \sum_{j\in\mathbb{Z}} \left(\mu_\rho(\eta_0\eta_j) - \mu_\rho(\eta_0)^2\right)$$

Example ASEP: $\cancel{j(\rho) = (q-p)}$

μ_ρ = Bernoulli with particle prob. $\frac{1+\rho}{2}$.

$$j(\rho) = (\cancel{q-p}\; p-q) \cdot \frac{1-\rho^2}{2} \quad \leftarrow \quad q\cdot\frac{1-\rho}{2}\cdot\frac{1+\rho}{2}\cdot(-2) + p\,\frac{1+\rho}{2}\cdot\frac{1-\rho}{2}\cdot(+2)$$

$$A(\rho) = 1-\rho^2 \qquad \mu_\rho(1) - \mu_\rho(\eta_0)^2 = 1-(\rho)^2$$

- Hydrodynamic theory asserts $\bar{h}(t,x) = \lim_{\varepsilon \to 0} \varepsilon h(\varepsilon^{-1}t, \varepsilon^{-1}x)$
  satisfies $\frac{\partial}{\partial t}\bar{h}(t,x) + j\left(\frac{\partial}{\partial x}\bar{h}(t,x)\right) = 0$ Hamilton-Jacobi flux j (integrated Burgers)
  subject to entropy condition.

Hopf-Lax-Oleinik variational representation is valid

For "step" or "wedge" initial data (i.e. $h(t,x) = |x|$)

$$\bar{h}(t,x) = t\,\phi(x/t)$$

$$\phi(y) = \sup_{|\rho|\le 1} (y\rho - j(\rho))$$

(sup over possible equilibriums ... microscopic HLO)

Example: ASEP

$$\phi(y) = (q-p)\,\frac{1+\left(\frac{y}{q-p}\right)^2}{2}, \quad |y| \le q-p, \qquad \phi(y) = |y|, \quad y > q-p$$

and $h(t,x) \approx t\,\phi(x/t)$ + fluctuations

What are the fluctuations and how do they depend on j and A ?

Key: $\lambda(\rho) := -j''(\rho)$

Conjecture (Krug, Meakin, Halpin-Healy '92, Spohn '12)

Fix y and let $p = \varphi'(y)$. If $A(p) < \infty$ and $\lambda(p) \neq 0$

$$h(yt, t) - t\phi(y) \approx C \cdot t^{1/3} X$$

$$C = -\left(-\tfrac{1}{2}\lambda A^2\right)^{1/3}$$

X ← universal random variable.

If $\lambda(p) \equiv 0$ $t^{1/4}$ scaling and Gaussian X.

~~If $j''(p) = 0$ but $j'''(p) \neq 0$ $\sqrt{t}(\log t)^{1/2}$ scale~~

Example: ASEP

$$C = -\left(\frac{1}{2}(q-p)\cdot(1-\rho^2)^2\right)^{1/3}$$

And $\lambda(p) \neq 0$ for $q \neq p$

But for $q = p$ $\lambda(p) \equiv 0$.

- This is certainly not the greatest generality in which the $t^{1/3}$ and universal scaling limit should arise.
- Also, there should be a larger universal object than just the one-point limit X.

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- KPZ fixed point

Renormalization operator

$$R_\varepsilon^{KPZ} h(t,x) = \varepsilon^{\overset{\text{"}b}{1/2}} h(\varepsilon^{\overset{\text{"}z}{-3/2}} t, \varepsilon^{-1} x) - \bar{h}_\varepsilon$$

asymptotic divergence in height

Under general hypotheses, we should have (think ASEP $p \neq q$)

$$\lim_{\varepsilon \to 0} R_\varepsilon^{KPZ} h(t,x) = h^{\ast}(t,x)$$

unique upto const. scaling depending on λ, A.

The fixed point of KPZ universality class

The BIG QUESTIONS:

1. Describe KPZ fixed point (Statistically, regularity prop. as a stochastic process)

2. Prove universality of KPZ fixed point.

In 0^{th} KPZ revolution, numerical statistics were computed via MC methods and the 1/3, 2/3 exponents observed numerically and roughly in some experiments.

- Edwards Wilkinson (EW) Fixed point (unstable....)

Corresponds with $b = 1/2$, $z = 2$

1. Solution to additive SHE

2. Well understood for $\lambda \equiv 0$ models

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- Rescaling KPZ equation

For $h(t,x)$ ← solution to KPZ eqn. with standard coeff. $h_\varepsilon(t,x) := \varepsilon^b h(\varepsilon^{-z}t, \varepsilon^{-1}x)$

$$\frac{\partial}{\partial t} h_\varepsilon = \frac{1}{2}\varepsilon^{2-z}\frac{\partial^2}{\partial x^2}h_\varepsilon + \frac{1}{2}\varepsilon^{2-z-b}\left(\frac{\partial}{\partial x}h_\varepsilon\right)^2 + \varepsilon^{b-\frac{z}{2}+\frac{1}{2}}\xi$$

KPZ scaling (Forster-Nelson-Stephen) says $b = 1/2$, $z = 3/2$

so at a formal level $h_0(t,x) = \mathcal{h}(t,x)$ solves

$$\frac{\partial}{\partial t}\mathcal{h} = \frac{1}{2}\left(\frac{\partial}{\partial x}\mathcal{h}\right)^2 \qquad \text{"Inviscid Burgers"}$$

Can not exactly be true since (1) No randomness (2) Doesn't preserve Brownian motion.

Believe the noise survives and becomes specific non-Gaussian noise,
But don't have rigorous results to this effect yet.

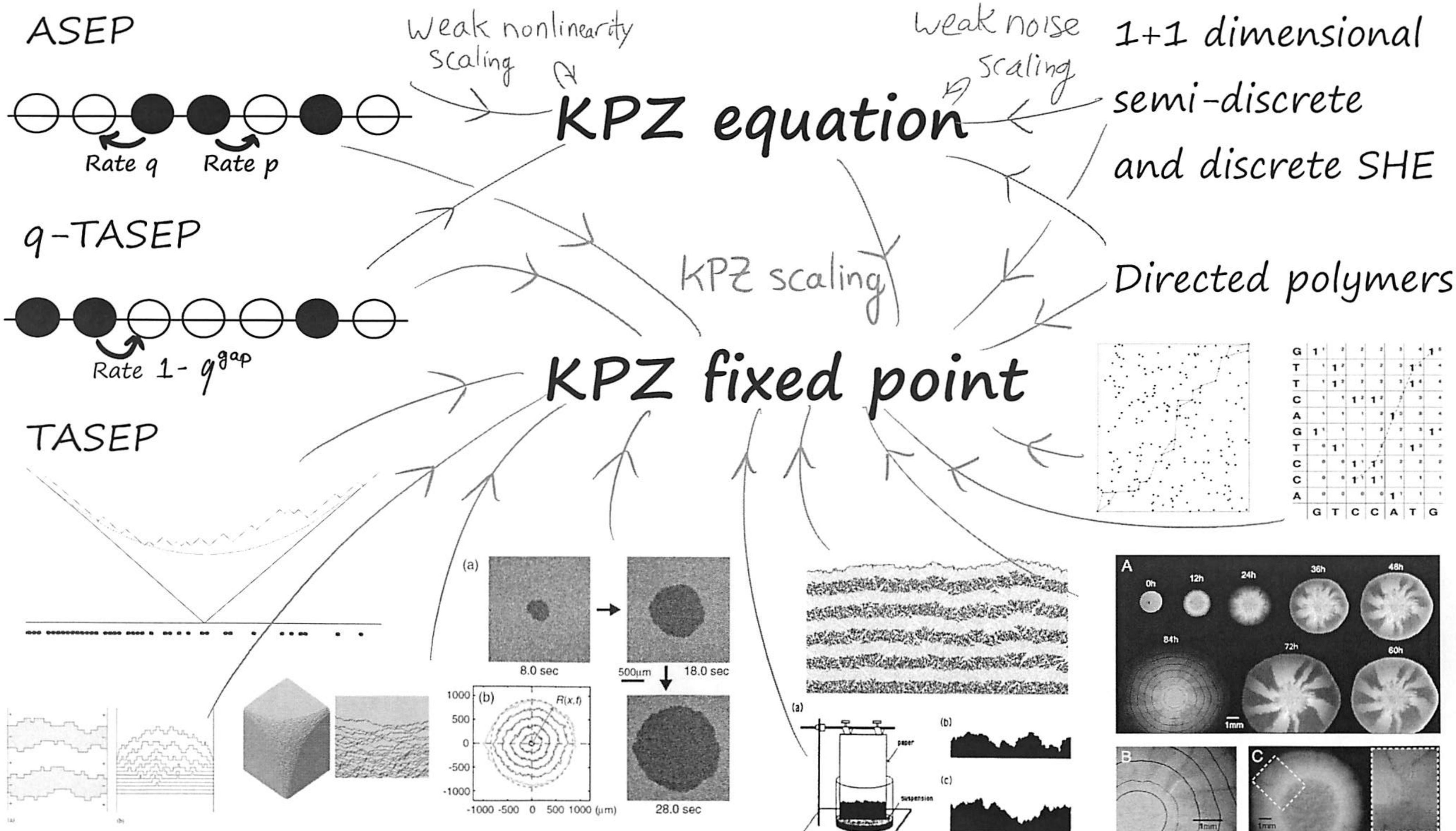
- Weak nonlinearity (asymmetry) scaling: $b = 1/2$, $z = 2$
  But scale non-linearity by $\varepsilon^{1/2}$.

- Weak noise (intermediate disorder) scaling: $b = 0$, $z = 2$
  But scale noise by $\varepsilon^{1/2}$

The KPZ equation is invariant under these two weak scalings ⇝ suggests

"Weak universality of KPZ equation"

>—— $\boxed{\text{Show Universality Figure on Computer}}$ ——<


ASEP
Rate q
Rate p
q-TASEP
Rate 1- q^gap
TASEP
Weak nonlinearity scaling
KPZ equation
Weak noise scaling
1+1 dimensional
semi-discrete
and discrete SHE
Directed polymers
KPZ scaling
KPZ fixed point
8.0 sec
18.0 sec
28.0 sec
500μm
R(x,t)
-1000 -500 0 500 1000 (μm)
0h
12h
24h
36h
48h
84h
72h
60h
1mm
G T C C A T G

- 1^{st} KPZ revolution '99 - '09

Mathematical proof of scaling exponents, plus discovery of exact distributions related to KPZ fixed point with various types of initial data.

Initiated with work of Baik-Deift-Johansson / Johansson

For TASEP ($q=1, p=0$) set $h_\varepsilon^{TASEP} := \varepsilon^{1/2} h^{TASEP}(\varepsilon^{-3/2} t, \varepsilon^{-1} x) - \frac{\varepsilon^{-1} t}{2}$

Theorem (Johansson '99): For $h(0,x) = |x|$,

$$\lim_{\varepsilon \to 0} \mathbb{P}\left(h_\varepsilon^{TASEP}(1,0) \geq -s\right) = F_{GUE}(2^{1/3} s)$$

(TW GUE)

where F_{GUE} is distribution of λ_{max} from $N \times N$ GUE as $N \to \infty$.

See Slide

Prahofer-Spohn extended to multi (space) points $h_\varepsilon^{TASEP}(1, x_i)$ and found Airy_2 process.

Various other initial data, models were analyzed to gain more statistical understand of fixed point.

Multi-time distributions remain beyond reach presently.

- Determinantal point processes

TASEP is one of a few growth models in KPZ class that can be analyzed via determinantal point processes (free fermions, nonintersecting paths, Schur processes)

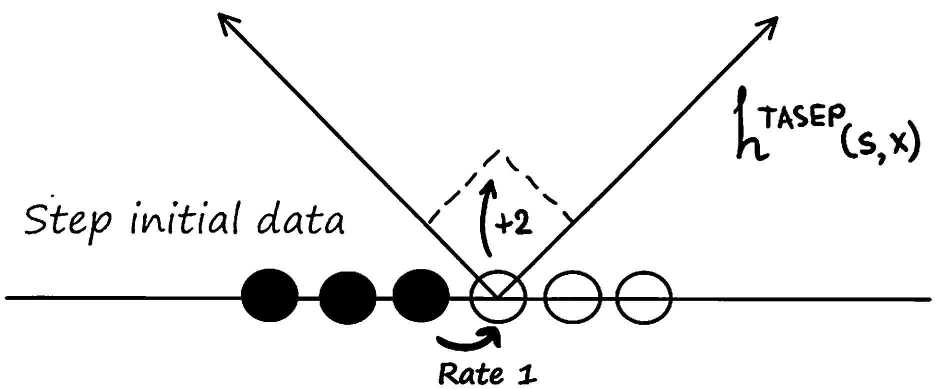
Other examples (basically only models solved in 1st KPZ revolution)

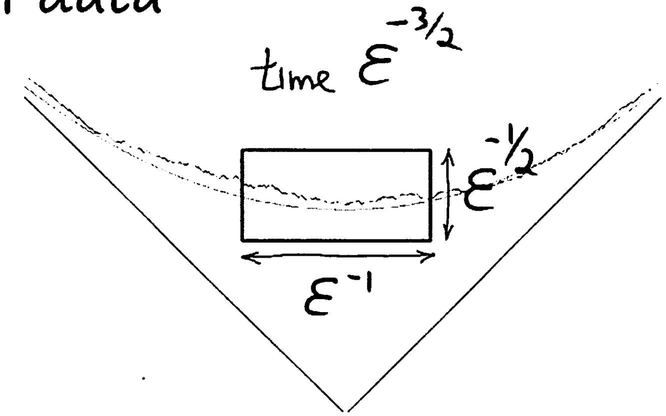
see slide

- Discrete time TASEPs with sequential/parallel updates
- PushASEP or long range TASEP
- Directed LPP in 1+1 dimension with geo/exp/Bernoulli weights
- Polynuclear growth process (longest increasing subsequence)

Suggested reading: Borodin - Gorin lecture notes.

TASEP with step initial data





$$h_\varepsilon(t,x) := \varepsilon^{1/2} h^{TASEP}(\varepsilon^{-3/2}t, \varepsilon^{-1}x) - \varepsilon^{-3/2}\frac{t}{2}$$

<u>Theorem [Johansson 1999]</u> *For TASEP with step initial data*

$$\lim_{\varepsilon \to 0} \mathbb{P}\{ h_\varepsilon(1,0) \geq -s \} = F_{GUE}(s)$$

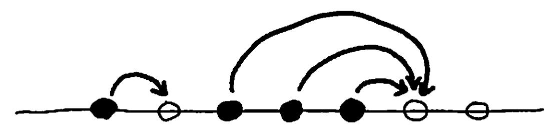
Tracy-Widom limit distribution for the largest eigenvalue of large Hermitian matrices

Limiting (joint) distributions for general t and x are conjecturally universal.

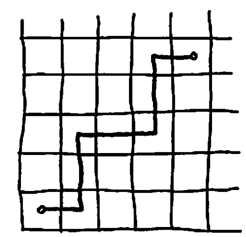
TASEP is one of a few growth models in the KPZ class that can be analyzed via the techniques of determinantal point processes (or free fermions, nonintersecting paths, Schur processes).

Other examples include

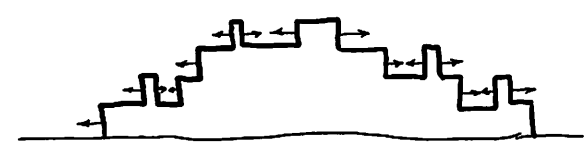
- Discrete time TASEPs with sequential/parallel update
- PushASEP or long range TASEP



- Directed last passage percolation in 2d with geometric/Bernoulli/exponential weights



- Polynuclear growth processes



Recent advances on the KPZ front:

1. Strong experimental evidence that real life systems follow the KPZ class universal laws [Takeuchi-Sano, 2010], [Takeuchi-Sano-Sasamoto-Spohn, 2011], [Yunker et al, 2012]
2. Direct well-posedness of the KPZ equation (see Martin and Massimiliano talks) and some evidence of weak universality of the equation [Hairer, 2011], [Jara-Goncalves, 2010], [Assing, 2011], [Bertini-Giacomin, 1997], [Amir-Corwin-Quastel, 2010], [Alberts-Khanin-Quastel, 2012]
3. Non-determinantal models whose large time behaviour has been analyzed (see also Patrik and Marton talks)

Non-determinantal **models whose limit behaviour has been analyzed:**

- **ASEP** [Tracy-Widom, 2009], [Borodin-C-Sasamoto, 2012]
- **KPZ equation / stochastic heat equation (SHE)**
  [Amir-C-Quastel, 2010], [Sasamoto-Spohn, 2010], [Dotsenko, 2010+], [Calabrese-Le Doussal-Rosso, 2010+], [Borodin-C-Ferrari, 2012]
- **q-TASEP** [Borodin-C, 2011+], [Borodin-C-Sasamoto, 2012]
- **Semi-discrete stochastic heat equation**
  [O'Connell, 2010], [Borodin-C, 2011, Borodin-C-Ferrari, 2012]
- **Fully discrete log-Gamma polymer (stochastic heat equation)**
  [C-O'Connell-Seppalainen-Zygouras, 2011] [Borodin-C-Remenik, 2012]
- **q-PushASEP** [Borodin-Petrov, 2013], [C-Petrov, 2013]

- 2^{nd} KPZ revolution

Slides :
- Recent Advances
- Non-determinantal asymptotics
- Scaling exponents
- Diagram of processes

Example outcome of 2^{nd} KPZ revolution

Theorem (Bertin-Giacomin '97, Amir-C-Quastel '10)
For ASEP with $q-p=\varepsilon^{1/2}$, started from $h^{ASEP}(0,x)=|x|$,

$$\varepsilon^{1/2} h^{ASEP}(\varepsilon^{-2}t, \varepsilon^{-1}x) - \varepsilon^{-1}\frac{t}{2} - \log\frac{\varepsilon^{-1/2}}{2} \underset{\varepsilon \downarrow 0}{\Longrightarrow} -h(t,x)$$

where $h(t,x)=\log Z(t,x)$, solution to SHE with $Z(0,x)=\delta_{x=0}$.

This "weak nonlinearity/asymmetry" convergence justifies defining: The Hopf-Cole solution to KPZ equation as $\log Z(t,x)$, with relevant initial data.

"Narrow wedge"

Theorem relies on microscopic Hopf-Cole transform (Gartner '88) and convergence at level of SHE (in general this lifting is not always possible)

- Exact one-point statistic for KPZ equation

Using Tracy-Widom '08-'09 formula for $\mathbb{P}(h^{ASEP}(t,x) \leq s)$

Theorem (Amir-C-Quastel '10)

$$\mathbb{P}\left(h(t,x) + \frac{x^2}{2t} + \frac{t}{24} \leq 2^{-1/3} s\right) = \int \frac{d\mu}{\mu} e^{-\mu} \det(I - K_{t,\mu})_{L^2(t^{-1/3}s,\infty)}$$

$$K_{t,\mu}(x,y) = \int_{-\infty}^{\infty} \frac{\mu}{\mu - e^{-t^{1/3} r}} Ai(x+r) Ai(y+r)\, dr$$

Corollary The KPZ equation is in the KPZ universality class

$$\lim_{\varepsilon \to 0} \mathbb{P}\left(\varepsilon^{1/2} h(\varepsilon^{-3/2} t, \varepsilon^{-1} x) + \frac{x^2}{2t} + \varepsilon^{-1} \frac{t}{24} \leq s\right) = F_{GUE}(s)$$

- Formula discovered independently and in parallel in work of Sasamoto-Spohn '10, though not completely rigorous.

- Via highly non-rigorous "replica method" also discovered by physicists (initial result was corrected) Dotsenko '10, Calabrese-Le Doussal-Rosso '10.

We now have other rigorous ways to prove this (see also Patrik's talk)

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- Solvability approaches for non-determinantal models

Basic observation is the existence of observables of these processes whose average can be computed explicitly (and concisely)

Two approaches

1. Many body systems (expectations solve closed systems) related to duality / replicas.

2. 2+1 extensions and Macdonald processes / gRSK (find extension of process into two spatial dimensions in which the measure becomes expressible nicely)

Approach 2 is more structural - in fact the integrable properties of the Macdonald polynomials lead to the probabilistic systems, observables and formulas.

Approach 1 is more elementary and seems to apply more generally.

- This is an example of integrable probability -

~~We will now start the course~~

I hope to convince you of the beauty and worth of integrable probability.

Discrete time q-TASEPs

q-pushASEP

q-TASEP

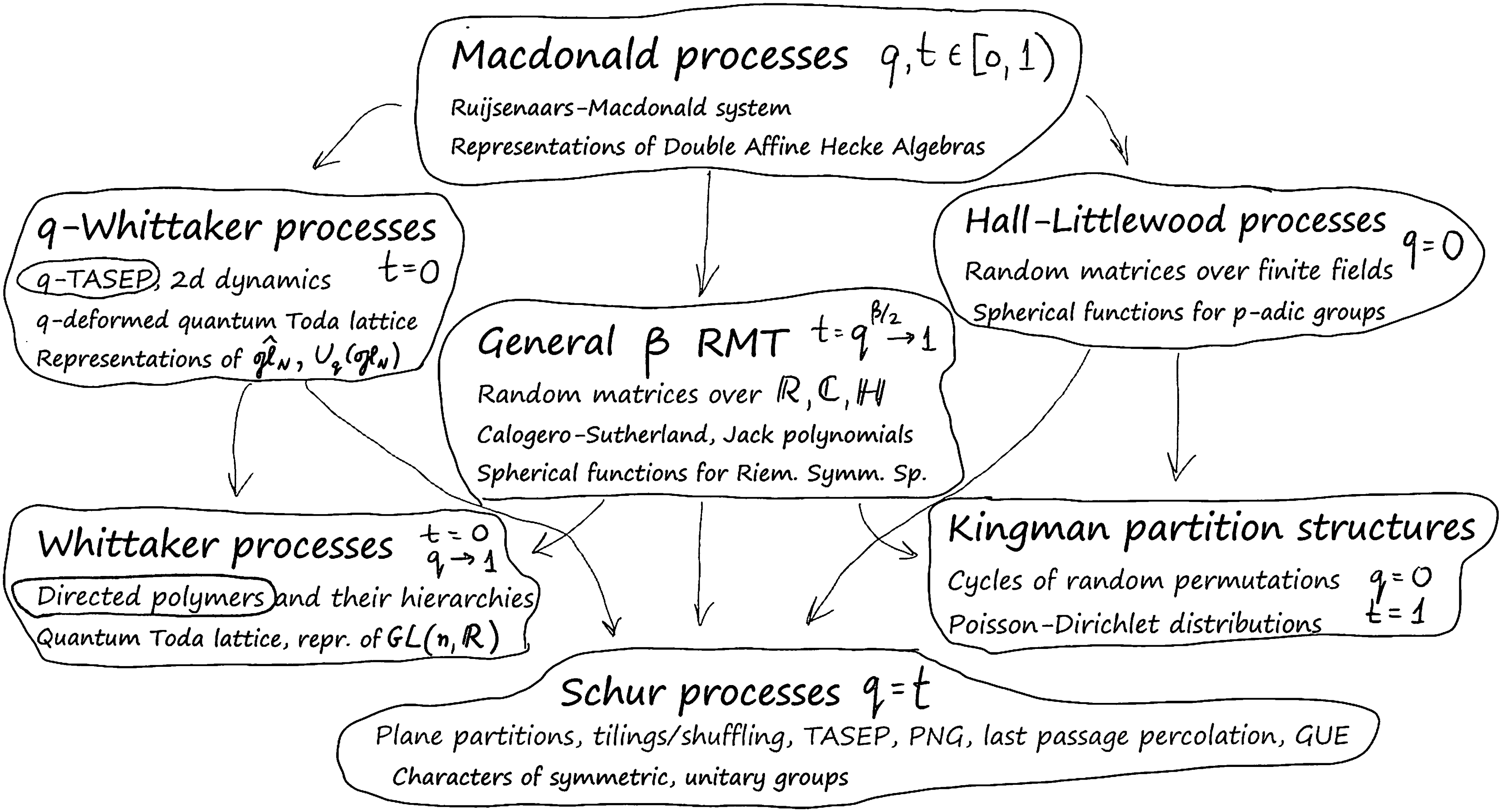
log-Gamma discrete polymer

ASEP

semi-discrete Brownian polymer

KPZ/SHE/continuous Brownian polymer

universal limits (Tracy-Widom distributions, Airy processes)


Macdonald processes $q, t \in [0,1)$
Ruijsenaars-Macdonald system
Representations of Double Affine Hecke Algebras
q-Whittaker processes $t=0$
q-TASEP, 2d dynamics
q-deformed quantum Toda lattice
Representations of $\hat{\mathfrak{gl}}_N$, $U_q(\mathfrak{gl}_N)$
General β RMT $t=q^{\beta/2} \to 1$
Random matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$
Calogero-Sutherland, Jack polynomials
Spherical functions for Riem. Symm. Sp.
Hall-Littlewood processes $q=0$
Random matrices over finite fields
Spherical functions for p-adic groups
Whittaker processes $t=0$, $q \to 1$
Directed polymers and their hierarchies
Quantum Toda lattice, repr. of $GL(n, \mathbb{R})$
Kingman partition structures
Cycles of random permutations $q=0$
Poisson-Dirichlet distributions $t=1$
Schur processes $q=t$
Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE
Characters of symmetric, unitary groups