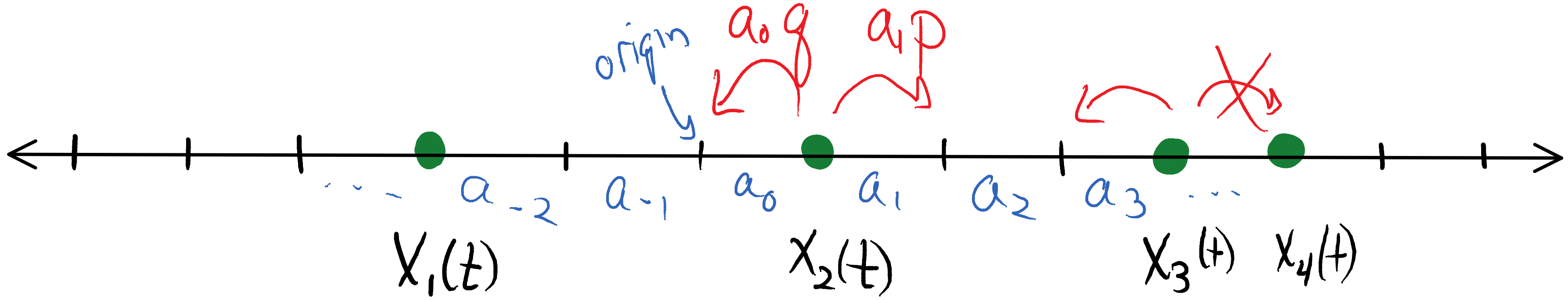
Asymmetric simple exclusion (particle) process



Particles attempt continuous time random walks, jumping left over bond $i \leftrightarrow i+1$ at rate $a_i q$ and right at rate $a_i p$. If the destination is occupied, the jump is suppressed.

State space for k particles: $W^k = \{ x_1 < x_2 < \cdots < x_k \} \subseteq \mathbb{Z}^k$.

Generator $(L^{k,\mathrm{part}} f)(\vec{x})$ for $\vec{x} \in W^k$.

e.g. k=1 $\quad (L^{1,\mathrm{part}} f)(x) = a_x p [f(x+1) - f(x)] + a_{x-1} q [f(x-1) - f(x)]$

Asymmetric simple exclusion (occupation) process

$\eta(t) = \{\eta_x(t)\}_{x\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$, $\eta_x(t) = \begin{cases} 1 & \text{particle at } x, \text{time } t \\ 0 & \text{otherwise} \end{cases}$

Dynamics: for each y $\begin{cases} \eta \mapsto \eta^{y,y+1} & \text{at rate } a_y p \text{ if } (\eta_y, \eta_{y+1}) = (1,0) \\ \eta \mapsto \eta^{y,y+1} & \text{at rate } a_y q \text{ if } (\eta_y, \eta_{y+1}) = (0,1) \end{cases}$

$$(L^{occ} f)(\eta) = \sum_{y\in\mathbb{Z}} a_y \left(p\,\eta_y(1-\eta_{y+1}) + q(1-\eta_y)\eta_{y+1}\right)\left[f(\eta^{y,y+1}) - f(\eta)\right]$$

Assume that $q \geq p \geq 0$ so $p/q =: \tau \leq 1$ $(p+q=1)$ and $c < a_x < c^{-1}$

Define: $N_x = N_x(\eta) = \sum_{y\leq x} \eta_y$

Theorem [Borodin-C-Sasamoto '12]: For any k>0, the ASEP particle process $\vec{x}(t)$ (with p<->q switched) and the ASEP occupation process $\eta(t)$ are dual with respect to

$$H(\eta,\vec{x}) = \prod_{i=1}^{k} \tau^{N_{x_i-1}(\eta)} \eta_{x_i}$$

(i.e. $\mathbb{E}^{\eta}\left(H(\eta(t),\vec{x})\right) = \mathbb{E}^{\vec{x}}\left(H(\eta,\vec{x}(t))\right)$ for all $\eta \in \{0,1\}^{\mathbb{Z}}$, $\vec{x} \in W^k$, $t \geq 0$)

If all bond jump rates parameters $a_i \equiv 1$ then the processes are also dual with respect to

$$G(\eta,\vec{x}) = \prod_{i=1}^{k} \tau^{N_{x_i}(\eta)}$$

Remarks on the duality.

- When p=q, the H-duality describes correlation functions and is much more general.
- When all $a_i \equiv 1$, H-duality shown previously [Schutz '97] via related quantum spin chain $\mathcal{U}_q(sl_2)$- symmetry.
- When k=1, the G-duality is Gartner's microscopic ASEP Hopf-Cole transform.

Proof: Directly from studying the effect of applying the Markov generators to the duality function.

From duality to determinants:

1. Duality lead to system of ODEs for $h(t,\vec{x}) := \mathbb{E}^{\eta}\left[\prod_{i=1}^{k} \tau^{N_{x_i-1}(\eta(t))} \eta_x(t)\right]$

2. For $a_i \equiv 1$ / step initial data, solve ODEs via a "nested contour integral ansatz" (relies on integrability)

3. Combine integral solutions to yield formula for $\mathbb{E}\left[\tau^{nN_x(\eta(t))}\right]$

4. Deform nested-contours to coincide and track residues

5. Form generating function (τ-Laplace transform) and identify Fredhold determinant (Mellin Barnes/Cauchy type).

$$\mathbb{E}\left[\frac{1}{(\zeta\tau^{N_x(t)};\tau)_\infty}\right] = \det(I+K_\zeta)$$

Let's focus on steps 1 and 2.

Duality provides a non-trivial coupled system of ODEs:

Since $\mathbb{E}^{\eta}\left[H(\eta(t), \vec{x})\right] = \mathbb{E}^{\vec{x}}\left[H(\eta, \vec{x}(t))\right] =: h(t, \vec{x})$

$$\frac{d}{dt} h(t, \vec{x}) = \left(L^{k, \text{part}}\right)^{*} h(t, \vec{x}) \quad , \quad h(0, \vec{x}) = H(\eta, \vec{x}).$$

But how to solve? For k>1 the generator depends on $\vec{x}$!

First idea (from Bethe, cf. Tracy-Widom ASEP papers):

Try to solve "free" system of ODEs on all of $\mathbb{Z}^k$ with boundary conditions on W^k.

Proposition: If $u: \mathbb{Z}^k \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ solves

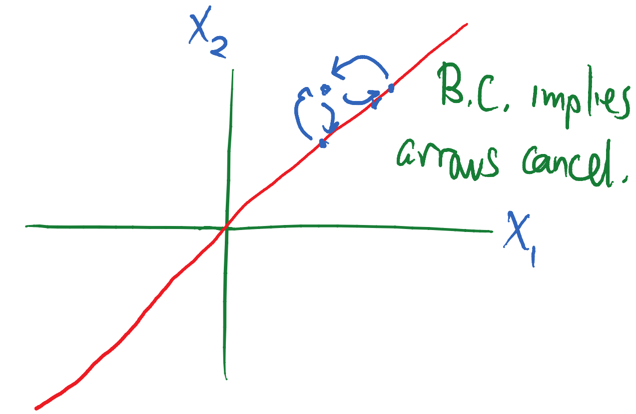
'Free' ODEs

1. $$\frac{d}{dt} u(t,\vec{x}) = \sum_{i=1}^{k} (L^{1,\text{part}})^*_i \, u(t,\vec{x})$$

Boundary condition

2. For all $\vec{x} \in \mathbb{Z}^k$: $x_{i+1} = x_i + 1$ for some i,

$$p\,u(t,\vec{x}_{i+1}^{-}) + q\,u(t,\vec{x}_i^{+}) = u(t,\vec{x})$$



Initial data

3. For all $\vec{x} \in W^k$, $u(0,\vec{x}) = H(\eta,\vec{x})$

Then for all $t \geq 0$, $\vec{x} \in W^k$, $h(t,\vec{x}) = u(t,\vec{x})$

(Note: Since system of ODEs is infinite, we must also impose an exponential growth condition; and we can weaken initial data to weakly converge, as is useful in our contour integral formulas we find)

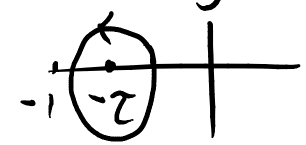
<u>Assume</u> from now on step initial condition ($\eta_x = \mathbb{1}_{x\geq 1}$) and $a_i \equiv 1$

How to solve this system of ODEs?

$$K=1: \quad h_z(t,x) := \exp\left\{-\frac{z(p-q)^2}{(1+z)(p+qz)}t\right\}\left(\frac{1+z}{1+z/\tau}\right)^{x-1}\frac{1}{\tau+z}$$

solves the "free" evolution eqn. for all $z \in \mathbb{C}/\{-\tau\}$.

$$u(t,x) = \mathbb{E}^{step}\left[\tau^{N_{x-1}(\eta(t))}\eta_x(t)\right] = \frac{1}{2\pi i}\oint h_z(t,x)\,dz$$

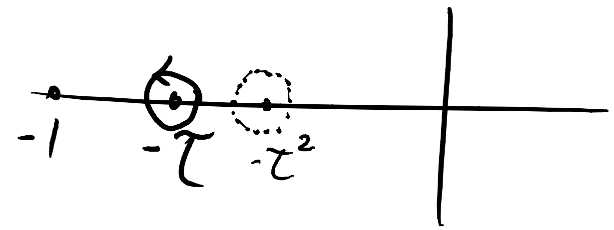


Proof: Check by residues that $u(0,x) = \tau^{x-1}\mathbb{1}_{x\geq 1}$

For $k>1$ we use an idea inspired from the theory of Macdonald processes -> "nested contour integral ansatz"

Theorem [Borodin-C-Sasamoto '12]: For all $k \geq 1$,

$$U(\vec{x};t) = \frac{\tau^{k(k-1)/2}}{(2\pi i)^k} \int \cdots \int \prod_{1\leq A<B\leq k} \frac{z_A - z_B}{z_A - \tau z_B} \prod_{i=1}^{k} h_{z_i}(t, x_i)\, dz_i$$

Where contour of integration is  to avoid poles of $z_A - \tau z_B$.

Restricting to $\vec{x} \in W^k$ yields:

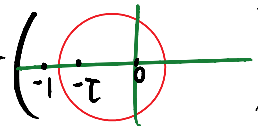
$$\mathbb{E}^{step}\left[\prod_{i=1}^{k} \tau^{N_{x_i - 1}(\eta(t))} \eta_x(t)\right]$$

Assume $x_2 = x_1 + 1$ and check boundary cond.
Try to apply it to integrand $\Rightarrow$ brings out factor $(z_1 - \tau z_2)$. Cancels with $\prod_{A<B}$ term.
What remains is $\iint (z_1 - z_2) G(z_1) G(z_2) = 0$.

Suitable combinations of $\mathbb{E}^{step}\left[\prod_{i=1}^{k} \tau^{N_{x_i-1}(\eta(t))}\eta_{x_i}(t)\right]$ yields $\mathbb{E}^{step}\left[\tau^{n N_x(\eta(t))}\right]$

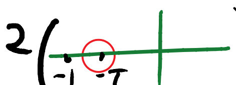
<u>Theorem</u> [Borodin-C-Sasamoto '12]: For step initial condition ASEP with $a_i \equiv 1$ and $p<q$ (hence $\tau = p/q < 1$, $\gamma = q-p > 0$)

$$\mathbb{E}\left[\frac{1}{(\zeta\tau^{N_x(\eta(t))};\tau)_\infty}\right] = \det(I+K_\zeta)_{L^2(\mathcal{C}_1)}$$



Mellin Barnes type

$$\mathbb{E}\left[\frac{1}{(\zeta\tau^{N_x(\eta(t))};\tau)_\infty}\right] = \frac{1}{(\zeta;\tau)_\infty}\det(I-\zeta\tilde{K})_{L^2(\mathcal{C}_2)}$$



Cauchy type

$$K_\zeta(w,w') = \frac{1}{2\pi i}\int \frac{\pi\, ds}{\sin(-\pi s)}(-\zeta)^s \frac{g(w)}{g(\tau^s w)}\frac{1}{w'-\tau^s w}$$

$$g(w) = e^{\gamma t \frac{\tau}{\tau+w}}\left(\frac{\tau}{\tau+w}\right)^x$$

$$\tilde{K}(w,w') = \frac{e^{\varepsilon'(w)t}}{\tau w - w'}$$

$$\varepsilon'(w) = -\frac{1}{q}\frac{w\gamma^2}{(1+w)(\tau+w)}$$

Corollary [Tracy-Widom '09, Borodin-C-Sasamoto '12]:

$$\lim_{t\to\infty} \mathbb{P}^{step}\left(\frac{N_0(\eta(t/\gamma)) - t/4}{t^{1/3}} \geq -r \right) = F_{GUE}(2^{4/3} r)$$

Recovering the celebrate Tracy-Widom / Johansson result.

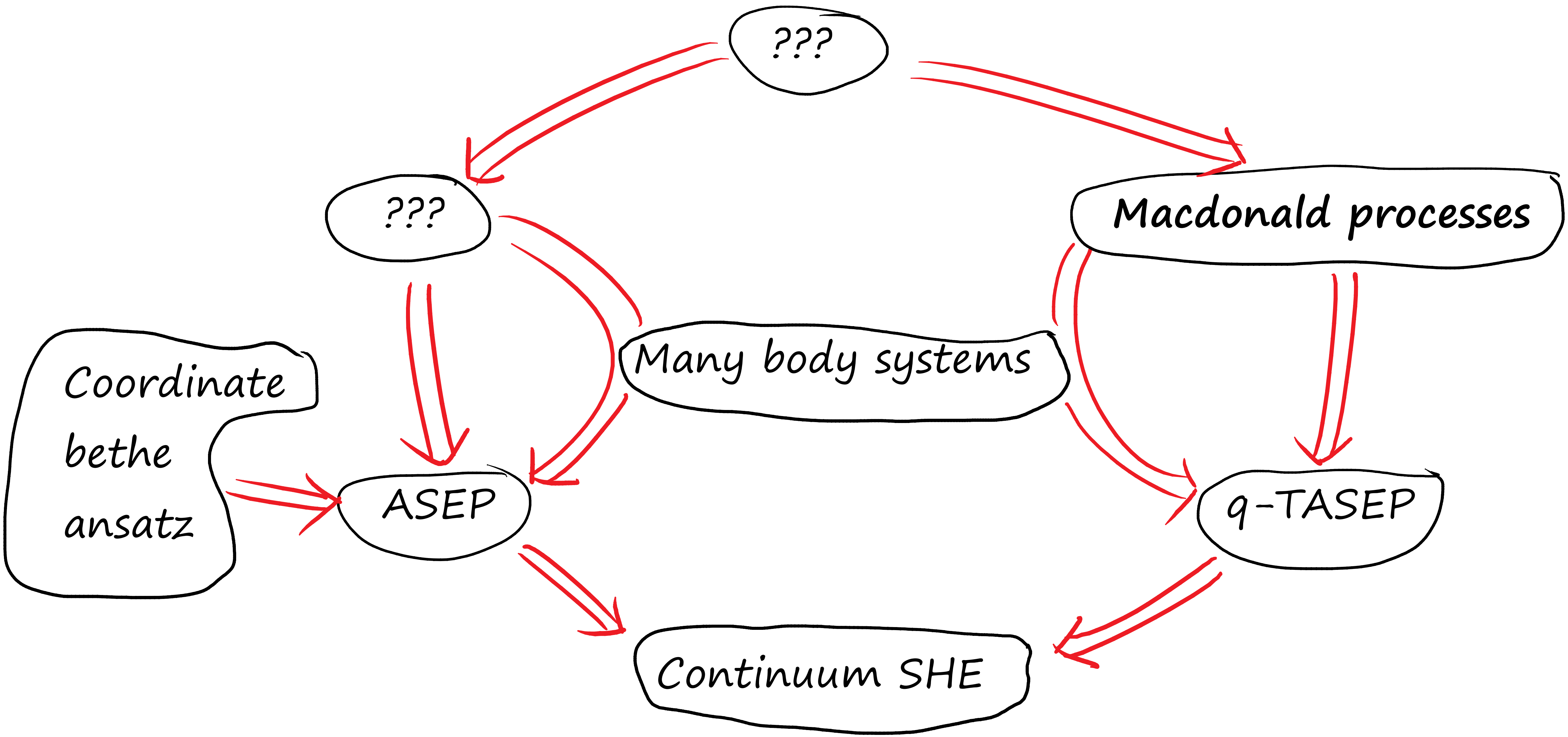
Remarks:

- Mellin Barnes Fredhold det. new and easy for asymptotics
- Inversion of Cauchy Fredholm det. equivalent to initial det. in [Tracy-Widom '09]
- Completely parallel to q-TASEP formulas

Coordinate approach of [Tracy-Widom '08-'09]:

- Study k particle ASEP and use coordinate Bethe ansatz (cf. [Schutz '97] for k=2) to compute Green's functions.
- Manipulate formulas to extra one-point marginal.
- Approach step initial condition by taking k to infinity and observe an integral transform of Cauchy type Fredholm det.
- Functional analysis to rework for asymptotic analysis.

Using k-particle Green's functions can write solution of duality ODEs as k! k-fold contour integrals [Imamura-Sasamoto '11]. Equivalence to nested formula is non-trivial.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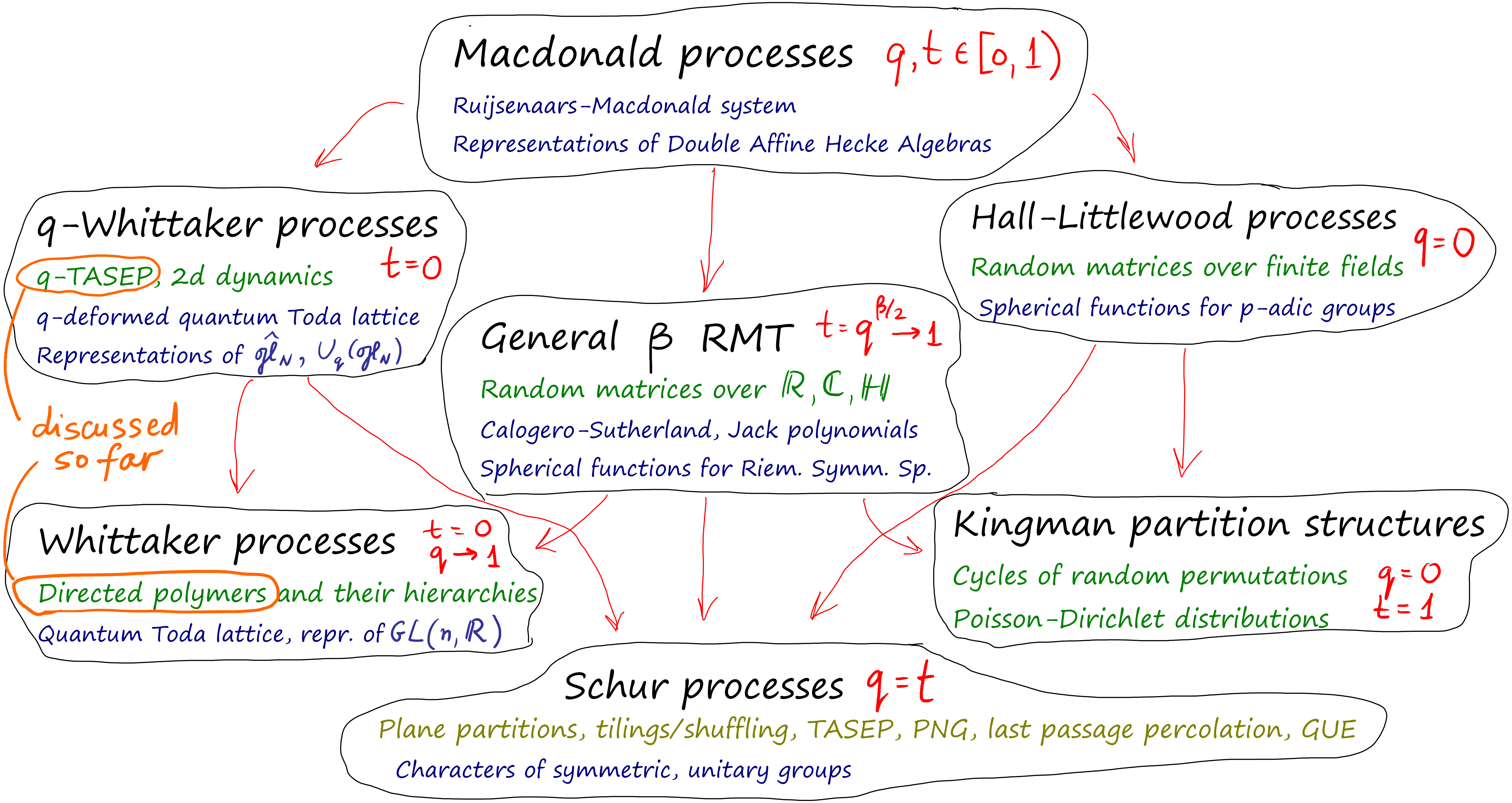
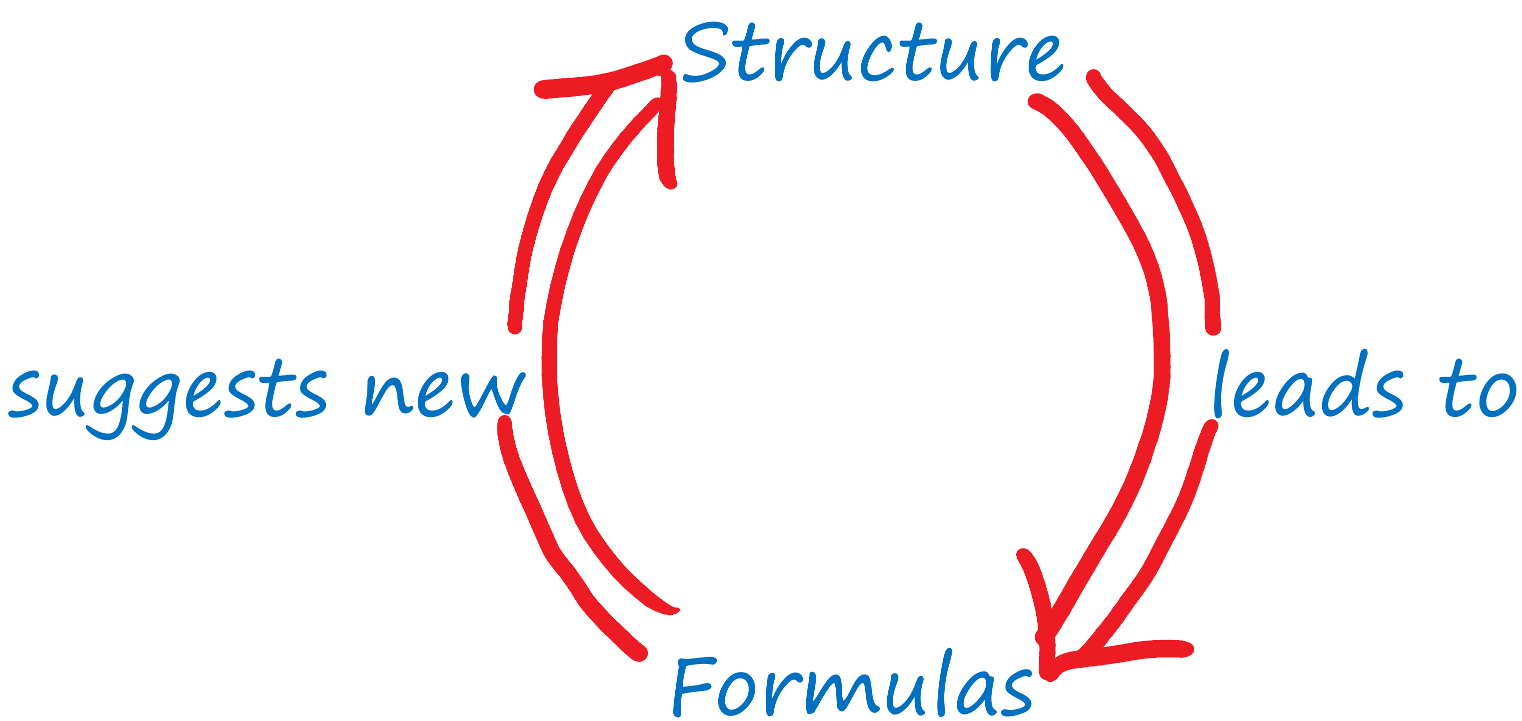
Macdonald processes
Many body systems
Coordinate
bethe
ansatz
ASEP
q-TASEP
Continuum SHE


Macdonald processes $q, t \in [0,1)$
Ruijsenaars-Macdonald system
Representations of Double Affine Hecke Algebras
q-Whittaker processes $t=0$
q-TASEP, 2d dynamics
q-deformed quantum Toda lattice
Representations of $\widehat{\mathfrak{gl}}_N$, $U_q(\mathfrak{gl}_N)$
discussed so far
General β RMT $t=q^{\beta/2}\to 1$
Random matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$
Calogero-Sutherland, Jack polynomials
Spherical functions for Riem. Symm. Sp.
Hall-Littlewood processes $q=0$
Random matrices over finite fields
Spherical functions for p-adic groups
Whittaker processes $t=0$ $q\to 1$
Directed polymers and their hierarchies
Quantum Toda lattice, repr. of $GL(n,\mathbb{R})$
Kingman partition structures
Cycles of random permutations $q=0$ $t=1$
Poisson-Dirichlet distributions
Schur processes $q=t$
Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE
Characters of symmetric, unitary groups

Macdonald processes: a source of (many parameter) integrable probabilistic systems. Specializations and degenerations include q-TASEP, continuum/semi-discrete/discrete SHE, KPZ equation

ASEP does not fit. But it does share certain parallel formulas



(Ascending) Macdonald processes are probability measures on interlacing triangular arrays (Gelfand-Tsetlin patterns)

$$
\begin{array}{ccccccccc}
\lambda^{(N)}_N & & \lambda^{(N)}_{N-1} & & \cdots & & \lambda^{(N)}_2 & & \lambda^{(N)}_1 \\
& \lambda^{(N-1)}_{N-1} & & \lambda^{(N-1)}_{N-2} & & \cdots & & \lambda^{(N-1)}_1 & \\
& & \ddots & & \cdots & & ⋰ & & \\
& & & \lambda^{(2)}_2 & & \lambda^{(2)}_1 & & & \\
& & & & \lambda^{(1)}_1 & & & &
\end{array}
$$

$$\lambda^{(m)}_j \in \mathbb{Z}_{\geq 0}$$

Macdonald polynomials

$$\mathbb{P}\left(\lambda^{(N)}\right) = \frac{P_{\lambda^{(N)}}(a_1, \ldots, a_N)\, Q_{\lambda^{(N)}}(b_1, \ldots, b_M)}{\Pi(a_1, \ldots, a_N; b_1, \ldots, b_M)}$$

$\mathbb{P}$ = $M_N(a_1, \ldots, a_N; b_1, \ldots, b_M)$

Π: normalization constant

$a_1, \ldots, a_N$ and $b_1, \ldots, b_M$: two groups of parameters

At $q=t$, $P_\lambda = Q_\lambda = s_\lambda$; recover Schur measure [Okounkov '01] which is a discrete random matrix ensemble;
and Schur Process [Okounkov-Reshetikhin '03] which is a discrete version of GUE minor process.

Think of Macdonald measure as a (q,t)-deformed discrete random matrix eigenvalue type ensemble (and analogously Mac. process).

BUT: For general $q \neq t$ this is NOT DETERMINANTAL

Macdonald polynomials $P_\lambda(x_1,\dots,x_N)\in\mathbb{Q}(q,t)[x_1,\dots,x_N]^{S(N)}$ with partitions $\lambda=(\lambda_1\geq\lambda_2\geq\dots\geq\lambda_N\geq 0)$ form a basis in symmetric polynomials in N variables over $\mathbb{Q}(q,t)$. They diagonalize

$$(\mathcal{D}_1 f)(x_1,\dots,x_N)=\sum_{i=1}^{n}\prod_{j\neq i}\frac{t x_i-x_j}{x_i-x_j}\, f(x_1,\dots,q x_i,\dots,x_N)$$

with (generically) pairwise different eigenvalues

$$\mathcal{D}_1 P_\lambda=(q^{\lambda_1}t^{N-1}+q^{\lambda_2}t^{N-2}+\dots+q^{\lambda_N})P_\lambda .$$

They have many remarkable properties that include orthogonality (dual basis Q_λ), simple reproducing kernel (Cauchy type identity), Pieri and branching rules, index/variable duality, explicit generators of the algebra of (Macdonald) operators commuting with $\mathcal{D}_1$, etc.

Reproducing kernel (Cauchy type identity)

$$\Pi(a_1,\dots,a_N;b_1,\dots,b_M) := \sum_{\lambda^{(N)}} P_{\lambda^{(N)}}(a_1,\dots,a_N)\, Q_{\lambda^{(N)}}(b_1,\dots,b_M)$$

$$= \prod \frac{(t a_i b_j; q)_\infty}{(a_i b_j; q)_\infty}$$

If $b_i \equiv \frac{\gamma}{M}$ and $M \to \infty$ then $\Pi(a;b) \to e^{\gamma a_1} \cdots e^{\gamma a_N}$ (Plancherel)

At q=t reduces to Schur function Cauchy identity

$$\sum_{\lambda} S_\lambda(a) S_\lambda(b) = \prod_{i,j} \frac{1}{1-a_i b_j}$$

We are able to do two basic things:

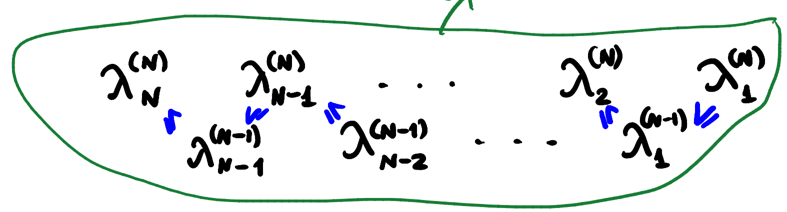
- Construct explicit Markov operators that map Macdonald processes to Macdonald processes (with new parameters)
- Evaluate averages of a rich class of observables

The integrable structure of Macdonald polynomials directly translates into probabilistic content.

By working at a high combinatorial level we avoid analytic issues (eventually need to work hard to take various limits).

Dynamics on Gelfand–Tsetlin patterns comes from an idea of [Diaconis–Fill '90]; in Schur process case [Borodin–Ferrari '08]

Branching rule: $P_{\lambda^{(N)}}(a_1,\dots,a_{N-1},a_N) = \sum_{\lambda^{(N-1)} \preceq \lambda^{(N)}} P_{\lambda^{(N-1)}}(a_1,\dots,a_{N-1})\, P_{\lambda^{(N)}/\lambda^{(N-1)}}(a_N)$



Skew Macdonald polynomial

$$P_{\lambda/\mu}(u) = \begin{cases} \psi_{\lambda/\mu}\, u^{|\lambda|-|\mu|}, & \lambda \succeq \mu \\ 0, & \text{else} \end{cases} \qquad (\psi_{\lambda/\mu} \in \mathbb{Q}(q,t))$$

Combinatorial expansion shows for positive a's, $P_\lambda(a_1,\dots,a_k) \geq 0$

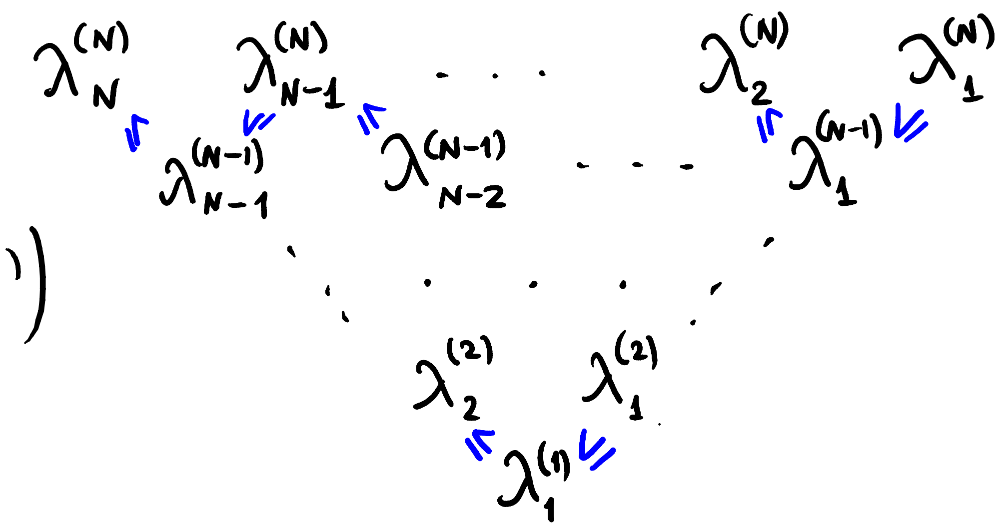
Markov kernel (stochastic link) from level N to $N-1$

$$\Lambda^{N}_{N-1}\left(\lambda^{(N)},\lambda^{(N-1)}\right) := \frac{P_{\lambda^{(N-1)}}(a_1,\dots,a_{N-1})\, P_{\lambda^{(N)}/\lambda^{(N-1)}}(a_N)}{P_{\lambda^{(N)}}(a_1,\dots,a_N)}$$

maps $M_N(a_1,\dots,a_N;b_1,\dots,b_M)$ to $M_{N-1}(a_1,\dots,a_{N-1};b_1,\dots,b_M)$.

Trajectory of this Markov chain defines the Macdonald process

$$M_{[1,N]}(a_1,\dots,a_N;b_1,\dots,b_M)\left(\lambda^{(N)},\dots,\lambda^{(1)}\right)$$
$$:= M_N\left(\lambda^{(N)}\right)\Lambda^{N}_{N-1}\left(\lambda^{(N)},\lambda^{(N-1)}\right)\cdots\Lambda^{2}_{1}\left(\lambda^{(2)},\lambda^{(1)}\right)$$



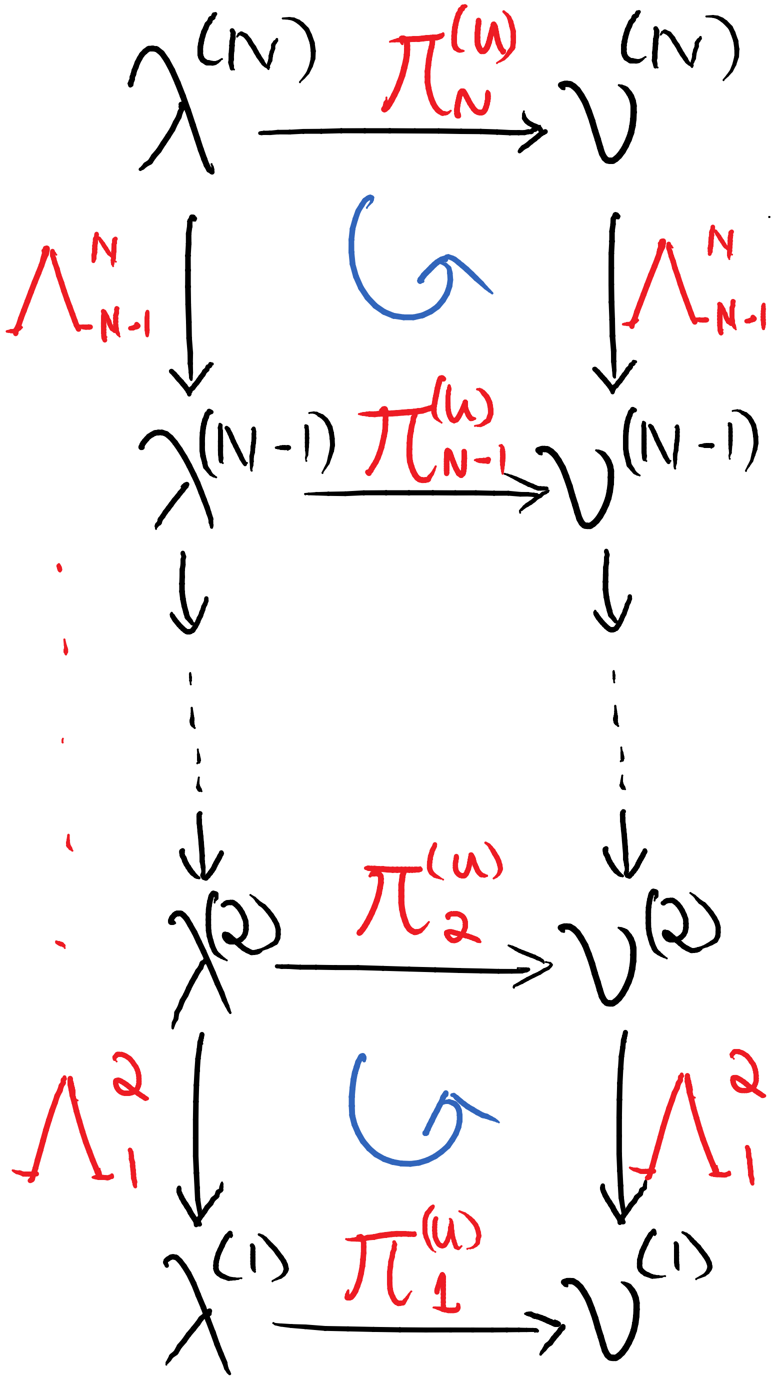
Markov kernel from level N to level N

$$\pi_N^{(u)}\left(\lambda^{(N)}, \nu^{(N)}\right) := \frac{P_{\nu^{(N)}}(a_1,\dots,a_N)}{P_{\lambda^{(N)}}(a_1,\dots,a_N)} \cdot \frac{Q_{\nu^{(N)}/\lambda^{(N)}}(u)}{\Pi(a_1,\dots,a_N;u)}$$

maps $M_N(a_1,\dots,a_N;b_1,\dots,b_M)$ to $M_N(a_1,\dots,a_N;b_1,\dots,b_M,u)$.

Note: $$\sum_{\nu} \frac{Q_{\nu/\lambda}(u)}{\Pi(a_1,\dots,a_N;u)} P_\nu(a_1,\dots,a_N) = P_\lambda(a_1,\dots,a_N)$$

so $P_\nu(a_1,\dots,a_N)$ has eigenvalue 1 and is positive inside $\nu \succ \lambda$ and zero outside: (q,t)-deformed Dyson Brownian motion



Multivariate Markov kernel

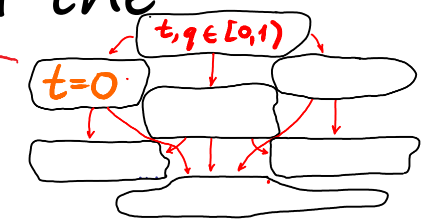
$$\mathbb{P}^{(u)}(\lambda,\nu) := \pi^{(u)}(\lambda^{(1)},\nu^{(1)}) \prod_{k=2}^{N} \frac{\pi_k^{(u)}(\lambda^{(k)},\nu^{(k)})\,\Lambda^k_{k-1}(\nu^{(k)},\nu^{(k-1)})}{(\pi_k^{(u)}\Lambda^k_{k-1})(\lambda^{(k)},\nu^{(k-1)})}$$

sequentially updates GT-pattern, mapping

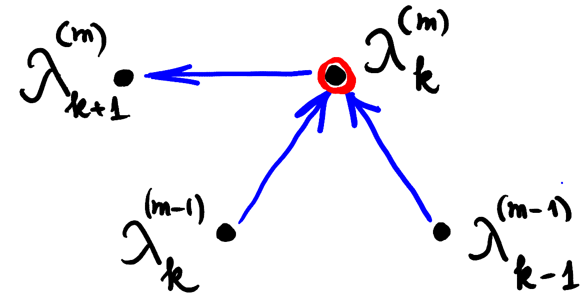
$\mathbb{M}_{[1,N]}(a_1,\dots,a_N;b_1,\dots,b_M)$ to $\mathbb{M}_{[1,N]}(a_1,\dots,a_N;b_1,\dots,b_M,u)$

Other dynamics may preserve class of Macdonald processes (Borodin's talk)

Here is an example of a Markov process preserving the class of the q-Whittaker processes (Macdonald processes with $t=0$).



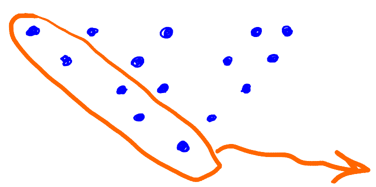
Each coordinate jumps by 1 to the right independently of the others with

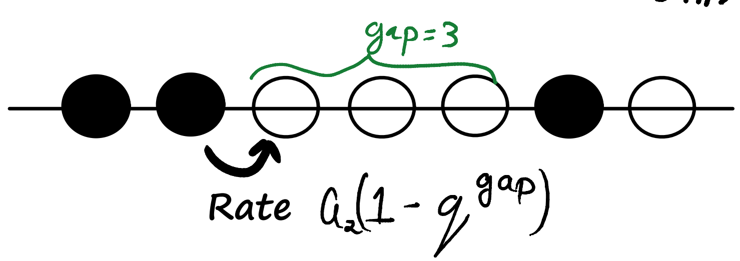


$$\mathrm{rate}\left(\lambda_k^{(m)}\right) = a_m \frac{\left(1-q^{\lambda_{k-1}^{(m-1)}-\lambda_k^{(m)}}\right)\left(1-q^{\lambda_k^{(m)}-\lambda_{k+1}^{(m)}+1}\right)}{\left(1-q^{\lambda_k^{(m)}-\lambda_k^{(m-1)}}\right)}$$

Simulation

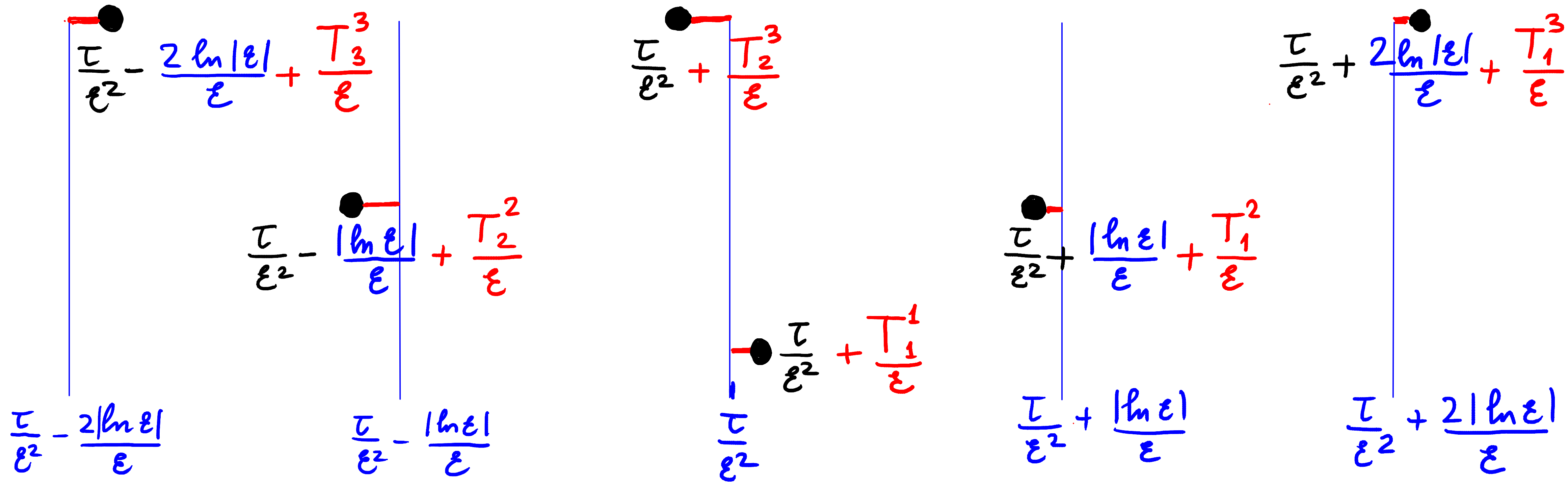
The set of coordinates $\left\{\lambda_m^{(m)}-m\right\}_{m\geq 1}$ forms q-TASEP





[O'Connell-Pei '12] give different dynamics with q-TASEP marginal

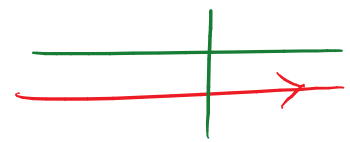
As $q=e^{-\varepsilon}\to 1$, at large times τ/ε^2, with zero initial conditions, low rows of the triangular array behave as



The real array $\{T_j^m\}_{1\le j\le m}$ is distributed according to the Whittaker process, and T_1^N or $-T_N^N$ is distributed as $\log Z(\tau,N)$. The Whittaker process and its connection to polymers is due to [O'Connell, '09].

[O'Connell '09] proved a Laplace transform formula

$$\mathbb{E}\left[e^{-\varsigma Z(\tau,N)}\right] = \int \cdots \int \prod_{j,k=1}^{N} \Gamma(i\nu_j) \prod_{j=1}^{N} \varsigma^{-i\nu_j} e^{-\frac{\tau}{2}\nu_j^2} \, m_N(d\nu)$$



Skylanin measure

$$m_N(d\nu) = \frac{d\nu_1 \cdots d\nu_N}{(2\pi i)^N N!} \prod_{j\neq k}^{N} \frac{1}{\Gamma(i\nu_k - i\nu_j)}$$

Initially unclear how to take asymptotics of this.

[Borodin-C '11] develop general machinery to compute observables; leads to Fredholm determinant formula.

[Borodin-C-Remenik '12] show equivalence of two formulas.

Evaluation of averages is based on the following observation.
Let $\mathcal{D}$ be an operator that is diagonalized by the Macdonald polynomials (for example, a product of Macdonald operators),

$$\mathcal{D} P_\lambda = d_\lambda P_\lambda .$$

Applying it to the Cauchy type identity $\sum_\lambda P_\lambda(a) Q_\lambda(b) = \Pi(a;b)$ we obtain

$$\mathbb{E}[d_\lambda] = \frac{\mathcal{D}^{(a)} \Pi(a;b)}{\Pi(a;b)} .$$

If all the ingredients are explicit (as for products of Macdonald operators), we obtain meaningful probabilistic information. Contrast with the lack of explicit formulas for the Macdonald polynomials.

Macdonald difference operators $\{D_N^r\}_{1\le r\le N}$

$$D_N^r := t^{r(r-1)/2} \sum_{\substack{I\subset\{1,\ldots,N\}\\ |I|=r}} \prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I} T_{q,x_i}$$

$(T_{q,x_i} f)(x_1,\ldots,x_N) = f(x_1,\ldots q x_i,\ldots x_N)$

Commuting operators all diagonalized by $\{P_\lambda\}_{\ell(\lambda)\le N}$

$$D_N^r P_\lambda(x_1,\ldots,x_N) = e_r(q^{\lambda_1}t^{N-1}, q^{\lambda_2}t^{N-2},\ldots, q^{\lambda_N}t^0)$$

$e_r(y_1,\ldots,y_N) = \sum_{i_1<\cdots<i_r} y_{i_1}\cdots y_{i_r}$ "elementary symmetric polynomials

Expectations characterize Macdonald meas. (q=t ->cor kernel):

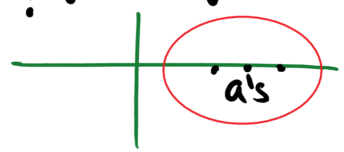
$$\mathbb{E}\left[\prod_{i=1}^{k} e_{r_i}(q^{\lambda_i}t^{N-i})\right] = \frac{D_N^{r_1}\cdots D_N^{r_k}\,\Pi(a_1,\ldots a_N; b_1,\ldots,b_M)}{\Pi(a_1,\ldots a_N; b_1,\ldots,b_M)}$$

Note $\Pi(a_1,\dots,a_N; b_1,\dots,b_M) = \Pi(a_1; b_1,\dots,b_M)\cdots\Pi(a_N; b_1,\dots,b_M)$

Encode products of difference operators as contour integrals

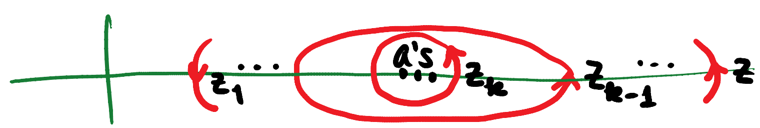
<u>Proposition</u> [Borodin-C '11]: For nice $F(u_1,\dots,u_N) = f(u_1)\cdots f(u_N)$

$$(D_N^r F)(\vec{a}) = \frac{F(\vec{a})}{(2\pi i)^r r!}\int\cdots\int \det\left(\frac{1}{tz_k - z_\ell}\right)_{k,\ell=1}^{r} \prod_{j=1}^{r}\left(\prod_{m=1}^{N}\frac{tz_j - a_m}{z_j - a_m}\right)\frac{f(qz_j)}{f(z_j)}\,dz_j$$



Here is another example for powers of first diff. op. at $t=0$

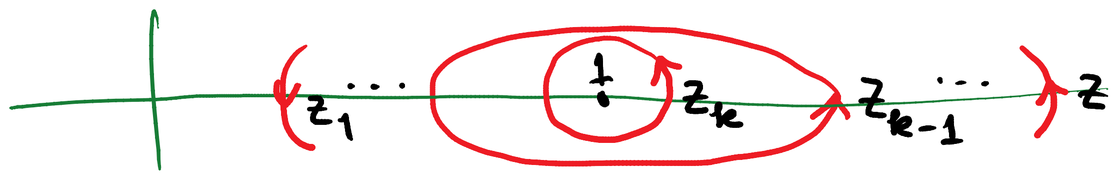
$$((D_N^1)^k F)(\vec{a}) = \frac{(-1)^k}{(2\pi i)^k}\int\cdots\int \prod_{1\le A<B\le k}\frac{z_A - z_B}{z_A - q z_B}\prod_{j=1}^{k}\left(\prod_{m=1}^{N}\frac{a_m}{a_m - z_j}\right)\frac{f(qz_j)}{f(z_j)}\frac{dz_j}{z_j}$$



Taking $t=0$ and the observables corresponding to products of the first order Macdonald operators on different levels results in the integral representation for the q-moments of the q-TASEP particle positions (the eigenvalues are $q^{\lambda^{(N_j)}} = q^{x_{N_j}(t)+N_j}$)

$$\mathbb{E}\left[q^{(x_{N_1}(t)+N_1)+\ldots+(x_{N_k}(t)+N_k)}\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint \cdots \oint \prod_{A<B} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{e^{(q-1)t z_j}}{(1-z_j)^{N_j}} \frac{dz_j}{z_j}$$

$(N_1 \geq N_2 \geq \ldots \geq N_k)$



Commutation relation implies the q-TASEP many body system

$$\left[(D_N^1)^k, p_1\right] = (1-q^k)(D_{N-1}^1 - D_N^1)(D_N^1)^{k-1}$$

We saw how these led to q-Laplace transform of $q^{\lambda^{(N_j)}} = q^{x_{N_j}(t)+N_j}$

Using another operator (attributed to Noumi) diagonalized by the P_λ, [Borodin-C-Gorin-Shakirov '13] prove (q,t)-Laplace transform Fredholm determinant formula:

$$\mathbb{E}\left[\prod_{i=1}^{N} \frac{(\zeta q^{\lambda_i} t^{N-i+1};q)_\infty}{(\zeta q^{\lambda_i} t^{N-i};q)_\infty}\right] = \det(I + K_\zeta) \quad \text{Mellin Barnes type}$$

$$= \frac{1}{(\zeta;q)_\infty}\det(I+\zeta\tilde{K}) \quad \text{Cauchy type}$$

At $t=0$ these reduce to those we have already seen.

One last example of an expectation at $t=0$ is

$$\mathbb{E}\left[q^{k(\lambda_N+\cdots+\lambda_{N-r+1})}\right]=\frac{(-1)^{rk}q^{r^2\frac{k(k-1)}{2}}}{(2\pi i)^{rk}(r!)^k}\int\cdots\int\prod_{1\le A<B\le k}\prod_{i,j=1}^{r}\frac{z_{A,i}-z_{B,j}}{z_{A,i}-qz_{B,j}}$$

$$\times\prod_{A=1}^{k}\left(\prod_{i\neq j}^{r}\frac{z_{A,i}-z_{A,j}}{-z_{A,j}}\right)\left(\prod_{j=1}^{r}\prod_{m=1}^{N}\frac{a_m}{a_m-z_{A,j}}\,e^{(q-1)\gamma z_{A,j}}\frac{dz_{A,j}}{z_{A,j}}\right)$$

Limits to moments of r-path polymer partition function [O'Connell '09, O'Connell-Warren '11] (O'Connell/Warren talk) or further to LPP / sum of bottom r eigenvalues (Adler talk)

Some problems:

- Multipoint distribution for q-TASEP / ASEP etc.?
- r-path polymer partition function distribution?
- In general, what replaces determinantal structure?
- Develop Bethe ansatz at higher level in hierarchy?
- Analog of Macdonald processes about ASEP?
- Develop theory for other root systems / symmetries?

To summarize:

- Studied various integrable probabilistic systems in KPZ class
- Many body system approach: General (q-TASEP, ASEP) but provides no way of knowing which models / observables fit.
- Macdonald processes: Structural properties of Macdonald poly. give rise to models, observables and formulas (no ASEP).
- Geometric RSK: In special cases (e.g. Log-gamma polymer) yields further probabilistic content of Whittaker processes, different GT-dynamics and Gibbs property (Hammond talk)
- More cases of Macdonald processes to be investigated!