Integrability in the Kardar-Parisi-Zhang universality class

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An "integrable probabilistic system" has two properties:

- Exact and concise formulas for expectations of a rich class of interesting observables
- Scaling limits of systems, observables and formulas provide exact descriptions of large universality classes

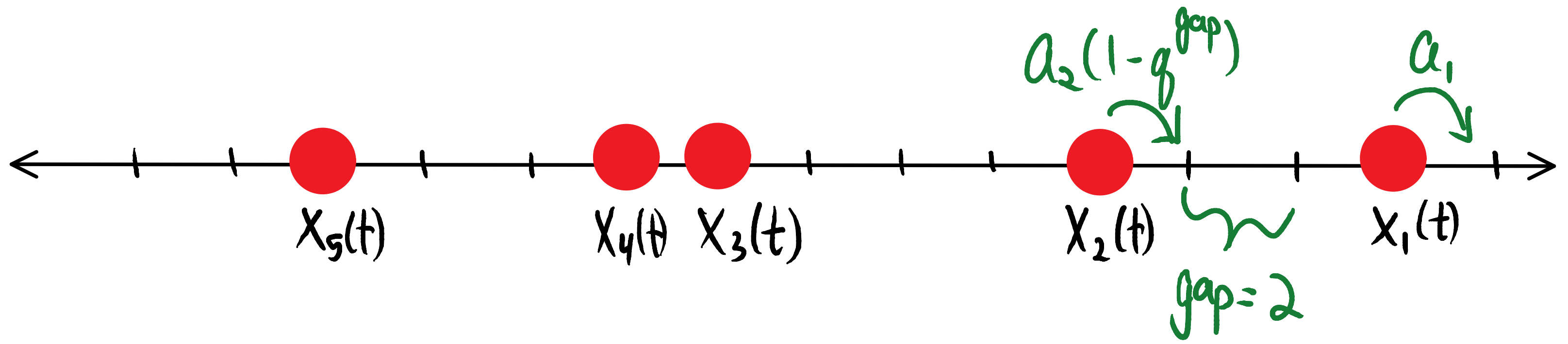
We will focus on systems related to KPZ universality class.

The primary source of exact solvability here comes from representation theory and integrable systems.

Plan for the lectures:

- Quantum many body system approach
    - q-TASEP (-> semi-discrete SHE -> SHE)
    - ASEP
- Symmetric function theory approach
    - Macdonald processes
    - (Briefly) Geometric RSK correspondence, Gibbs properties

q-TASEP:



$q \in (0,1)$

$a_i \in (0,\infty)$

Restrict to N particle state space

$$\mathbb{X}^N = \left\{ \vec{x} := (x_0, x_1, \ldots, x_N) \in \{\infty\} \times \mathbb{Z}^N : \infty = x_0 > x_1 > x_2 > \cdots > x_N \right\}$$

Generator acts on $f : \mathbb{X}^N \to \mathbb{R}$ as

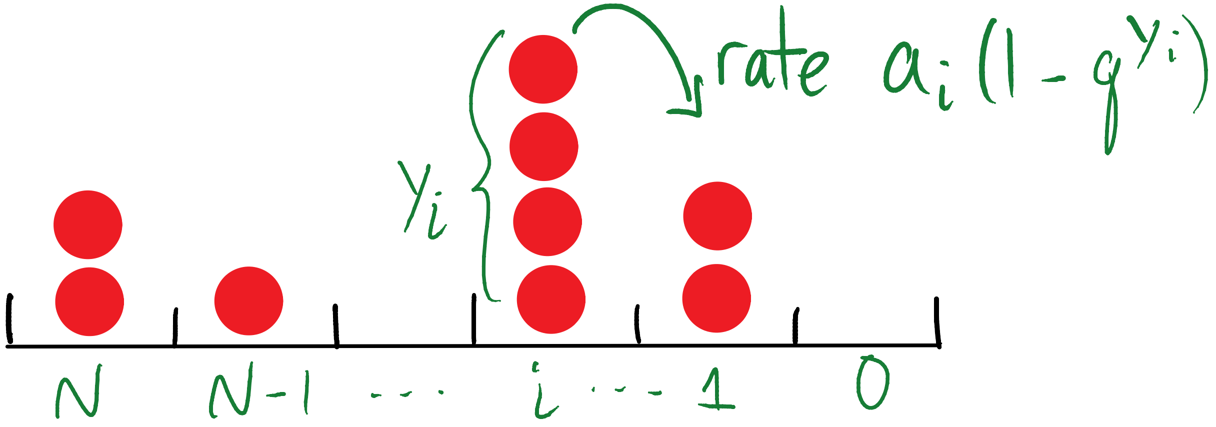
$= (x_0, x_1, \ldots, x_i + 1, \ldots, x_N)$

$$\left(L^{q\text{-TASEP}} f\right)(\vec{x}) = \sum_{i=1}^{N} a_i \left(1 - q^{x_{i-1} - x_i - 1}\right)\left(f(\vec{x}_i^{+}) - f(\vec{x})\right)$$

Natural initial condition is <u>step</u> where $x_i(0) = -i \quad , i \geq 1$

(When q=0, we recover the usual TASEP)

q-TAZRP:



N+1 site state space $\mathbb{Y}^N = \{\vec{y} = (y_0, y_1, \dots, y_N) \in \mathbb{Z}_{\geq 0}^{\{0,1,\dots,N\}}\}$

$\mathbb{Y}^N_k = \{\vec{y} \in \mathbb{Y}^N : \sum y_i = k\}$

Generator acts on $h : \mathbb{Y}^N \to \mathbb{R}$ as

$= (y_0, \dots, y_{i-1}+1, y_i - 1, \dots, y_N)$

$$(L^{q\text{-TAZRP}} h)(\vec{y}) = \sum_{i=1}^{N} a_i (1-q^{y_i}) \left(h(\vec{y}^{\,i,i-1}) - h(\vec{y}) \right)$$

[Sasamoto-Wadati '98] stochastic representation of q-Bosons

[Balazs-Komjathy-Seppalainen '08] stationary 1/3 exponent

__Duality__: Suppose $x(t) \in X$ and $y(t) \in Y$ independent Markov processes and $H : X \times Y \to \mathbb{R}$. Then $x(t)$ and $y(t)$ are dual with respect to H if for all x, y, and t

$$\mathbb{E}^{x}\left[H(x(t), y)\right] = \mathbb{E}^{y}\left[H(x, y(t))\right].$$

- Duality leads to hidden evolution equations for expectations of observables corresponding to the duality function.

Theorem [Borodin-C-Sasamoto '12]: q-TASEP $\vec{x}(t) \in \mathbb{X}^N$ and q-TAZRP $\vec{y}(t) \in \mathbb{Y}^N$ are dual with respect to

$$H(\vec{x},\vec{y}) = \prod_{i=0}^{N} q^{(x_i+i)y_i}$$

(convention that if $y_0 > 0$, $H \equiv 0$)

Proof: Suffices to show that

$$\begin{array}{ccc}
L^{q\text{-TASED}} H(\vec{x},\vec{y}) & \overset{?}{=} & L^{q\text{-TAZRP}} H(\vec{x},\vec{y}) \\
\| & & \| \\
\sum_{i=1}^{N} a_i \left(1 - q^{x_{i-1}-x_i-1}\right)\left(q^{y_i}-1\right) \prod_{j=0}^{N} q^{(x_j+j)y_j} & = & \sum_{i=1}^{N} a_i \left(1-q^{y_i}\right)\left(q^{x_{i-1}-x_i-1}-1\right) \prod_{j=0}^{N} q^{(x_j+j)y_j}
\end{array}$$

$\square$

Purpose of duality (for us):

If $\vec{y} = (0,0,\dots,0,k)$ then

$$h(t;\vec{y}) := \mathbb{E}^{\vec{x}}\left[H(\vec{x}(t),y)\right] = \mathbb{E}^{\vec{x}}\left[q^{k(x_N(t)+N)}\right]$$

Duality implies that for $\vec{x}$ fixed, $h(t;\vec{y})$ solves the

<u>True evolution equation</u>:

$$\begin{cases} \frac{d}{dt} h(t;\vec{y}) = L^{q\text{-TAZRP}} h(t;\vec{y}) \\ h(0;\vec{y}) = H(\vec{x},\vec{y}) \quad \left[= h_0(\vec{y})\right] \end{cases}$$

True evolution equation splits according to number of particles

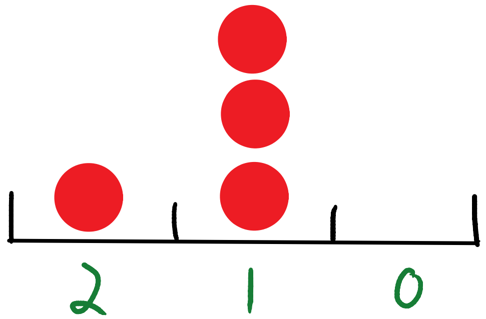
$$W^k_{\geq 0} := \{ \vec{n} = (n_1, \dots, n_k) \in \mathbb{Z}^k_{\geq 0} : n_1 \geq n_2 \geq \cdots \geq n_k \geq 0 \}$$

Encode $\vec{y} \in \mathbb{Y}^N_k$ by an ordered list of particle locations

$$\begin{tikzcd}
\mathbb{Y}^N_k \ni & \vec{y} \arrow[r, mapsto] & \vec{n}(\vec{y}) & \in W^k_{\geq 0} \\
 & \vec{y}(\vec{n}) & \vec{n} \arrow[l, mapsto] &
\end{tikzcd}$$

Example: $N = 2$, $k = 4$

$$\vec{y} = (0, 3, 1) \longleftrightarrow \vec{n} = (1, 1, 1, 2)$$



We can encode true evolution equation in the $\vec{n}$ coordinates by writing $g(t;\vec{n}) := h(t;\vec{y}(\vec{n}))$, $g_0(\vec{n}) := h_0(\vec{y}(0))$

- $k=1$: single particle, so $\vec{n}=(n)$, then

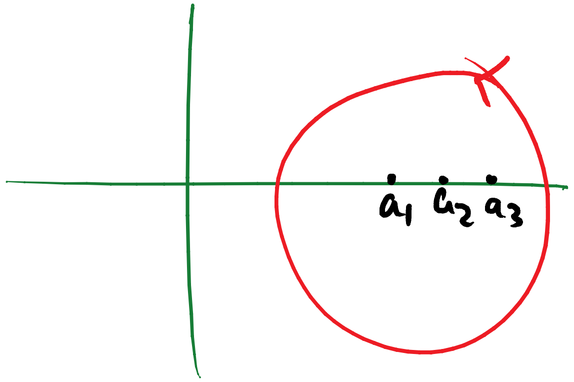
$$\begin{cases} \frac{d}{dt} g(t;n) = a_n(1-g)\nabla g(t;n) \\ g(t;0) \equiv 0 \\ g(0;n) = g_0(n) \end{cases}$$

$$\left[(\nabla f)(n) := f(n-1) - f(n) \right]$$

For step initial data $X_i + i = 0$ so $H(\vec{x},\vec{y}) \equiv 1$ and so too $g_0 \equiv 1$

Claim: $\mathbb{E}^{\text{step}}\left[q^{X_n(t)+n}\right] = g(t;n) = \frac{-1}{2\pi i}\oint g_z(t;n)\,\frac{dz}{z}$

where $g_z(t;n) = \prod_{m=1}^{n} \frac{a_m}{a_m - z}\, e^{(q-1)tz}$



Proof: Check free equation, zero boundary condition, and initial data. $\square$

- k=2: two particles, so $\vec{n} = (n_1 \geq n_2)$
  - ○ If $n_1 > n_2$

acts as ∇ on n_i coordinate

$$\frac{d}{dt} g(t;\vec{n}) = \sum_{i=1}^{2} a_{n_i} (1-q) \nabla_i \, g(t;\vec{n})$$

  - ○ If $n_1 = n_2$

$$\frac{d}{dt} g(t;\vec{n}) = a_{n_2} (1-q^2) \nabla_2 \, g(t;\vec{n})$$

Not constant coefficient, so unclear how to solve...

- k>2: there are different equations for each type of clustering (i.e., many body interactions)

Proposition: (Free evolution eqn with k-1 boundary conditions):

If $u : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}^{k} \longrightarrow \mathbb{R}$ solves

- For all $\vec{n} \in \mathbb{Z}_{\geq 0}^{k}$, $t \geq 0$,

Free evolution eqn $$\frac{d}{dt} u(t;\vec{n}) = \sum_{i=1}^{k} a_{n_i} (1-q) \nabla_i u(t;\vec{n})$$

- For all $\vec{n} \in \mathbb{Z}_{\geq 0}^{k}$ such that $n_i = n_{i+1}$

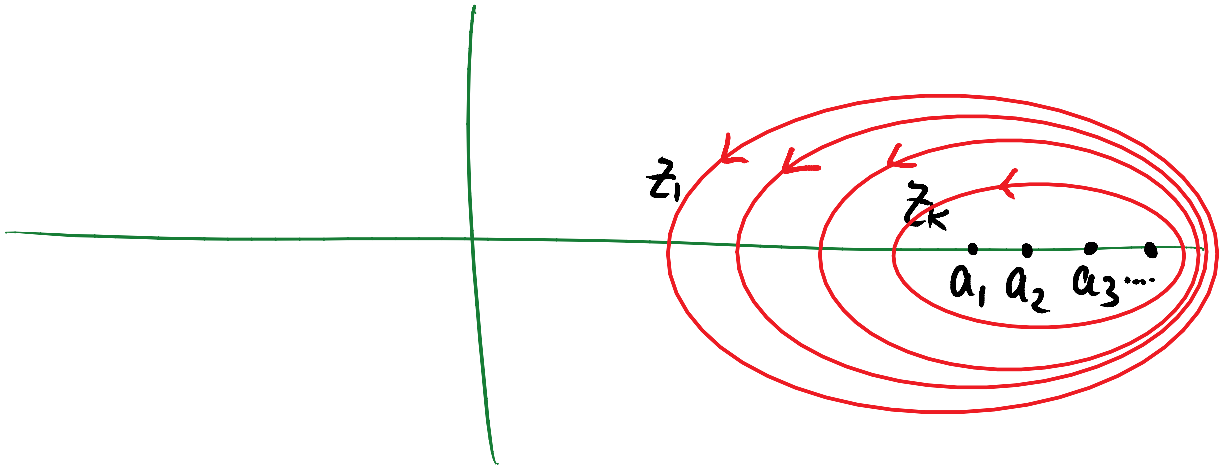
Boundary conditions $$(\nabla_i - q \nabla_{i+1}) u(t;\vec{n}) = 0$$

- For all $\vec{n} \in \mathbb{Z}_{\geq 0}^{k}$ such that $n_k = 0$, $u(t;\vec{n}) \equiv 0$
- For all $\vec{n} \in W_{\geq 0}^{k}$, $u(0;\vec{n}) = g_0(\vec{n})$

Then, restricted to $\vec{n} \in W_{\geq 0}^{k}$, $g(t;\vec{n}) = u(t;\vec{n})$.

Theorem: For step initial condition (i.e., $g_0(\vec{n}) \equiv 1$) we have

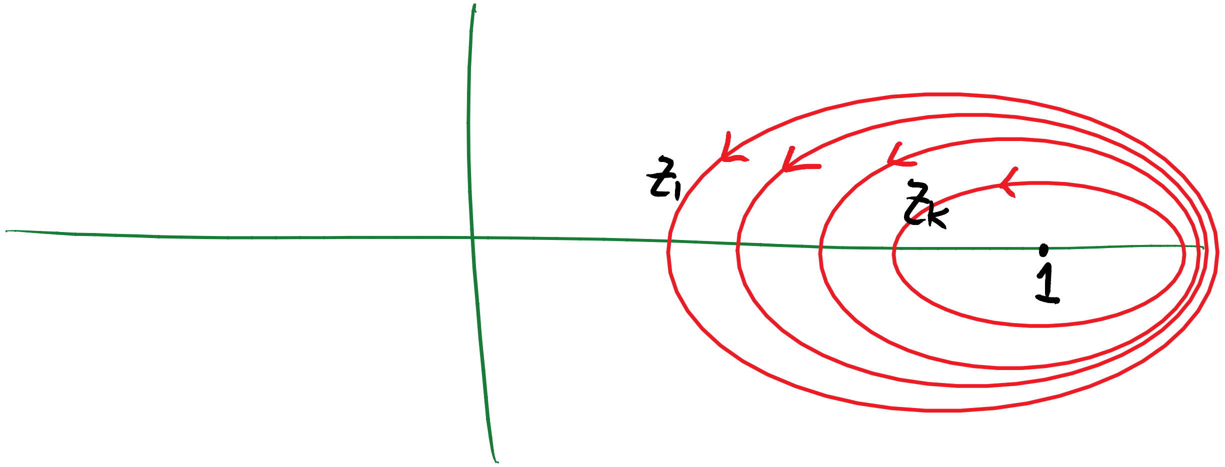
$$\mathcal{U}(t;\vec{n}) = \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k} \oint\cdots\oint \prod_{1\leq A<B\leq k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} g_{z_j}(t;n_j)\frac{dz_j}{z_j}$$



Proof: Only new aspect is boundary condition. Applied to integrand brings out factor of $z_i - q z_{i+1}$. Contour symmetry and integrand asymmetry shows integral is zero. □

Implies joint moment formulas. For example, if all $a_i \equiv 1$

$$\mathbb{E}^{\text{step}}\left[q^{k(x_n(t)+n)} \right] = \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k} \oint \cdots \oint \prod_{1 \le A < B \le k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{e^{(q-1)t z_j}}{(1-z_j)^n} \frac{dz_j}{z_j}$$



Success in using moments to asymptotically study one-point distribution, though multi-point distributions remain open

True evolution equation also equivalent to a certain

q-deformed discrete delta Bose gas

$$\frac{d}{dt} g(t;\vec{n}) = H g(t;\vec{n})$$

with Hamiltonian

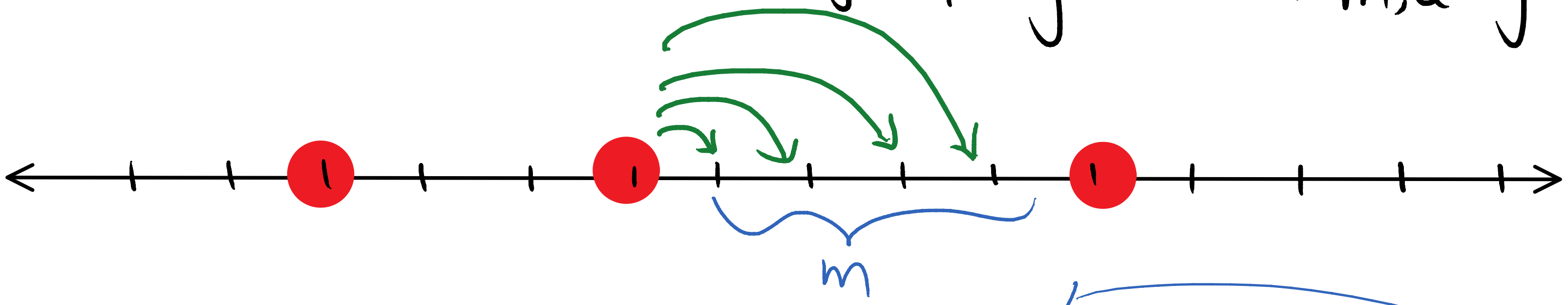
$$H = (1-q)\left[\sum_{j=1}^{k} \nabla_j + (1-q^{-1}) \sum_{1\leq i<j\leq k} \delta_{n_i=n_j} \, q^{j-i} \nabla_j \right]$$

subject to Bosonic symmetry and zero boundary condition

Integrability (equiv. to free eqn with k-1 B.C.s) not obvious for this system (Note: not all delta Bose gases are integrable)

(Parallel) Geometric discrete time q-TASEP [Borodin-C '13]:

$$\mathbb{P}(\text{jump} = j) = P_{m,\alpha}(j)$$



$q \in (0,1)$, $\alpha \in (0,1)$

$$P_{m,\alpha}(j) = \alpha^{j} (\alpha;q)_{m-j} \frac{(q;q)_m}{(q;q)_{m-j}(q;q)_j} \mathbb{1}_{0 \le j \le m}$$

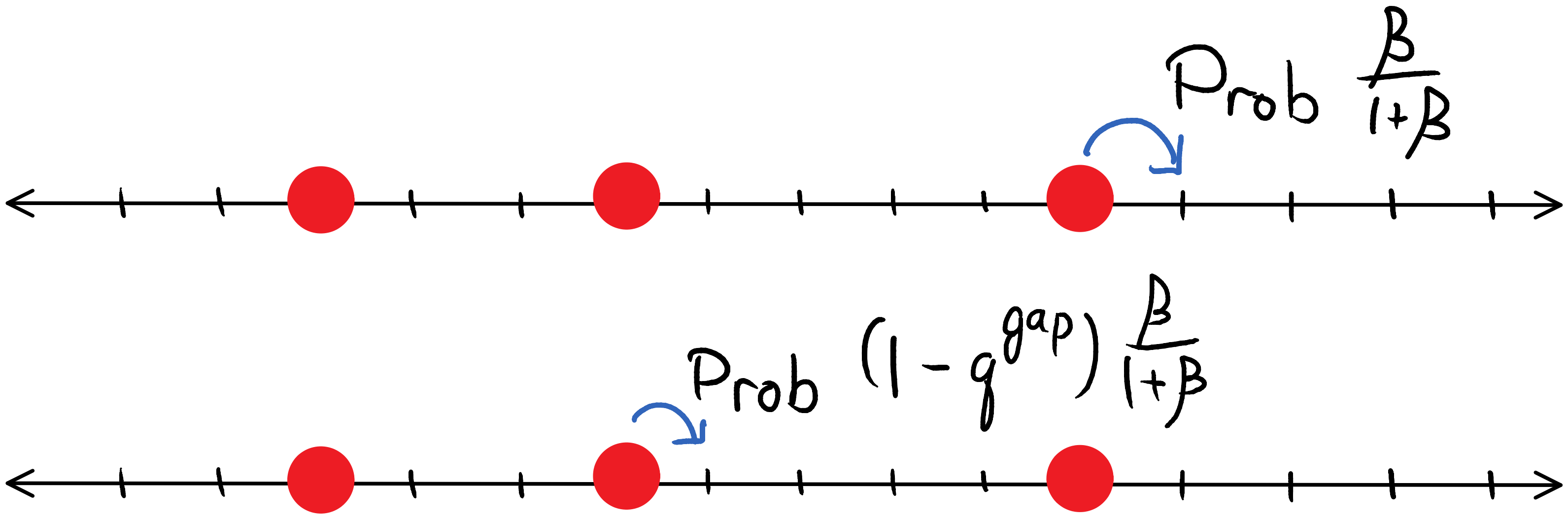
$$\left[(a;q)_n = \prod_{j=0}^{n-1} (1 - aq^{j}) \right]$$

At q=0 -> parallel geometric TASEP with blocking

[Warren-Windridge '09]

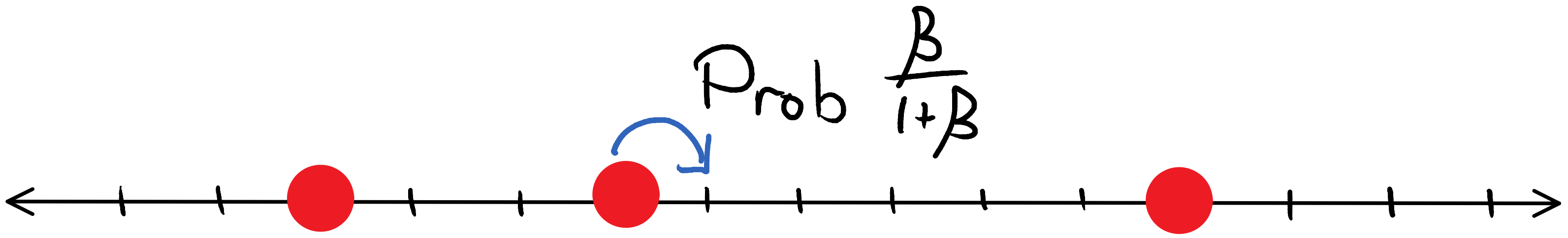
(Sequential) Bernoulli discrete time q-TASEP [Borodin-C '13]:

1)



$q\in(0,1)$, $\beta\in(0,\infty)$

2)



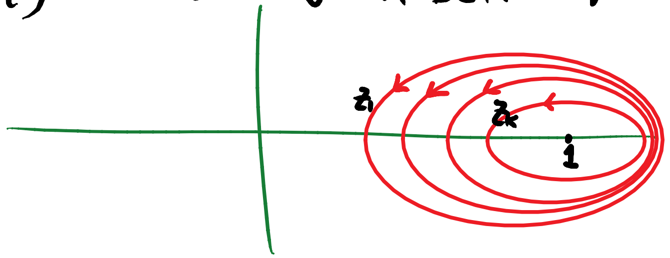
3) …

At q=0 -> sequential Bernoulli TASEP [Borodin-Ferrari '08]

q-TASEP joint moments satisfy various many body systems

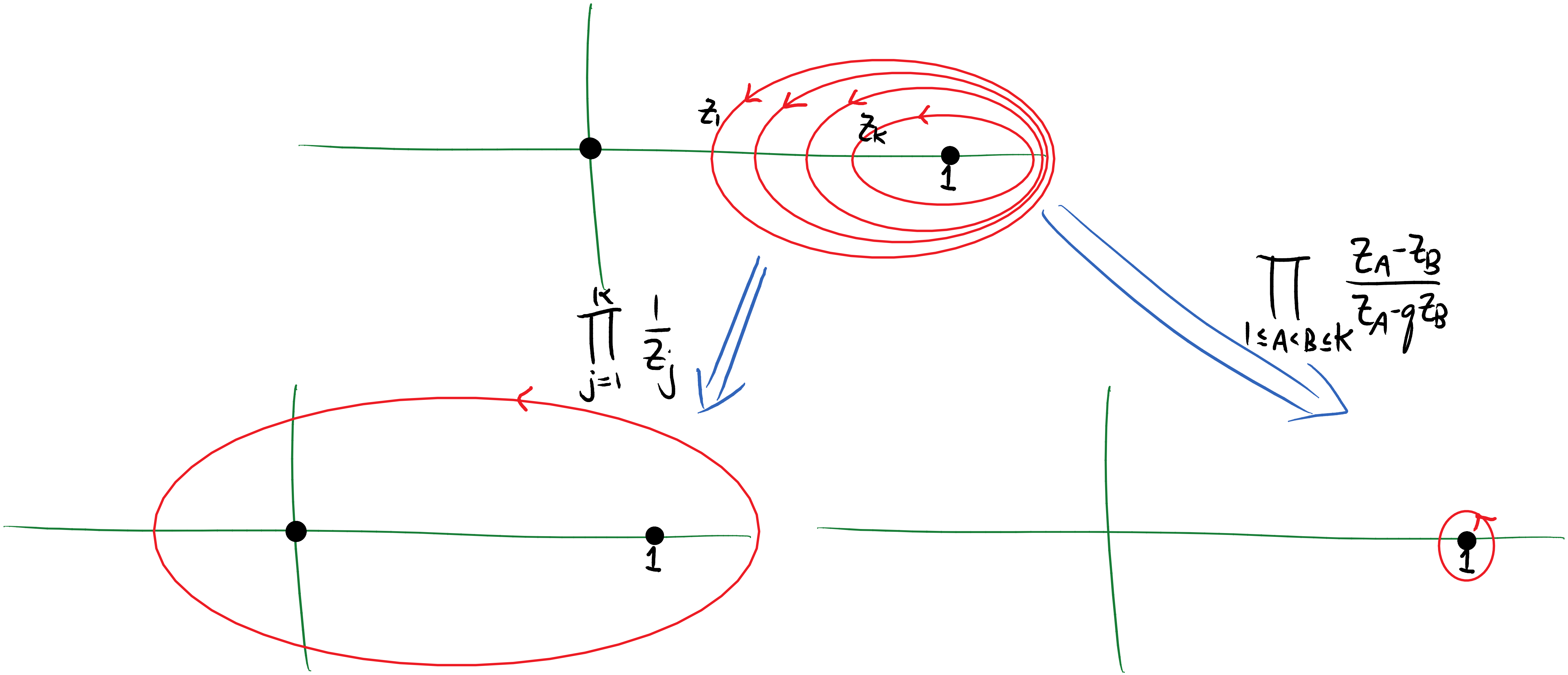
Theorem [Borodin-C '13]: For $n_1 \geq n_2 \geq \cdots \geq n_k > 0$

$$\mathbb{E}^{\text{step}}\left[\prod_{j=1}^{k} q^{x_{n_j}(t)+n_j}\right] = \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k} \oint \cdots \oint \prod_{1\leq A<B\leq k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{1}{(1-z_j)^{n_j}} \frac{f(qz_j)}{f(z_j)} \frac{dz_j}{z_j}$$



$$f(z) = \begin{cases} e^{tz}, & \text{Poissonian continuous } q\text{-TASEP} \\ \left(\frac{1}{(\alpha z;q)_\infty}\right)^t, & \text{Geometric discrete } q\text{-TASEP} \\ (1+\beta z)^t, & \text{Bernoulli discrete } q\text{-TASEP} \end{cases}$$

Nested contours -> fixed contours



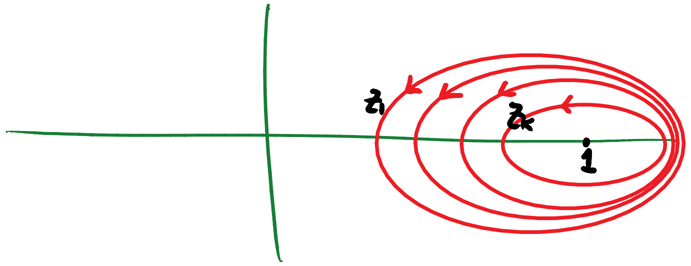
$$\prod_{j=1}^{k} \frac{1}{z_j}$$

$$\prod_{1\le A<B\le k} \frac{z_A - z_B}{z_A - q z_B}$$

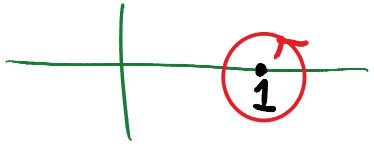
Must account for residues from poles crossed in deformation

Proposition [Borodin-C '11]: For "nice" f(x)

$$\frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k} \oint \cdots \oint \prod_{1 \le A < B \le k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{1}{(1-z_j)^{n_j}} \frac{f(q z_j)}{f(z_j)} \frac{dz_j}{z_j}$$



$$= \sum_{\substack{\lambda \vdash k \\ \lambda = 1^{m_1} 2^{m_2} \cdots}} \frac{(1-q)^k}{m_1! m_2! \cdots} \cdot \frac{1}{(2\pi i)^{\ell(\lambda)}} \oint \cdots \oint \det\left[\frac{1}{w_i q^{\lambda_i} - w_j}\right]_{i,j=1}^{\ell(\lambda)} E_{\vec{n}}(w_1, q w_1, \ldots, q^{\lambda_1 - 1} w_1, w_2, \ldots, q^{\lambda_2 - 1} w_2, \ldots, w_{\ell(\lambda)}, \ldots, q^{\lambda_{\ell(\lambda)} - 1} w_{\ell(\lambda)}) \, dw$$



where

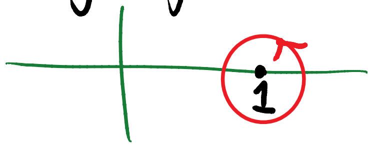
$$E_{\vec{n}}(z_1, \ldots, z_k) = \prod_{j=1}^{k} \frac{f(q z_j)}{f(z_j)} \cdot \sum_{\sigma \in S_k} \prod_{k \ge B > A \ge 1} \frac{z_{\sigma(A)} - q z_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}} \prod_{j=1}^{k} \frac{1}{(1 - z_{\sigma(j)})^{n_j}}$$

For all $n_j \equiv n$, $E_{\vec{n}}$ simplifies to

$$E_n(z_1,\ldots,z_k) = \prod_{j=1}^{k} \frac{f(qz_j)}{f(z_j)} \frac{1}{(1-z_j)^n} \cdot \underbrace{\sum_{\sigma\in S_k} \prod_{k\ge B>A\ge 1} \frac{z_{\sigma(A)}-qz_{\sigma(B)}}{z_{\sigma(A)}-z_{\sigma(B)}}}_{C_k = \frac{(q;q)_k}{(1-q)^k} =: k_q!}$$

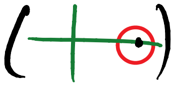
Conclusion: for step initial condition q-TASEP

$$\mathbb{E}^{\text{step}}\left[q^{k(x_n(t)+n)}\right] = k_q! \sum_{\substack{\lambda\vdash k\\ \lambda=1^{m_1}2^{m_2}\cdots}} \frac{(1-q)^k}{m_1!m_2!\cdots} \cdot \frac{1}{(2\pi i)^{\ell(\lambda)}} \oint\cdots\oint \det\left(\frac{1}{w_i q^{\lambda_i}-w_j}\right)_{i,j=1}^{\ell(\lambda)} \prod_{j=1}^{\ell(\lambda)} \frac{f(q^{\lambda_j}w_j)}{f(w_j)} \left(\frac{1}{(w_j;q)_{\lambda_j}}\right)^n$$



Moment generating function -> Fredholm determinant

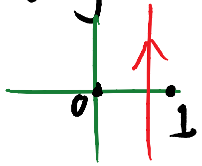
$$G(\zeta) := \sum_{k\geq 0} \mathbb{E}^{\text{step}}\left[q^{k(x_n(t)+n)}\right] \frac{\zeta^k}{k_q!} = \det\left(I + K_\zeta\right)_{L^2(\;)}$$



$$K_\zeta(w,w') = \sum_{\lambda=1}^{\infty} \left[(1-q)\zeta\right]^\lambda \frac{f(q^\lambda w)\,(q^\lambda w;q)_\infty^N}{f(w)\,(w;q)_\infty^N} \cdot \frac{1}{wq^\lambda - w'}$$

"Mellin Barnes" representation suitable for asymptotics

$$K_\zeta(w,w') = \frac{1}{2\pi i}\int \frac{\pi}{\sin(-\pi s)} \left[-(1-q)\zeta\right]^s \frac{f(q^s w)\,(q^s w;q)_\infty^N}{f(w)\,(w;q)_\infty^N} \cdot \frac{1}{wq^s - w'}\, ds$$



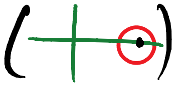
Moments of $q^{x_n(t)+n} \leq 1$, so they characterize the distribution.

q-deformed exponential [Hahn '49]:

$$e_q(x) := \frac{1}{((1-q)x;q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k}{k_q!} \qquad \left(e_q(x) \xrightarrow{q\uparrow 1} e^x \right)$$

Theorem [Borodin-C '11+]:

$$\mathbb{E}^{\text{step}}\left[e_q\left(\zeta q^{x_n(t)+n}\right)\right] = \sum_{k=0}^{\infty} \mathbb{E}^{\text{step}}\left[q^{k(x_n(t)+n)}\right]\frac{\zeta^k}{k_q!} = \det\left(I + K_\zeta\right)_{L^2(}$$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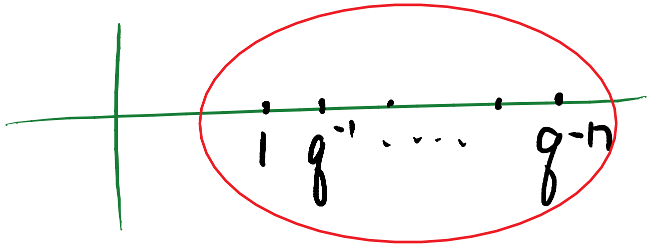
The above can be seen as a rigorous q-deformed discrete version of the polymer replica trick.

q-deformed Laplace transform has simple inversion formula.

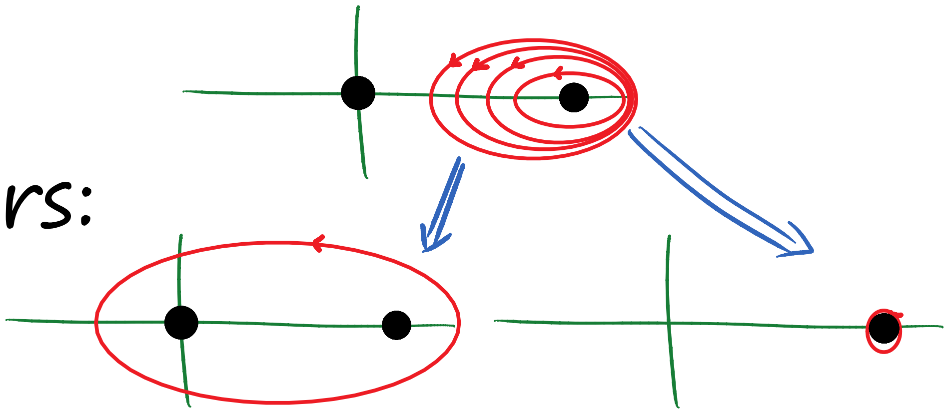
For $f \in \ell_1(\mathbb{N})$ and $z \in \mathbb{C}/q^{-\mathbb{N}}$ define $\hat{f}^q(z) := \sum_{n=0}^{\infty} \frac{f(n)}{(zq^n;q)_\infty}$

Proposition [Borodin-C '11, Bangerezako '09]:

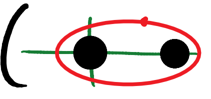
$$f(n) = -q^n \frac{1}{2\pi i} \int (q^{n+1}z;q)_\infty \, \hat{f}^q(z) \, dz$$



We could have chosen the large contours:



Theorem [Borodin C '11+]:

$\mathbb{E}^{\text{step}}\left[e_q\left(\zeta q^{x_n(t)+n}\right)\right] = \frac{1}{((1-q)\zeta;q)_\infty} \det(I+\zeta K)_{L^2(}$  $)$

$$K(w,w') = \frac{1-q}{(1-w)^N} \frac{f(qw)}{f(w)} \frac{1}{qw'-w}$$

"Cauchy" type formula simpler than "Mellin Barnes"; but apparently harder for asymptotic analysis.

q-TASEP satisfies:

$$\begin{cases} d\,q^{x_n(t)+n} = (1-q)\nabla q^{x_n(t)+n}\,dt + q^{x_n(t)+n}\,d\underset{\substack{\uparrow \\ \text{Martingale}}}{M_n}(t) \\ q^{x_n(0)+n} \equiv 1 \ (\text{step}), \quad q^{x_0(t)+0} \equiv 0 \ (x_0 = \infty) \end{cases}$$

<u>Theorem [Borodin-C '11]</u>: For q-TASEP with step init. cond. scale $q = e^{-\varepsilon}$, $t = \varepsilon^{-2}\tau$, $x_n(t) = \varepsilon^{-2}\tau - (n-1)\varepsilon^{-1}\log\varepsilon^{-1} - \varepsilon^{-1}F_\varepsilon(\tau,n)$ and call $z_\varepsilon(\tau,n) = \exp\left\{-\frac{3\tau}{2} + F_\varepsilon(\tau,n)\right\}$. Then as $\varepsilon \searrow 0$, $z_\varepsilon(\cdot,\cdot) \Rightarrow z(\cdot,\cdot)$ where z solves the <u>semi-discrete SHE</u>:

$$\begin{cases} dz(\tau,n) = \nabla z(\tau,n)\,d\tau + z(\tau,n)\,d\underset{\nwarrow \text{ ind. BM's}}{B_n}(\tau) \\ z(0,n) = \mathbb{1}_{n=0}, \quad z(\tau,0) \equiv 0 \end{cases}$$

This leads to a rigorous derivation of $\mathbb{E}\left[e^{-\zeta Z(\tau,n)}\right] = \det(I + \tilde{K}_\zeta)$ and proof that logarithm of semi-discrete SHE has GUE Tracy-widom scaling limit under $\tau^{1/3}$ scaling (Ferrari's talk)

Under weak noise scaling [Alberts-Khanin-Quastel '12] the semi-discrete SHE converges weakly to the continuum SHE [Moreno Flores-Remenik-Quastel '13]:

$$\partial_t Z(t,x) = \tfrac{1}{2}\partial_x^2 Z(t,x) + Z(t,x)\underset{\nwarrow\text{(space time white noise)}}{\xi(t,x)}, \qquad Z(0,x) = \delta_{x=0}$$

Thus a second proof of SHE Laplace transform Fredholm det.

[Sasamoto Spohn '10, Amir-C-Quastel '10, Calabrese-Le Doussal-Rosso '10, Dotsenko '10]

Feynman-Kac representation leads to semi-discrete polymer [O'Connell-Yor '01] and continuum random polymer.

Replica method [Molchanov '86, Kardar '87] shows that joint moments $\mathbb{E}\left[\prod_{j=1}^{k} Z(\tau, n_j)\right]$, $\mathbb{E}\left[\prod_{j=1}^{k} Z(t, x_i)\right]$ satisfy delta Bose gases

$$H = \sum_{j=1}^{k} \nabla_j + \sum_{1 \le i < j \le k} \mathbb{1}_{n_i = n_j} \quad , \quad H = \frac{1}{2} \sum_{j=1}^{k} \partial_{x_j}^2 + \sum_{1 \le i < j \le k} \delta_{x_i = x_j}$$

Both can we written as free evolution eqn. with k-1 B.C.'s and solved by limits of the q-TASEP nested contour formulas.

However, these moments grow like e^{ck^2}, e^{ck^3} and hence do not characterize the distribution of Z (replica trick).

General coupling const. version ($c<0$ repulsive, $c>0$ attractive)

$$\begin{cases} \partial_t u(t;\vec{x}) = \sum_{j=1}^{k} \partial_{x_j}^2 u(t;\vec{x}), \\ (\partial_{x_i} - \partial_{x_{i+1}} - c) u(t;\vec{x}) \Big|_{x_i \nearrow x_{i+1}} = 0, \quad u(0;\vec{x}) = \delta_{\vec{x}=0} \end{cases}$$

is solved (in $x_1 < x_2 < \cdots < x_k$) by the nested contour integral formula

$$u(t;\vec{x}) = \frac{1}{(2\pi i)^k} \int \cdots \int \prod_{1 \le A < B \le k} \frac{z_A - z_B}{z_A - z_B - c} \prod_{j=1}^{k} e^{x_j z_j + \frac{t}{2} z_j^2} dz_j$$

where z_j is integrated over $\alpha_j + i\mathbb{R}$, with $\alpha_1 > \alpha_2 + c > \alpha_3 + 2c > \cdots$

See [Heckman-Opdam '97] Plancherel theorem for delta Bose gas; ideas trace back to [Harish Chandra '58] (Borodin's talk)

SHE in a half space corresponds to symmetric noise $\xi(t,x)=\xi(t,-x)$

Joint moments satisfy delta Bose gas on $x_1 < x_2 < \cdots < x_k < 0$

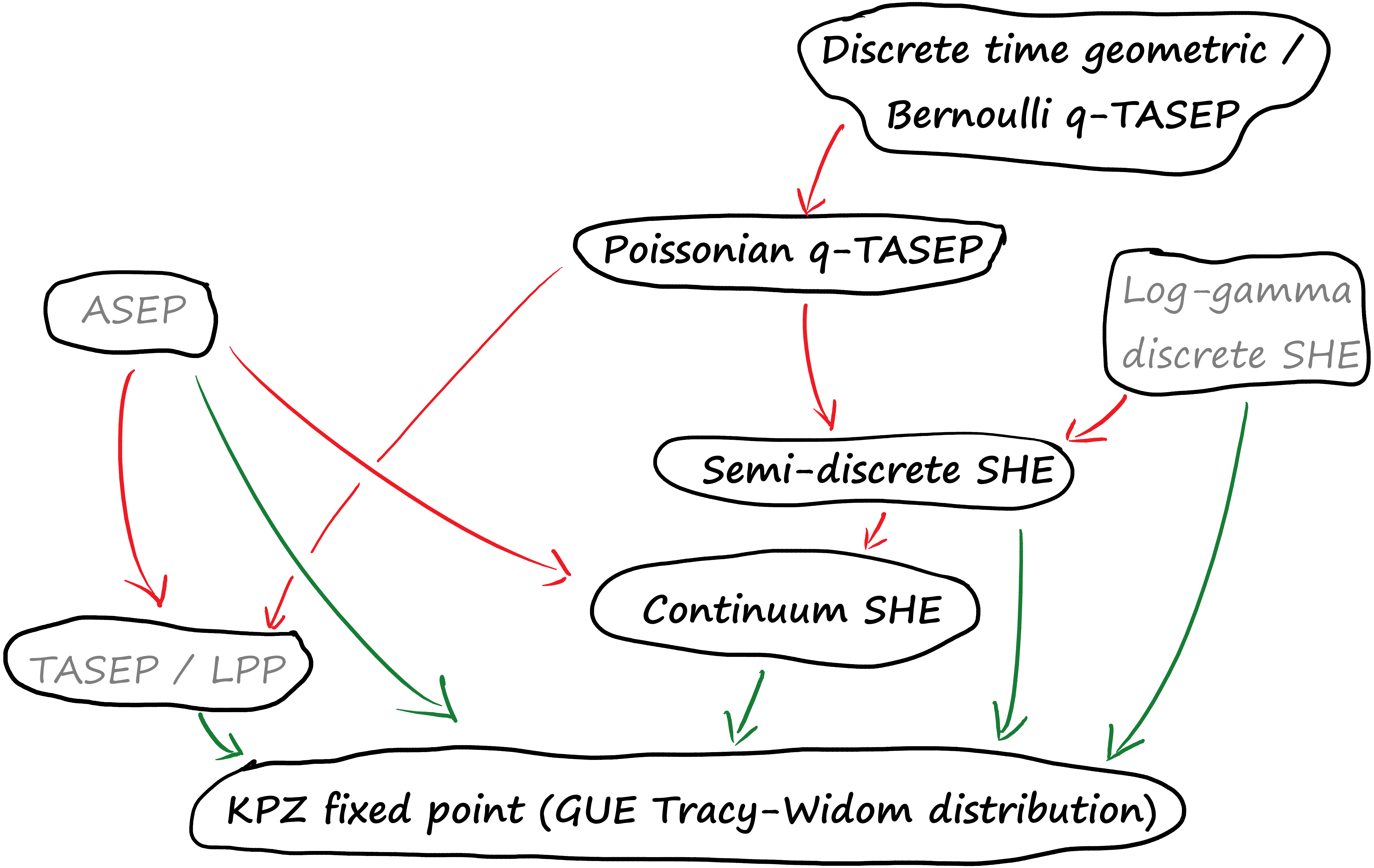
+ extra boundary condition $\frac{d}{dx_k} u(t;\vec{x}) = 0$. (Le Doussal's talk)

Theorem [Borodin-C '13]: Solved by nested contour integral

$$u(t;\vec{x}) = \frac{1}{(2\pi i)^k} \int \cdots \int \prod_{1 \le A < B \le k} \frac{z_A - z_B}{z_A - z_B - 1} \cdot \frac{z_A + z_B}{z_A + z_B - 1} \prod_{j=1}^{k} e^{x_j z_j + \frac{t}{2} z_j^2} dz_j$$

where z_j is integrated over $\alpha_j + i\mathbb{R}$, with $\alpha_1 > \alpha_2 + 1 > \alpha_3 + 2 > \cdots,\ \alpha_k = 0$

(Non-rigorously) yields Fredholm Pfaffian formula for Laplace transform of symmetric SHE (cf. [Le Doussal-Gueudre '12])



Discrete time geometric / Bernoulli q-TASEP

Poissonian q-TASEP

ASEP

Log-gamma discrete SHE

Semi-discrete SHE

Continuum SHE

TASEP / LPP

KPZ fixed point (GUE Tracy-Widom distribution)

Summary:

- Duality for q-TASEP leads to unexpected many body systems for expectations of certain observables
- Free evolution equation with k-1 boundary condition form can be solved via a "nested contour integral ansatz"
- Moment formulas combine into generating function (q-Laplace transform) as Fredholm determinants
- Mellin Barnes Fredholm determinant good for asymptotics
- Degenerates to semi-discrete and continuum replica trick
- Many body system approach is general, but not structural

Two questions:

- What is the structural origin of q-TASEP, the special $q^{x_n(t)+n}$ observables and the nested contour integral formulas?
- Are there any parallel systems of q-TASEP which are solvable via the many body system approach?

These questions will draw us to the study of

- Macdonald processes (+ directed polymer / geometric RSK)
- ASEP

Many body systems approach reveals parallel formulas.

Is there a higher structure which accounts for this?