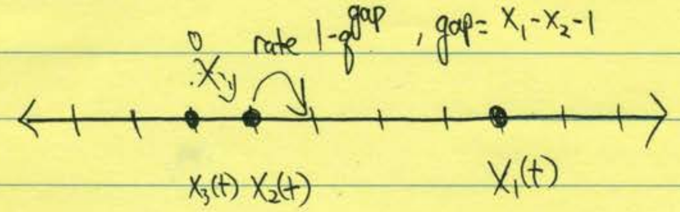
Duality, replicas and the SHE

References:
- From duality to determinants for q-TASEP and ASEP
- Free energy fluctuations for directed polymers in random media in 1+1 dimensions
- Two ways to solve ASEP
- Macdonald processes

Recall q-TASEP on $\mathbb{Z}$



$q \in (0,1)$

Restrict to N particles in which case state space is

$$\mathbb{X}^N := \{ \vec{x} = (x_0, x_1, \ldots, x_N) \in \{\infty\} \times \mathbb{Z}^N : \infty = x_0 > x_1 > x_2 > \cdots > x_N \}$$

(here x_0 is a "virtual particle at ∞" so x_1 always jumps at rate 1)

- Aside on general continuous time Markov processes

↳ Defined via a semigroup $\{S_t\}_{t \geq 0}$ with $S_{t_1} S_{t_2} = S_{t_1+t_2}$

and $S_0 = Id$,

↳ For $f : \mathbb{X} \to \mathbb{R}$, $\quad \mathbb{E}^x[f(x(t))] = (S_t f)(x)$

where $\mathbb{X}$ is state space and $\mathbb{E}^x[f(x(t))]$ is expectation wrt $x(0) = x$.

↳ Generator $L := \lim_{t \to 0} \frac{S_t - I}{t}$ captures the entire

process since $S_t = e^{tL} = \sum_{k=0}^{\infty} \frac{(tL)^k}{k!}$

↳ Follows that $\frac{d}{dt} S_t = S_t L = L S_t$ so that for $f : \mathbb{X} \to \mathbb{R}$.

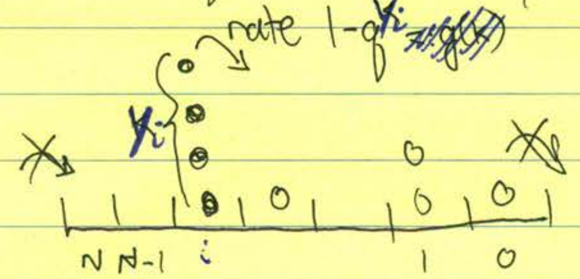
$$\frac{d}{dt} \mathbb{E}^x[f(x(t))] = \mathbb{E}^x[L f(x(t))] = L\, \mathbb{E}^x[f(x(t))]$$

Returning to q-TASEP observe that for $f: \mathbb{X}^N \to \mathbb{R}$,

$$\left(L^{q\text{-TASEP}} f\right)(\vec{x}) := \sum_{i=1}^{N} \left(1 - q^{x_{i-1} - x_i - 1}\right) \left[f(\vec{x}_i^{+}) - f(\vec{x}) \right]$$

$x_i^+ = (x_1, \ldots, x_i + 1, \ldots x_N)$.

Consider q-deformed totally asymmetric zero range process (q-TAZRP)



$\mathbb{Y}^N$ state space, $\vec{y} = (y_0, \ldots, y_N) \in (\mathbb{Z}_{\geq 0})^{\{0,1,\ldots,N\}}$

$y_0 = 1,\ y_1 = 2, \ldots$

For $h: \mathbb{Y}^N \to \mathbb{R}$, generator of $\vec{y}(t)$ acts as

$$\left(L^{q\text{-TAZRP}} h\right)(\vec{y}) := \sum_{i=1}^{N} \left(1 - q^{y_i}\right) \left[h(\vec{y}^{\,i,i-1}) - h(\vec{y}) \right]$$

$\vec{y}^{\,i,i-1} = (y_0, y_1, \ldots, y_{i-1}+1, y_i - 1, \ldots y_N)$.

<u>Remarks</u>
- This type of q-TAZRP (jump rate) arose in Sasamoto-Wadati as certain representation of the q-Boson Hamiltonian
- The gaps of q-TASEP evolve according to q-TAZRP jump rule with different boundary conditions.
- For q-TASEP with doubly infinite number of particles, gaps evolve as q-TAZRP on $\mathbb{Z}$. This has product meas. invariant dist.
- For this "equilibrium" q-TAZRP, Balazs-Komjathy-Seppalainen provided $t^{1/3}$ KPZ exponent for current fluctuations using second class particles and coupling methods

<u>Duality of Markov processes</u>: Suppose $x(t)$, $y(t)$ are independent Markov processes with state spaces X, Y. Let $H : X \times Y \to \mathbb{R}$ bdd measurable.

Then $x(t)$ and $y(t)$ are <u>dual wrt H</u> if $\forall x, y, t$

$$\mathbb{E}^{x}\left[H(x(t), y)\right] = \mathbb{E}^{y}\left[H(x, y(t))\right].$$

<u>Theorem:</u> q-TASEP $\vec{x}(t) \in \mathbb{X}^{N}$ and q-TAZRP $\vec{y}(t) \in \mathbb{Y}^{N}$ are dual wrt

$$H(\vec{x}, \vec{y}) = \prod_{i=0}^{N} q^{(x_i + i) y_i}$$

(with convention that b/c $x_0 = \infty$, $H = 0$ if $y_0 > 0$ else $H = \prod_{i=1}^{N} q^{(x_i+i)y_i}$

<u>Proof:</u> By $LS_t = S_t L$ it suffices to prove that $\forall \vec{x}, \vec{y}$,

$$L^{q\text{-TASEP}} \underset{\text{acts here}}{H(\vec{x}}, \vec{y}) = L^{q\text{-TAZRP}} H(\vec{x}, \underset{\text{acts here}}{\vec{y}})$$

$$\begin{array}{ccc} \| & & \| \\ \sum_{i=1}^{N} \left(1 - q^{x_{i-1} - x_i - 1}\right)\left[H(\vec{x}_i^{+}, \vec{y}) - H(\vec{x}, \vec{y})\right] & & \sum_{i=1}^{N} \left(1 - q^{y_i}\right)\left[H(\vec{x}, \vec{y}^{\,i,i-1}) - H(\vec{x}, \vec{y})\right] \\ \| & & \| \end{array}$$

$$\sum_{i=1}^{N} \left(1 - q^{x_{i-1} - x_i - 1}\right)\left(q^{y_i} - 1\right) \prod_{j=0}^{N} q^{(x_j + j) y_j} = \sum_{i=1}^{N} \left(1 - q^{y_i}\right)\left(q^{x_{i-1} - x_i - 1} - 1\right) \prod_{j=0}^{N} q^{(x_j + j) y_j}$$

Our present motivation

<u>Remark</u> If $\vec{y} = (0, 0, \dots, k)$ then

$\mathbb{E}^{x}\left[H(\vec{x}(t), \vec{y})\right] = \mathbb{E}^{\vec{x}}\left[q^{k(x_N(t) + N)}\right]$ and if $\vec{x} = (-1, -2, \dots, -N)$ <u>step initial data</u>

then $\mathbb{E}^{\vec{x}}\left[q^{k(x_N(t)+N)}\right] = \mathbb{E}_{MM(1,\dots,1;(0;0;t))}\left[q^{k\lambda_N}\right]$ (generally get multilevel moments)

<u>Prop</u> Fix $\vec{x} \in \mathbb{X}^N$ (initial data)

(A) (True evolution equation) If $h : \mathbb{R}_{\geq 0} \times Y^N \to \mathbb{R}$ solves

1) For all $\vec{y} \in \mathbb{Y}^N$ and $t \in \mathbb{R}_{\geq 0}$

$$\frac{d}{dt} h(t;\vec{y}) = L^{q\text{-TAZRP}} h(t;\vec{y})$$

2) For all $\vec{y} \in \mathbb{Y}^N$ such that $y_0 > 0$, $h(t;\vec{y}) \equiv 0$

3) For all $\vec{y} \in \mathbb{Y}^N$, $h(0;\vec{y}) = H(\vec{x},\vec{y}) = h_0(\vec{y})$

Then for all $\vec{y} \in \mathbb{Y}^N$ and $t \in \mathbb{R}_{\geq 0}$, $\mathbb{E}^{\vec{x}}[H(\vec{x}(t),\vec{y})] = h(t;\vec{y})$.

Proof (on separate board): By duality $\mathbb{E}^{x}[H(\vec{x}(t),\vec{y})] = \mathbb{E}^{\vec{y}}[H(\vec{x},\vec{y}(t))]$ so

$$\frac{d}{dt}\mathbb{E}^{\vec{x}}[H(\vec{x}(t),y)] = \frac{d}{dt}\mathbb{E}^{\vec{y}}[H(\vec{x},\vec{y}(t))] = L^{q\text{-TAZRP}}\mathbb{E}^{\vec{y}}[H(\vec{x},\vec{y}(t))]$$

by $\frac{d}{dt}S_t = LS_t$.

with initial value $H(\vec{x},\vec{y})$, and boundary condition from def^{n} of H.

Uniqueness follows because $L^{q\text{-TAZRP}}$ preserves the number of particles hence restricts to finite systems of coupled ODEs indexed by the number of particles in $\vec{y}$ (i.e. $\sum y_i$). For each of these finite coupled ODEs, standard uniqueness theorems for ODEs apply. Particle count preservation important! (i.e. hierarchy closes) $\square$

With this in mind the following naturally indexes $\vec{y}$'s in a given level.

Def$^{\underline{n}}$: For $k \geq 1$

$$W^{k} = \{ \vec{n} = (n_1, \dots, n_k) \in (\mathbb{Z}_{\geq 0})^k \text{ such that } n_1 \geq n_2 \geq \dots \geq n_k \geq 0 \}$$

For $k = \sum_{i=0}^{N} y_i$ with $\vec{y} \in \mathbb{Y}^N$ we associate a vector $\vec{n} = \vec{n}(y) \in W^k$

Where i shows up in $\vec{n}$ with multiplicity y_i (i.e. $\vec{n}$ lists in order the particle locations of $\vec{y}$) and to $\vec{n}$ we associate $\vec{y} = \vec{y}(\vec{n}) \in \mathbb{Y}^N$ with

~~convention $y_0(\vec{n}) = 0$~~ (i.e. $\vec{y}$ is a particle configuration of particles located at $\vec{n}$)

Example: $N = 3$ $\quad y_0 = 0, y_1 = 2, y_2 = 0, y_3 = 1$

Then $k = 3$ and $n_1 = 3$, $n_2 = n_3 = 1$. This $\vec{n}$ has two clusters of elements and all n's $\leq N$ (as is necessarily the case)

Def$^{\underline{n}}$ For $f: \mathbb{N} \to \mathbb{R}$, $(\nabla f)(n) = f(n-1) - f(n)$

and for $f: \mathbb{N}^k \to \mathbb{R}$, $(\nabla_i f)(\vec{n}) = f(\vec{n}_i^-) - f(\vec{n})$

where $\vec{n}_i^- = (n_1, \dots, n_i - 1, \dots, n_k)$

Example $k=1$ so $\vec{n} = n$ and $\vec{y}$ has one non zero entry which is 1.

$$H(\vec{x},\vec{y}) = q^{x_n+n}.$$

How to understand the true evolution equation.

$$d q^{x_n+n} = \underbrace{\left(q^{(x_n+1)+n} - q^{x_n+n}\right)}_{\text{change}} \cdot \underbrace{\left(1 - q^{x_{n-1}-x_n-1}\right)}_{\text{rate}} dt + \underbrace{q^{x_n+n}\, dM_n(t)}_{\text{Martingale}}$$

$$= (q-1)\left(q^{x_n+n} - q^{x_{n-1}+n-1}\right) dt + q^{x_n+n}\, dM_n(t)$$

So

$$\frac{d}{dt}\mathbb{E}^{\vec{x}}\left[q^{x_n(t)+n}\right] = \underbrace{(1-q)\nabla}_{\text{q-TAZRP on one particle subspace}} \mathbb{E}^{\vec{x}}\left[q^{x_n(t)+n}\right]$$

acts on n above.

equivalent to consequence of duality for $k=1$

Notice that due to boundary condition that $\mathbb{E}[q^{x_0(t)}] = 0$

the above hierarchy is closed: n depends on n and $n-1$.

The initial data for above ODEs depends on what $\vec{x}$ is.

Example $k=2$ so $\vec{n} = (n_1 \geq n_2)$

- If $n_1 > n_2$ then in dt only one particle (n_1 or n_2) will jump.

This means $$\frac{d}{dt}\mathbb{E}^{\vec{x}}\left[q^{\sum_{i=1}^{k} x_{n_i}+n_i}\right] = \sum_{i=1}^{k}(1-q)\nabla_i \mathbb{E}^{\vec{x}}\left[q^{\sum_{i=1}^{k} x_{n_i}+n_i}\right]$$

- If $n_1 = n_2 = n$ then

$$d\, q^{\sum_{i=1}^{k} x_{n_i}+n_i} = d\, q^{2(x_n+n)} = \left(q^{2(x_n+1+n)} - q^{2(x_n+n)}\right)\left(1-q^{x_{n-1}-x_n-1}\right)dt + d\text{Mart}$$

$$= (1-q^2)\left[q^{(x_n+n)+(x_{n-1}+n-1)} - q^{(x_n+n)+(x_n+n)}\right] + d\text{Mart}$$

$$= (1-q^2)\,\nabla_2\, q^{\sum_{i=1}^{k} x_{n_i}+n_i} + d\text{Mart}$$

choose to stay closed in $W^k_{\geq 0}$

equivalent to consequence of duality for $k=2$

so $$\frac{d}{dt}\mathbb{E}^{\vec{x}}\left[q^{\sum_{i=1}^{k} x_{n_i}+n_i}\right] = (1-q^2)\nabla_2 \mathbb{E}^{\vec{x}}\left[q^{\sum_{i=1}^{k} x_{n_i}+n_i}\right]$$

Having/solving a non-constant coeff. system is not easy.

Free evolution eqn $\frac{d}{dt} = (1-q)\nabla_1 + (1-q)\nabla_2$ matches true evolution equation except possibly at $n_1=n_2$'s $= (1-q^2)\nabla_2$. Take the difference of RHS gives $(1-q)\nabla_1 + (q^2-q)\nabla_2$. If we find a ~~function~~ solution to free evolution on $\mathbb{Z}^2$ for which this difference is 0 always, then it will also solve the true equation when restricted to $W^2_{\geq 0}$.

Prop (Continuation)

(B) (Free evolution equation with boundary condition)

If $u : \mathbb{R}_{\geq 0} \times (\mathbb{Z}_{\geq 0})^k \to \mathbb{R}$ solves

(1) For all $\vec{n} \in (\mathbb{Z}_{\geq 0})^k$ and $t \in \mathbb{R}_{\geq 0}$

$$\frac{d}{dt} u(t;\vec{n}) = \sum_{i=1}^{k} (1-q) \nabla_i u(t;\vec{n})$$ "free evolution equation"

(2) For all $\vec{n} \in (\mathbb{Z}_{\geq 0})^k$ such that $n_i = n_{i+1}$ for some i

$$\nabla_i u(t;\vec{n}) - q \nabla_{i+1} u(t;\vec{n}) = 0$$ "Boundary condition"

(3) For all $\vec{n} \in (\mathbb{Z}_{\geq 0})^k$ such that $n_k = 0$, $u(t;\vec{n}) = 0$.

(4) For all $\vec{n} \in W^k$ $\quad u(0;\vec{n}) = H(\vec{x}, \vec{y}(\vec{n}))$

Then for all $\vec{y} \in \mathbb{Y}^N$ with $k = \sum_{i=1}^{N} y_i$, $\mathbb{E}^{\vec{x}}[H(\vec{x}(t), \vec{y})] = u(t;\vec{n}(y))$.

Remark: • The fact that for $k > 2$ the boundary condition only involves 2-body interactions is the hallmark of integrability.

• This idea of trying to represent a k-dim Hamiltonian in terms of a constant coeff factorized solution with $k-1$ boundary conditions is old, going back to Bethe's 1931 solution to the Heisenberg spin chain.

Prop (continuation)

(C) Schrödinger equation (imaginary time) with Bosonic Hamiltonian.

If $V : \mathbb{R}_{\geq 0} \times (\mathbb{Z}_{\geq 0})^k \to \mathbb{R}$ solves

(1) For all $\vec{n} \in (\mathbb{Z}_{\geq 0})^k$, $t \in \mathbb{R}_{\geq 0}$

$$\frac{d}{dt} V(t;\vec{n}) = H V(t;\vec{n})$$

$$H = (1-q)\left[\sum_{i=1}^{k} \nabla_i + (1-q^{-1}) \sum_{1 \leq i < j \leq k} \delta_{n_i = n_j} q^{j-i} \nabla_i \right]$$

(2) For all $\sigma \in S_k$, $V(t;\sigma\vec{n}) = V(t;\vec{n})$

(3) For all $\vec{n} \in (\mathbb{Z}_{\geq 0})^k$ such that $n_k = 0$, $V(t;\vec{n}) = 0$

(4) For all $\vec{n} \in W^k_{>0}$ $V(0;\vec{n}) = H(\vec{x} ; \vec{y}(\vec{n}))$

Then for all ~~$\vec{n} \in W^k_{>0}$ $V(0;\vec{n}) = H(\vec{x};\vec{y}(\vec{n}))$~~
$y \in Y^n$ such that $k = \sum y_i$, $\mathbb{E}^{\vec{x}}[H(\vec{x}(t),\vec{y})] = V(t;\vec{n}(\vec{y}))$

Proof (for $k=2$ ~~...~~): For $n_1 \neq n_2$ H coincides with true evolution eqn.

For $n_1 = n_2 = n$, $\frac{d}{dt} V(n,n) = (1-q)\nabla_1 + (1-q)\nabla_2 + (1-q)(1-q^{-1})q\nabla_1$

$$= (q-q^2)\nabla_1 + (1-q)\nabla_1$$

But by symmetry of V,
$\nabla_1 V = \nabla_2 V$ so

$$= \left[(q-q^2) + (1-q)\right] \nabla_1$$

$$= (1-q^2)\nabla_1$$ True evolution equation on $W^k_{>0}$ $\square$.

- So, in order to compute $\mathbb{E}^{\vec{x}}\left[q^{\sum_{i=1}^{k} x_{n_i}(t)+n_i}\right]$ is suffices to solve any of the systems of equations in (A), (B) or (C).

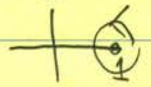
- We will now assume step initial data for q-TASEP so that $\vec{x}: x_n = -n\ ,\ n \geq 1$ or equivalently $H(\vec{x},\vec{y}) \equiv 1$.

- We solve (B) since it is simple to find general solutions to free evolution equation, and then impose boundary condition via superpositions of these. This too is an idea going back to Bethe.

- Observe that $\forall z \in \mathbb{C}/\{1\}$, $g_z(t,n) := \dfrac{e^{(q-1)tz}}{(1-z)^{n}}$ solves $\frac{d}{dt} g_z(t,n) = (1-q)\nabla g_z(t,n)$

- Lets address $k=1$ first $\rightarrow$ no 2 body boundary condition but need to ~~check~~ match initial data and boundary condition at zero. <u>Take superposition</u>

Define $u(t;n) = \dfrac{-1}{2\pi i} \displaystyle\int g_z(t;n) \frac{dz}{z}$



[Exercise: Prove that ~~for n≠0~~, $u(t:0) = 0$ and that for $n \geq 1$ $u(0:n) \equiv 1$.

Hence $u(t;n) = \mathbb{E}_{\text{step}}\left[q^{x_n(t)+n}\right]$.

How to solve for general K (with step initial data for $\vec{x}$)?

Inspired by formulas for $n_i \equiv n$ which were from Macdonald first difference operator at parameter $(q,0)$ we made following guess.

Later we realized that this can be shown directly from diff. operator and Macdonald polynomial branching rules, and in fact the whole q-TASEP duality is a consequence of a certain commutation relation between the diff operator and ~~multipl certain~~ another operator involving sym. functions.

<u>Thm</u> (B) with $\vec{x}_n = -n$ is solved by

$$u(t;\vec{n}) = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \int \cdots \int \prod_{1\le A<B\le k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} g_{z_i}(t;n_j) \frac{dz_j}{z_j}$$

Z_A contains $\{q z_B\}_{B>A}$ and 1, but not 0.

<u>Cor</u> $\mathbb{E}_{\text{step}}\left[q^{\sum_{i=1}^{k} x_{n_i}(t)+n_i}\right] = \mathbb{E}_{MM(1,\dots;(0,0,t))}\left[q^{\sum_{i=1}^{k} \lambda^{n_i}_{n_i}}\right] = u(t;\vec{n})$

where $\vec{n} = (n_1 \ge n_2 \ge \cdots \ge n_k) \in W^k_{>0}$.

This with $n_i \equiv n$ served as starting point for distribution of $x_n(t)$ formula.
Open problem to work out multipoint distribution of q-TASEP.

Proof: Exercise to check (1), (3), (4) (hint: (1) follows Leibnitz rule and (3), (4) require residue calculus)

Check (2) that for $n_i = n_{i+1}$ $\nabla_i u(t;\vec{n}) - q\nabla_{i+1} u(t;\vec{n}) = 0$.

For concreteness take $i=1, i+1=2$. The ∇_1 acts just on $g_{z_1}(t;\vec{n}_1)$ as

$$\nabla_1 g_{z_1}(t;\vec{n}_1) = -z_1 g_{z_1}(t;n_1) \quad , q\nabla_2 g_{z_2}(t;n_2) = -qz_2\, g_{z_2}(t;n_2)$$

so applying $\nabla_1 - q\nabla_2$ simply introduces extra factor $(z_1 - qz_2)$ in integrand. Cancels denominator and integrand becomes antisymmetric in z_1 and z_2 and since no pole exists at $z_1 = qz_2$, we can rewrite as

$$\iint G(z_1)G(z_2)(z_1 - z_2) = 0. \quad \square$$

The ultimate output (for all $n_i \equiv n$) was following

$$\mathbb{E}_{\text{step}}\left[\frac{1}{(\zeta q^{x_n(t)+n};q)_\infty}\right] = \det(I + K_\zeta)_{L^2(C_1)}$$

$$K_\zeta(w,w') = \frac{1}{2\pi i}\int_{-i\infty+1/2}^{i\infty+1/2} \frac{\pi}{\sin(-\pi s)}(-\zeta)^s \frac{g(w)}{g(q^s w)} \frac{ds}{q^s w - w'}$$

$$g(w) = \frac{e^{-tw}}{(w;q)_\infty}$$

Will show how q-TASEP and formulas degenerates to various limits

(1) Semi-discrete SHE (2) Continuum SHE

(1) Semi-discrete SHE

From duality we saw $$d q^{x_n(t)+n} = (1-q)\nabla q^{x_n(t)+n} + q^{x_n(t)+n} dM_n(t)$$

$$q^{x_n(0)+n} \equiv 1, \quad q^{x_0(t)} \equiv 0.$$

[Claim that under suitable scaling as $q \nearrow 1$ and n fixed, this goes to...]

Def^n $Z: \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ solves the semi-discrete SHE with initial data $Z_0(n)$ if $\forall n$, $$dZ(\tau; n) = \nabla Z(\tau; n) + Z(\tau; n)\, dB_n(\tau)$$

$$Z(0; n) = Z_0(n), \quad Z(\tau; 0) \equiv 0$$

where B_i are independent standard Brownian motions and the above system is Ito SDE's for which uniqueness follows since its a closed system

↳ Equivalent to partition function for O'Connell-Yor semidiscrete directed polymer (we will see later)

<u>Theorem</u> Consider q-TASEP with step initial data and set

$$q = e^{-\varepsilon}, \quad t = \varepsilon^{-2}\tau, \quad x_n(t) = \varepsilon^{-2}\tau - (n-1)\varepsilon^{-1}\log\varepsilon^{-1} - \varepsilon^{-1}F_\varepsilon(\tau;n).$$

Let $Z_\varepsilon(\tau;n) := \exp\left\{\frac{-3\tau}{2} + F_\varepsilon(\tau;n)\right\}$ then for any $N \geq 1, T > 0$

as $\varepsilon \to 0$ law of $\{Z_\varepsilon(\tau;n) : \tau \in [0,T], 1 \leq n \leq N\}$ converges to

the law of $\{Z(\tau;n) : \tau \in [0,T], 1 \leq n \leq N\}$ with $Z_0(n) = \delta_{n=1}$.

Proof: (Heuristic as in duality paper – full proof in Macdonald process paper via different approach ... someone should make this rigorous)

- Initial data: $Z_\varepsilon(0;n) = \varepsilon^{n-1}e^{\varepsilon n} \longrightarrow \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$

- Dynamics : $dF_\varepsilon(\tau;n) = F_\varepsilon(\tau;n) - F_\varepsilon(\tau - d\tau;n)$

$$= \varepsilon^{-1}d\tau - \varepsilon\left[x_n(\varepsilon^{-2}\tau) - x_n(\varepsilon^{-2}\tau - \varepsilon^{-2}d\tau)\right]$$

q-TASEP jump rate in rescaled time variables is

$$1 - q^{x_{n-1}(t) - x_n(t) - 1} = 1 - \varepsilon e^{F_\varepsilon(\tau;n-1) - F_\varepsilon(\tau;n)} + O(\varepsilon^2)$$

so in time $\varepsilon^{-2}d\tau$ (by convergence of Poisson jump process to BM)

$$\varepsilon\left[x_n(\varepsilon^{-2}\tau) - x_n(\varepsilon^{-2}\tau - \varepsilon^{-2}d\tau)\right] = \varepsilon^{-1} - e^{F_\varepsilon(\tau;n-1) - F_\varepsilon(\tau;n)}d\tau - dB_n(\tau)$$

$B_n(\tau) - B_n(\tau - d\tau)$.

<u>Exercise</u>: Let P be a poisson point process of intensity 1 on $\mathbb{R}_{\geq 0}$ and let $N(t) = \#\{\text{points of } P \leq t\}$. Prove that

$$\frac{N(t)-t}{\sqrt{t}} \underset{t\to\infty}{\Longrightarrow} N(0,1) \quad \text{and} \quad \frac{N(st)-st}{\sqrt{t}} \underset{t\to\infty}{\Longrightarrow} B(s)$$

in terms of finite dimensional distributions and (harder) ~~as a process on~~ or that its law on $C[0,T]$ goes to that of B.

Thus $$dF_\varepsilon(\tau;n) = e^{F_\varepsilon(\tau,n-1)-F_\varepsilon(\tau,n)}dt + dB_n(\tau) + o(1)$$

~~Taking ε to 0 gives equation for~~

Exponentiating, Itô's lemma (explain in exercises?) gives

$$d\exp\{F_\varepsilon(\tau;n)\} = \left(\tfrac{1}{2}\exp\{F_\varepsilon(\tau;n)\} + \exp\{F_\varepsilon(\tau;n-1)\}\right)d\tau + \exp\{F_\varepsilon(\tau,n)\}dB_n + o(1)$$

with Z_ε defined as above we find

$$dZ_\varepsilon(\tau,n) = \nabla Z_\varepsilon(\tau,n) + Z_\varepsilon(\tau,n)\,dB_n(\tau) + o(1)$$

and as $\varepsilon \to 0$ we recover the claimed formula. $\square$

The convergence theorem implies weak convergence of $q^{x_n(t)}$ to $Z(\tau;n)$ as $q \nearrow 1$.

On account of this, the q-Laplace transform of $q^{x_n(t)}$ converges to the Laplace transform of $Z(\tau;n)$ and we conclude

Thm For u with $\mathrm{Re}\, u \geq 0$,

$$\mathbb{E}\left[e^{-ue^{\frac{3\tau}{2}} Z(\tau,n)}\right] = \det(I + K_u)_{L^2(C_0)}$$

$$K_u(v,v') = \frac{1}{2\pi i}\int_{-i\infty+1/2}^{i\infty+1/2} \frac{\pi}{\sin(-\pi s)} \frac{g(v)}{g(v+s)} \frac{ds}{v+s-v'}, \quad g(z) = \Gamma(z)^n u^{-z} e^{-tz^2/2}$$

Corollary (In Macdonald paper for $\kappa > \kappa^*$ and with B-C-F for all $\kappa > 0$)

Call $F(\tau;n) = \frac{3\tau}{2} + \log Z(\tau,n)$ (the limit of $F_\varepsilon(\tau,n)$)

then for all $\kappa > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{F(\kappa n;n) - n\bar{f}_\kappa}{n^{1/3}} \leq r\right) = F_{GUE}\left(\left(\frac{\bar{g}_\kappa}{2}\right)^{-1/3} r\right)$$

where $\bar{f}_\kappa = \inf_{t>0}(\kappa t - \Psi(t))$, $\Psi(t) = (\log\Gamma)'(t)$ digamma function

and $\bar{g}_\kappa = -\Psi''(\bar{t}_\kappa)$, $\bar{t}_\kappa = \arg\inf$ above.

Remarks

- Such asymptotics should hold for q-TASEP directly, though this has not been performed.
- The proof does not require inverting Laplace transform because for well choosen u

$$e^{-ue^{\frac{3\tau}{2}}Z(\tau,n)} = e^{-e^{n^{1/3}A_n}}, \quad A_n = \frac{F(\kappa n;n) - n\bar{f}_\kappa}{n^{1/3}} - r$$

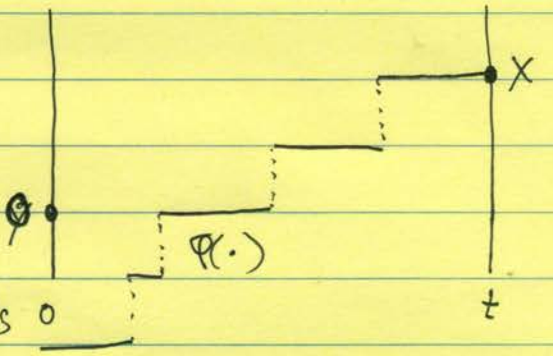
Note that $e^{-e^{\lambda x}} \xrightarrow[\lambda\to\infty]{} \mathbb{1}_{x<0}$, hence (with a little work to make rigorous) expectations of above (i.e. Laplace trans) converges to limit of probability of $A_n<0$.

- $\bar{f}_\kappa$ is LLN for $F(\kappa n;n)$ and was conjectured in O'Connell Yor and proved in O'Connell-Moriarty. Digamma function is flux in equilibrium of this system so this matches with limit shape type results for TASEP, $\bar{g}_\kappa$ is convexity scaling.
- A $t^{2/3}$ variance upper bound previously proved by Seppalainen and Valko using different techniques.
- ~~[illegible]~~ $\lim_{k\to\infty} F(\kappa;n)$ evolves according to continuous limit of TASEP with GUE eigenvalue relation.

It is possible to take limits of q-TASEP duality, moments and many body systems. Alternatively we now see how they arise naturally.

Aside on <u>Feynman-Kac representation</u>:

- Consider homogeneous Markov process generator L and deterministic potential $V(t,x)$.
- Solve $\frac{d}{dt} Z(t,x) = (LZ)(t,x) + V(t,x)Z(t,x)$; $Z(0,x) = Z_0(x)$.
- Define L-heat kernel $p(t,x)$ as sol^n with $Z_0(x) = \delta_{x=0}$
- Probabilistic interpretation for $L = \nabla$



$P(\cdot)$ is Markov process with $\mathcal{E}$ and with generator L run backwards from time t to 0.

$$p(t,x) = \mathcal{E}^{P(t)=x}\left[\delta_{P(0)=0} \right]$$

For $V \equiv 0$, by superposition / linearity of expectation

$$Z(t,x) = \mathcal{E}^{P(t)=x}\left[Z_0(P(0)) \right].$$

• When V is turned on, Duhamel's principle shows

($\int_{-\infty}^{\infty} dy$ or relevant summation)

$$Z(t,x) = \int_{-\infty}^{\infty} p(t,x-y)\, Z_0(y)\, dy + \int_0^t ds \int_{-\infty}^{\infty} dy\; p(t-s,y-x)\, Z(s,y)\, V(s,y)$$

Apply this identity again to $Z(s,y)$ etc yields series which sums to

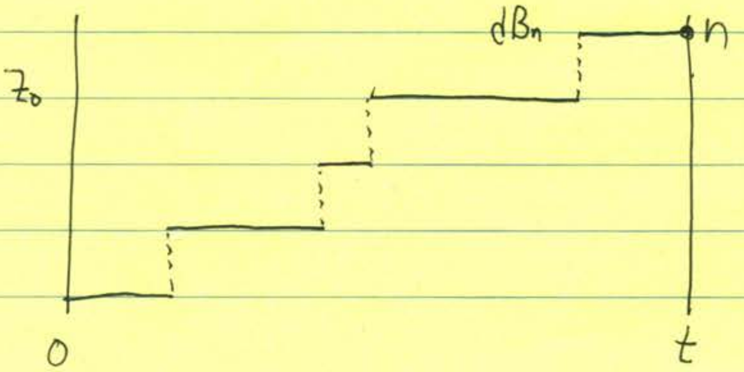
Exercise

$$\boxed{Z(t,x) = \mathcal{E}^{\varphi(t)=x}\left[e^{\int_0^t V(s,\varphi(s))\,ds}\, Z_0(\varphi(0)) \right]}$$

If V is random, care must be taken in defining the multiple stochastic integrals (e.g. V whitenoise must avoid diagonal). This generally leads to a correction to the exponential which goes by Wick or Girsanov exponential.

For $Z(\tau;n)$, $V(\tau;n) = dB_n(\tau)$ and $L = \nabla$ so

$$Z(\tau;n) = \mathcal{E}^{\varphi(\tau)=n}\left[\exp\left\{ \int_0^t \left(dB_{\varphi(s)}(s) - \frac{ds}{2}\right)\right\} Z_0(\varphi(0)) \right]$$



O'Connell-Yor polymer.

partition function

The following is the replica method (not trick...later) and in this context certainly goes back to Molchanov (1986) / Kardar (1987) if not earlier.

- Consider $Z(\tau;n_1), \dots, Z(\tau;n_k)$. We can represent each one via path integral wrt $\varphi_1, \dots, \varphi_k$ (independent replicas) so that

$$\mathbb{E}\Big[\prod_{i=1}^{k} Z(\tau;n_i)\Big] = \mathbb{E}\Big[\prod_{i=1}^{k} \mathcal{E}^{\varphi_i(\tau)=n_i}\Big[e^{\int_0^\tau dB_{\varphi_i(s)}(s)-\frac{ds}{2}} Z_0(\varphi_i(0))\Big]\Big]$$

Interchange $\mathbb{E}$ with $\mathcal{E}$'s yielding

$$= \mathcal{E}^{\varphi_1(\tau)=n_1} \cdots \mathcal{E}^{\varphi_k(\tau)=n_k}\Big[\mathbb{E}\Big[e^{\sum_{i=1}^{k}\int_0^\tau dB_{\varphi_i(s)}(s)-\frac{ds}{2}}\Big] \cdot \prod_{i=1}^{k} Z_0(\varphi_i(0))\Big]$$

Claim
$$\mathbb{E}\Big[e^{\sum_{i=1}^{k}\int_0^\tau dB_{\varphi_i(s)}(s)-\frac{ds}{2}}\Big] = e^{\int_0^\tau \sum_{i<j} \delta_{\varphi_i(s)=\varphi_j(s)}\,ds}$$

Pf: Use ind^t of B_i's and ind^t increments. Consider at time s a cluster of size c of paths all equal. Then

$$\mathbb{E}\, e^{c(dB(s)-\frac{ds}{2})} = \mathbb{E}\Big[e^{c(\overset{N(0,\sigma)}{X}-\frac{\sigma^2}{2})}\Big] = e^{\sigma^2 c(c-1)/2}$$
$$= e^{\frac{c(c-1)}{2}ds}$$
$$= e^{\sum_{i<j}\delta_{\varphi_i(s)=\varphi_j(s)}}$$

over cluster c.

The claim shows that

$$\mathbb{E}\left[\prod_{i=1}^{k} Z(\tau,n_i)\right] = \mathcal{E}^{\vec{q}(\tau)=\vec{n}}\left[e^{\int_0^\tau \sum_{i<j} \delta_{q_i(s)=q_j(s)}\, ds} \prod_{i=1}^{k} Z_0(q_i(0))\right]$$

generator   potential   initial data

By (deterministic) Feynman-Kac this solves

$$\frac{d}{d\tau} V(\tau;\vec{n}) = H V(\tau;\vec{n}) \quad , H = \sum_{i=1}^{k} \nabla_i + \sum_{i<j} \delta_{n_i=n_j}$$

subject to Bosonic symmetry, zero boundary condition and Z_0 initial data

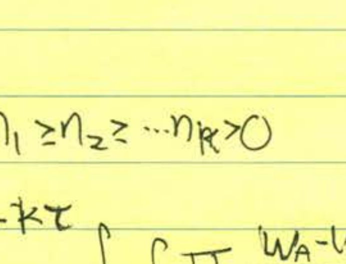
This is the limit of system (C) and can also be rewritten as in (B) as free evolution wrt $\sum \nabla_i$ and a boundary condition that when $n_i = n_{i+1}$,

$$(\nabla_i - \nabla_{i+1} - 1) V = 0 .$$

Cor: For $Z_0(n) = \delta_{n=1}$ and $n_1 \geq n_2 \geq \cdots n_k > 0$

$$\mathbb{E}\left[Z(\tau,n_1)\cdots Z(\tau,n_k)\right] = \frac{e^{-k\tau}}{(2\pi i)^k} \int\cdots\int \prod_{1\le A<B\le k} \frac{w_A - w_B}{w_A-w_B-1} \prod_{j=1}^{k} \frac{e^{\tau w_j}}{w_j^{n_j}}\, dw_j$$

with w_A containing 0 and $\{w_B+1\}_{B>A}$



Application : Compute k^{th} moment Lyapunov exponent

$$\gamma_k := \lim_{n\to\infty} \frac{\log \mathbb{E}[Z(n,n)^k]}{n}$$

Cor: $\gamma_k = \inf_{z\in(0,\infty)} H_k(t)$, $H_k(t) = \frac{k(k-3)}{2} + kz - \log\frac{\Gamma(z+k)}{\Gamma(z)}$

Physic's replica trick goes back to 1968 paper of Kac (as I know)

Basic form : Compute LLN for $\log Z(n,n)$:

$$\tilde{\gamma}_1 = \lim_{n\to\infty} \mathbb{E}\left[\frac{\log Z(n,n)}{n}\right]$$

For $z \in \mathbb{C}\setminus\mathbb{R}_-$ deterministic

Exercise: $\log z = \lim_{k\to 0} \frac{z^k - 1}{k}$

Apply as $\tilde{\gamma}_1 = \lim_{n\to\infty} \frac{1}{n} \lim_{k\to 0} \frac{\mathbb{E}[Z^k]-1}{k}$, By Cor $\mathbb{E}[Z^k] \approx e^{n\gamma_k}$

$$= \lim_{n\to\infty} \frac{1}{n} \lim_{k\to 0} \frac{n\gamma_k}{k}$$

$$= \lim_{k\to 0} \inf_{t\in(0,\infty)} \frac{H_k(t)}{k} \qquad \text{L'Hopital}$$

$$= \frac{-3}{2} + \inf_{t\in(0,\infty)} \left(t - \Psi(t)\right) \quad , \; \Psi(t) = (\log\Gamma)'(t)$$

Which matches $-\frac{3}{2}$ + LLN for $F(n,n)$ we saw earlier.

More advanced version

Given $\mathbb{E}[Z^k_{(\tau,n)}]$ try to recover $\mathbb{E}[e^{-uZ(\tau,n)}]$ via

$$\mathbb{E}\left[e^{-uZ(\tau,n)}\right] = \sum_{k\geq 0} \frac{(-u)^k\,\mathbb{E}[Z(\tau,n)^k]}{k!}$$

From γ_k's we know $\mathbb{E}[Z(\tau,n)^k] \approx e^{ck^2}$ so RHS makes no sense.

How can this be. We computed $\mathbb{E}[e^{-uZ(\tau,n)}]$ rigorously as limit of q-Laplace tranform for q-TASEP, and we computed $\mathbb{E}[Z(\tau,n)^k]$ rigorously as limit of $\mathbb{E}[(q^{x_n(t)+n})^k]$.

Answer: ~~At large constant is~~ $q^{x_n(t)+n} \to 0$ so we rescaled leading to unbounded $Z(\tau,n)$. Even though a series has a limit, and the terms have pointwise limits, it need not mean that the series limit = limit of the series.

Example $\frac{1}{1-z} = G(z) = 1 - z + z^2 \cdots$ as $z \to 1$

$$\begin{array}{ccc} \| & & = \\ \frac{1}{2} & & = 1 - 1 + 1 - 1 \cdots \end{array}$$

Of course, the situation is much more complicated due to infinite range rearrangements in various terms to sum series

Finally, semidiscrete SHE limits to SHE under weak noise scaling which due to the Brownian scaling of noise, can be absorbed into time

Thm [Moreno-Flores, Remenik, Quastel]

$$\frac{Z(\sqrt{tn}+x;n)}{C(n,t,x)} \underset{n\to\infty}{\Longrightarrow} Z(t,x) \quad \text{solves} \quad \partial_t Z = \tfrac{1}{2}\partial_x^2 Z + Z\xi$$
$$Z(0,x) = \delta_{x=0}$$

$$C(n,t,x) = \exp\left(n + \frac{\sqrt{tn}+x}{2} + x\sqrt{\tfrac{n}{t}}\right)\left(\tfrac{t}{n}\right)^{n/2}$$

Consequences

- Limit of Laplace transform gives second rigorous derivation of $\mathbb{E}[e^{-uZ(t,x)}]$ [Borodin-C-Ferrari], see also ACQ

- Limit of moment formulas suggests sol^n to continuum δ Bose gas (connection to SHE moments not totally rigorous, though certainly true)

Def^n: Weyl chamber $W^n = \{x_1 < \cdots < x_n\}$. A function $U: W^k \to \mathbb{R}$ solves δ-Bose gas with coupling constant $c \in \mathbb{R}$ and u_0 initial data if.

(1) For $x \in W^k$, $\partial_t u = \frac{1}{2}\partial_x^2 u$

(2) For $x \in \partial W^k$, $(\partial_{x_i} - \partial_{x_{i+1}} - c)u\big|_{x_{i+1}=x_i+0} = 0$

(3) $\forall f \in L^2(W^k) \cap C_b(\overline{W}^k)$, as $t \to 0$

$$\int_{W^k} f(x)\,u(x;t)\,dx \to \int_{W^k} f(x)\,u_0(x)\,dx.$$

<u>Prop</u> [Borodin-C] For $u_0(x) = \delta_{x=0}$ and all $c \in \mathbb{R}$,

$$u(x;t) = \frac{1}{(2\pi i)^k} \int \cdots \int \prod_{1 \le A < B \le k} \frac{z_A - z_B}{z_A - z_B - c} \prod_{j=1}^{k} e^{x_j z_j + \frac{t}{2} z_j^2} \, dz_j$$

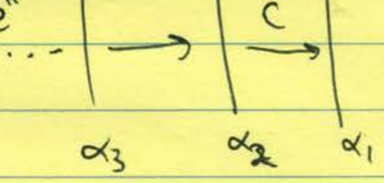
z_j on $\alpha_j + i\mathbb{R}$

$\alpha_1 > \alpha_2 + c > \alpha_3 + 2c > \cdots$

Solves δ-Bose gas.

<u>Example</u>

- For $c < 0$ "repulsive" all $\alpha_i \equiv 0$ works
- For $c > 0$ "attractive" ...



$c = 1$ Corresponds to SHE moments and "nested contour integral ansatz".

Non-rigorous derivation (initially wrong) of the Laplace transform from moments (which now grow like e^{ck^3}) by Dotsenko and Calabrese-Le Doussal-Rosso. at roughly same time as ACQ (Amir-C-Quastel) and Sasamoto-Spohn

Spectral approach: (Leib-Liniger $c<0$ 1963, McGuire $c>0$ 1964, Oxford, Heckman-Opdam)

- Find functions $\{\Phi^c(\vec{x};\vec{z})\}_{\vec{z}\in I}$ s.t.

$$\frac{1}{2}\partial_x^2 \Phi^c(\vec{x};\vec{z}) = E(\vec{z})\,\Phi^c(\vec{x};\vec{z})$$

$$(\partial x_i - \partial x_{i+1} - c)\,\Phi^c(\vec{x};\vec{z})\Big|_{x_{i+1}=x_i+0} = 0$$

- Determine $I' \subseteq I$ such that $\{\Phi^c(\vec{x};\vec{z})\}_{\vec{z}\in I'}$ form a complete basis, and determine Plancherel measure $d\mu_p(\vec{z})$ such that $u(t,\vec{x}) = \int d\mu_p(\vec{z})\,\Phi(\vec{x};\vec{z})\left(\int u_0(\vec{y})\,\overline{\Phi^c(\vec{y};\vec{z})}\,d\vec{y}\right) e^{\frac{t}{2}E(\vec{z})}$.

The Φ's and μ_p come out of the type of residue calculations and ~~[illegible]~~ I' depends significantly on $c>0$ or $c<0$ on account of the residue structure.

$$\Phi^c(\vec{x};\vec{z}) = \frac{1}{k!}\sum_{\sigma\in S_k} \mathrm{sgn}(\sigma) \prod_{A>B} \frac{z_{\sigma(A)} - z_{\sigma(B)} - c}{z_{\sigma(A)} - z_{\sigma(B)}} \prod_{j=1}^{k} e^{x_j z_{\sigma(j)}}$$

$$E(\vec{z}) = \sum z_j^2$$

I' for $c<0$ is $(iR)^k$
for $c>0$ is the full residue subspace from before.

From spectral approach (Bethe ansatz) one can recover the nested contour formula, but only by reverse engineering an involved combinatorial expansion. One in the nested form, clear how two different types of Fredholm det. arise. Also clear how Plancherel measure arises.

Additionally, when in a non-self-adjoint case (semi discrete, q-TASEP or ASEP) then eigenfunctions are not clearly orthonormal, so Bethe ansatz and Plancherel not clear how to apply for our purposes.

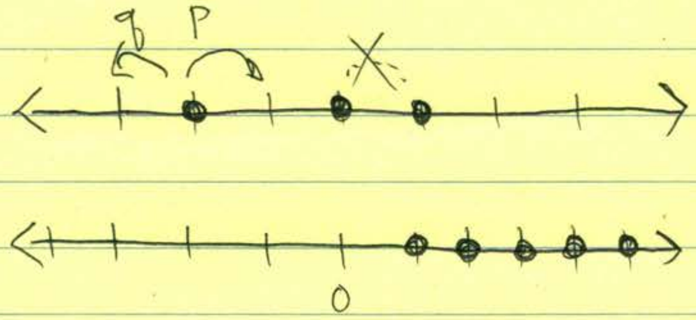
Two ways to solve ASEP

- Tracy-Widom 2008-2009 Bethe ansatz (coordinate approach)
- Borodin-C-Sasamoto 2012 Nested contour integral ansatz (Duality approach)

While coordinate approach has (presently) no parallel in other KPZ class non-determinantal yet exactly solvable systems, duality (or equiv replicas) is prevasive revealing parallel formulas which beg for parallel structure.

Recall ASEP

step



$p+q=1$
$q-p=\gamma>0$
$p/q=\tau<1$

Occupation process: state space $Y=\{0,1\}^{\mathbb{Z}}$, $\eta=\{\eta_x\}_{x\in\mathbb{Z}}$ $\quad \eta_x=\begin{cases}1 & \text{particle}\\ 0 & \text{hole}\end{cases}$
$\eta(t)$

Coordinate process: k particles $X_k=\{x_1<\cdots<x_k\}\subseteq\mathbb{Z}^k$ with x_i = location of particle i
$\vec{x}(t)$

Step initial condition: $\eta_x(0)=\mathbb{1}_{x>0}$

Current $N_x(\eta)=\sum_{y\le x}\eta_y$ so $N_0(\eta(t))$ = # particles at or left of origin at time t.

Johansson $\gamma=1$

<u>Thm</u> (TW, BCS) For ASEP with step initial condition and $q>p$

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{N_0(t/\gamma) - t/4}{2^{-1/3} t^{1/3}} \geq -s \right) = F_{GUE}(s)$$

(1) Coordinate approach (Tracy-Widom 2008-2009)

Step 1: Consider k particle ASEP coordinate process and compute transition probability of starting at $\vec{y}$ and ending at $\vec{x}$ in time t.

Call this $P_{\vec{y}}(\vec{x};t)$ then it solves the evolution eqn

- $\frac{d}{dt} u(\vec{x};t) = (L^k)^* u(\vec{x};t)$

- $u(\vec{x};0) = \mathbb{1}_{\vec{x}=\vec{y}}$. (more generally $u_0(\vec{x})$)

Where $(L^k)^*$ is adjoint of generator of $\vec{x}(t)$.

Eg. for $k=1$

$$(L^1 f)(x) = q\left[f(x+1)\ f(x)\right] + p\left[f(x-1)\ f(x)\right]$$

$$((L^1)^* f)(x) = p\left[\qquad \right] + q\left[\qquad \right].$$

For $k>1$ not constant coefficient, so hard to solve.

Prop: If $v : \mathbb{Z}^k \times \mathbb{R}_+ \to \mathbb{R}$ solves "free evolution with boundary condition"

(1) $\frac{d}{dt} v(\vec{x};t) = \sum_{j=1}^{k} [(L')^*]_j \, v(\vec{x};t)$

2) For all $\vec{x}$: $x_{j+1} = x_j + 1$ and $t \geq 0$

$$p\, v(x_1, \dots, x_j, x_{j+1}-1, \dots ; t) + q\, v(x_1, \dots, x_j+1, x_{j+1}, \dots) - v(\vec{x};t) = 0$$

3) For all $\vec{x} \in X_k$, $v(\vec{x};0) = u_0(\vec{x})$

Then for all $t \geq 0$, $\vec{x} \in X_k$, $u(\vec{x};t) = v(\vec{x};t)$

(also a certain technical growth condition to ensure uniqueness)

Solution by Bethe ansatz ($k=2$ by Schütz, $k>2$ Tracy-Widom)

$$P_{\vec{y}}(\vec{x};t) = \int \cdots \int \sum_{\sigma \in S_k} \operatorname{sgn}\sigma \prod_{A>B} \frac{p + q\xi_{\sigma(A)}\xi_{\sigma(B)} - \xi_{\sigma(B)}}{p + q\xi_A \xi_B - \xi_B} \cdot \prod_{j=1}^{k} \xi_{\sigma(j)}^{x_j - y_{\sigma(j)} - 1} e^{\varepsilon(\xi_j)t} \, d\xi_j$$

$$\varepsilon(\xi) = p\xi^{-1} + q\xi - 1$$

(Briefly now) Using much combinatorics

Step 2: integrate out to get single particle transition formula.

Step 3: Manipulate so as to be able to take $k \to \infty$ and find for step

$$\mathbb{P}(N_0(t) = m) = \frac{-\tau^m}{2\pi i} \int \frac{\det(I - \xi K_1)_{L^2(C_R)}}{(\xi;\tau)_{m+1}} \, d\xi$$

ξ encloses q^{-k} $0 \leq k \leq m-1$, $K_1(\xi,\xi') = q \dfrac{e^{\varepsilon(\xi)t}}{p + q\xi\xi' - \xi}$

Step 4: Manipulate into form suitable for asymptotics.

Duality approach: Leads to two Fredholm det. characterizing N_0 distribution: One new and very good for asymptotics, second one equivalent (after change of variables) to Tracy-Widom's.

Recall $\vec{x}(t) \in X$, $\vec{\eta}(t) \in Y$ are dual wrt $H : X \times Y \to \mathbb{R}$ if $\forall \vec{x}, \eta, t$

$$\mathbb{E}^{\eta}\left[H(\vec{x}, \eta(t))\right] = \mathbb{E}^{\vec{x}}\left[H(\vec{x}(t) ; \eta)\right]$$

If for η fixed, we set $u_\eta(\vec{x} ; t) = \mathbb{E}^{\eta}\left[H(\vec{x} ; \eta(t))\right]$ then duality implies $\frac{d}{dt} u_\eta(\vec{x} ; t) = L u_\eta(\vec{x} ; t)$, $u_\eta(\vec{x} ; 0) = H(\vec{x}, \eta)$

Schutz 97: If $\vec{x}(t)$ is ASEP particle process with p, q reversed from earlier definition, and $\eta(t)$ is occupation process (independent) then these markov processes are dual wrt $H(\vec{x} ; \eta) = \prod_{j=1}^{k} \tau^{N_{x_j - 1}(\eta)} \eta_{x_j}$

<u>Remarks</u> • Schutz proof via quantum spin chain encoding of ASEP

• Borodin-C-Sasamoto direct proof from Markov generators, plus a second duality function.

Implies that for η = step initial condition $u_{step}(\vec{x};t)$ solves

$$\frac{d}{dt} u_{step}(x;t) = (L^k)^* u_{step}(x;t) \quad , u_{step}(x;0) = \mathbb{1}_{x_1 \geq 1} \prod_{j=1}^{k} \tau^{x_j - 1}$$

By appealing to free evolution w/ boundary condition and using (by guessing) a nested contour integral ansatz we find

$$u_{step}(\vec{x};t) = \frac{\tau^{k(k-1)/2}}{(2\pi i)^k} \int \cdots \int_{\circledcirc_\tau} \prod_{1 \leq A < B \leq k} \frac{z_A - z_B}{z_A - \tau z_B} \prod_{j=1}^{k} h_{x_j,t}(z_j)\, dz_j$$

$$h_{x,t}(z) = e^{\varepsilon'(z)t} \left(\frac{1+z}{1+z/\tau}\right)^{x-1} \frac{1}{1+z} \quad , \varepsilon'(z) = -\frac{z(p-q)^2}{(1+z)(p+qz)}$$

Note: One could have gotten from summing initial data over $P_y(x;t)$, but completely unclear how one would have guessed such a ~~n~~ nontrivial symmetrization.

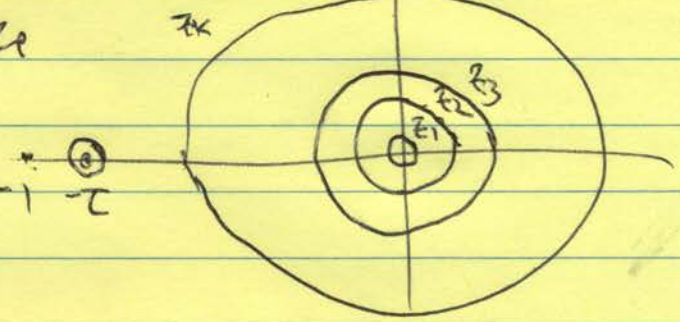
A suitable summation of $H(\vec{x};\eta)$ over $\vec{x}$ gives $\tau^{kN_x(\eta)}$

which, after summing the corresponding expectations gives

$$\mathbb{E}\left[\tau^{kN_0(t)}\right] = \frac{\tau^{k(k-1)/2}}{(2\pi i)^k}\int\cdots\int \prod_{1\le A<B\le k}\frac{z_A-z_B}{z_A-\tau z_B}\prod_{j=1}^{k} e^{\varepsilon(z_j)t}\frac{dz_j}{z_j}$$

where $N_0(t) = N_0(\eta(t))$ for step initial condition

and contour for z_j looks like



From here on out, the path to Fred. det and τ-laplace transform is essentially identical to q-TASEP case so we conclude with

<u>Thm</u> $$\mathbb{E}\left[\frac{1}{(\zeta\tau^{N_0(t)};\tau)_\infty}\right] = \det(I+K_\zeta)$$

$$K_\zeta(w,w') = \frac{1}{2\pi i}\int \frac{\pi}{\sin(-\pi s)}(-\zeta)^s\frac{g(w)}{g(\tau^s w)}\frac{ds}{w'-\tau^s w}, \quad g(w) = e^{\gamma t\frac{\tau}{\tau+w}}$$

from which GUE thm follows fairly easily.