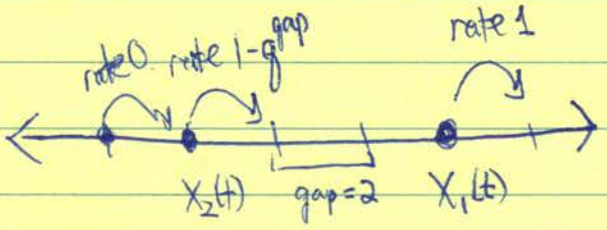
Macdonald processes:

Definition, ~~...~~ dynamics and computations

References: - Symmetric functions and hall polynomials, I.G. Macdonald
- Macdonald processes, Borodin - Corwin

q-TASEP on $\mathbb{Z}$



$q \in (0,1)$

- Each particle X_i jumps right at rate $1-q^{gap}$, $gap = X_{i-1}(t) - X_i(t) - 1$
- For $q \searrow 0$ becomes TASEP
- For $q \nearrow 1$ particles become very far appart ... interesting limit.
- Step initial data $X_i(0) = -i$ $i \geq 1$.

Goal: For step initial data, compute expression for distribution of $X_N(t)$ which allows asymptotics in N, t, q.

In fact, the real goal is to introduce Macdonald processes and observe how the integrable properties of Macdonald sym. poly gives rise to a large class of exactly solvable, probabilistic systems of phenomenological interest, q-TASEP being one example.

q-TASEP arises analogously to Mac. proc. as TASEP is to Schur, but the methods to compute are totally different!

A defining property of Schur sym. functions

<u>Def</u>^{n}: The dominance partial ordering on Young diagrams

$$\lambda \trianglerighteq \mu \quad \text{iff} \quad \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i \quad i \geq 1 \quad \text{and } |\lambda| = |\mu|$$

and $\lambda \triangleright \mu$ if $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$.

Exercise: Show this is only a partial order (hint: requires size of diagrams ≥ 6)

<u>Recall</u>: Λ has an $\mathbb{F}$-basis in the monomial symmetric functions.

The Schur sym. functions are uniquely defined by two properties

$$(1) \quad S_\lambda = m_\lambda + \sum_{\mu \triangleleft \lambda} K_{\lambda\mu} m_\mu \qquad K_{\lambda\mu} \in \mathbb{N} \text{ Kostka numbers.}$$

$$(2) \quad \langle S_\lambda, S_\mu \rangle = \delta_{\lambda=\mu}.$$

<u>Remarks</u>: (1) is true via combinatorial formula

(2) is true by definition of $\langle \cdot, \cdot \rangle$. However, we could

alternatively define $\langle \cdot, \cdot \rangle$ via $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda=\mu}$

with $z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$ $\quad \lambda = 1^{m_1} 2^{m_2} \ldots$

- In fact, the existence of such polynomials is amazing since Gram-Schmidt won't ~~order~~ produce the partial order in (1).

There are other choices of z_λ which produce interesting sym. functions (i.e. Hall-Littlewood, Jack, and on top Macdonald symmetric functions)

Macdonald symmetric functions

- Coefficients in $\mathbb{Q}[q,t]$ with q,t formal parameters (for our analytic purposes we will take $q,t \in (0,1)$)
- $\Lambda = \mathbb{Q}[q,t][x_1, x_2, \ldots]^{S(\infty)}$.
- Written $P_\lambda(x;q,t) = P_\lambda(x)$.

Uniquely defined by

(1) $P_\lambda(x) = m_\lambda(x) + \sum_{\mu \triangleleft \lambda} R_{\lambda\mu} m_\mu(x)$ $R_{\lambda\mu}$ are functions of λ, μ, q, t only

(2) $\langle P_\lambda, P_\mu \rangle = \langle P_\lambda, P_\lambda \rangle \delta_{\lambda=\mu}$ where the inner product $\langle \cdot, \cdot \rangle$ is defined via $\langle p_\lambda, p_\mu \rangle_{q,t} = z_\lambda(q,t)\, \delta_{\lambda=\mu}$ and

$$z_\lambda(q,t) = z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}, \quad z_\lambda \text{ as before.}$$

Define dual basis $Q_\lambda = P_\lambda / \langle P_\lambda, P_\lambda \rangle$ so that $\langle P_\lambda, Q_\mu \rangle = \delta_{\lambda=\mu}$.

Setting $q=t$ recover $P_\lambda = Q_\lambda = s_\lambda$

~~[illegible]~~ Cauchy identity

Orthogonality implies that

$$\sum_{\lambda\in\mathbb{Y}} P_\lambda(x)\,Q_\lambda(y) = \sum_\lambda \frac{P_\lambda(x)P_\lambda(y)}{Z_\lambda(q,t)}$$

$p_k(x)p_k(y)$

(exercise)
$$= \exp\left\{\sum_{k\geq 1}\frac{p_k(x)p_k(y)}{k}\cdot\frac{1-t^k}{1-q^k}\right\}$$

$$=: \Pi(x;y)$$ (replaces H from Schur case)

Identity holds for ~~spec~~ replacing x, y by two specializations.

If all ~~[illegible]~~ but finitely many $x_1,\dots,x_m \neq 0$, $y_1,\dots,y_n\neq 0$ (others 0) then

(exercise)
$$\exp\left\{\sum_{k\geq 1}\frac{p_k(x_1,\dots,x_m)\,p_k(y_1,\dots,y_n)}{k}\cdot\frac{1-t^k}{1-q^k}\right\}$$

$$= \prod_{i,j}\frac{(t x_i y_j;q)_\infty}{(x_i y_j;q)_\infty}$$ where $(a;q)_\infty = \prod_{i\geq 0}(1-q^i a)$, $(a;q)_k = \prod_{i=0}^{k-1}(1-q^i a)$

Note that setting $t=q$ we recover the Schur Cauchy identity.

Skew Macdonald functions

Def: For $\lambda \in \mathbb{Y}$ define $P_{\lambda/\mu}(x)$ via the coefficients in the expansion

$$P_\lambda(x,y) = \sum_{\mu \in \mathbb{Y}} P_{\lambda/\mu}(x)\, P_\mu(y)$$

and likewise $Q_{\lambda/\mu}(x)$ via

$$Q_\lambda(x,y) = \sum_{\mu \in \mathbb{Y}} Q_{\lambda/\mu}(x)\, Q_\mu(y).$$

Note $P_{\lambda/\mu} = \dfrac{\langle P_\lambda, P_\lambda \rangle}{\langle P_\mu, P_\mu \rangle} Q_{\lambda/\mu}$.

Clearly $P_{\lambda/\mu}$ is homogenious sym. poly. of degree $|\lambda| - |\mu|$ and hence zero if $|\lambda| < |\mu|$. More is true...

Skew Cauchy identity

$$\sum_{\mu\in\mathbb{Y}} P_{\mu/\lambda}(x)\, Q_{\mu/\nu}(y) = \Pi(x;y) \sum_{\varkappa\in\mathbb{Y}} Q_{\lambda/\varkappa}(y)\, P_{\nu/\varkappa}(x)$$

Chapmin-Kolmogorov

$$\sum_{\nu\in\mathbb{Y}} P_{\lambda/\nu}(x)\, P_{\nu/\mu}(y) = P_{\lambda/\mu}(x,y)$$

also holds with $P \leftrightarrow Q$.

Combinatorial formula

$$P_{\lambda/\mu}(x) = \sum_{T:\,sh(T)=\lambda/\mu} \psi_T\, x^T \quad , \quad Q_{\lambda/\mu}(x) = \sum_T \varphi_T\, x^T$$

- T : semi std Skew Young Tableaux of shape λ/μ is a sequence of Young diagrams

$$\mu = \lambda^{(0)} \prec \lambda^{(1)} \prec \lambda^{(2)} \prec \cdots \prec \lambda^{(\ell)} = \lambda$$ where $\lambda^{(i)}/\lambda^{(i-1)}$ horizonal strip, ℓ is arbitrary

- $\psi_T = \prod_{i\geq 1} \psi_{\lambda^{(i)}/\lambda^{(i-1)}}$ (same φ_T) and $x^T = x_1^{|\lambda^{(1)}|-|\lambda^{(0)}|}\, x_2^{|\lambda^{(2)}|-|\lambda^{(1)}|} \cdots\cdots$

(T ⇒ type of Tableaux)

$\psi_{\lambda/\mu}$ is independent of x's and given via explicit (complicated) formula.

$$\psi_{\lambda/\mu} = \prod_{1\leq i\leq j\leq \ell(\mu)} \frac{f(q^{\mu_i-\mu_j} t^{j-i})\, f(q^{\lambda_i-\lambda_{j+1}} t^{j-i})}{f(q^{\lambda_i-\mu_j} t^{j-i})\, f(q^{\mu_i-\lambda_{j+1}} t^{j-i})} \quad , \quad f(u) = \frac{(tu;q)_\infty}{(qu;q)_\infty}$$

horizontal strip

Similar definitions for $\varphi_{\lambda/\mu}$ and "dual" $\psi'_{\lambda/\mu}, \varphi'_{\lambda/\mu}$.

<u>Def^n</u>: A specialization $\rho: \Lambda \to \mathbb{C}$ is <u>Macdonald positive</u> if

$$P_{\lambda/\mu}(\rho) \geq 0 \quad \text{for all } \lambda, \mu \in \mathbb{Y}.$$

- Unlike Schur (or even Jack) case, the classification conjecture (of Kerov's) we now record has not been proved.

<u>Def^n</u>: The specialization $\rho = (\alpha;\beta;\gamma)$ with $\alpha = \{\alpha_1 \geq \alpha_2 \geq \dots \geq 0\}$, $\beta = \{\beta_1 \geq \beta_2 \geq \dots \geq 0\}$, $\gamma \geq 0$ and $\sum_{i=1} (\alpha_i + \beta_i) < \infty$ is defined via generating series

$$\exp\left\{ \sum_{k\geq 1} z^k \frac{p_k(\rho)}{k} \frac{1-t^k}{1-q^k} \right\} = e^{\gamma z} \prod_{i\geq 1} \frac{(t\alpha_i z; q)_\infty}{(\alpha_i z; q)_\infty} (1+\beta_i z) =: \Pi(z;\rho)$$

<u>Prop</u>: All such ρ are Macdonald ~~non-negative~~ positive

Pf: Pure α case follows combinatorial formula, β follows a duality, γ from limits.

<u>Conj (Kerov)</u>: This is all of the Macdonald ~~non-negative~~ positive specializations.

<u>Notation</u>

- Pure α (all but α's zero)
- Pure β (all but β's zero)
- Plancherel γ (all but γ zero)

<u>Examples</u>

- Pure α, $\alpha_1 = c$ all else 0

$$P_{\lambda/\mu}(\rho) = \begin{cases} \psi_{\lambda/\mu}\, c^{|\lambda|-|\mu|} & \lambda/\mu \text{ hor. strip} \\ 0 & \text{else.} \end{cases}$$

- Pure β, $\beta_1 = c$ all else 0

$$P_{\lambda/\mu}(\rho) = \begin{cases} \varphi'_{\lambda/\mu}\, c^{|\lambda|-|\mu|} & \lambda/\mu \text{ vert. strip} \\ 0 & \text{else} \end{cases}$$

- Plancherel more complicated... won't directly appear

Macdonald measure / process

- (q,t)-generalization of Schur case first mentioned in Forrester-Rains '05, with various degenerations also discussed in Vuletic '09, Fulman '02, Okounkov, O'Connell '12

Def^{n}: Macdonald process is measure on $\lambda^{(1)} \supseteq \mu^{(1)} \subseteq \lambda^{(2)} \supseteq \mu^{(2)} \subseteq \lambda^{(3)} \cdots \subseteq \lambda^{(N)}$

specified by Macdonald positive $\rho_0^+, \rho_1^-, \rho_1^+, \ldots, \rho_{N-1}^-, \rho_{N-1}^+, \rho_N^-$ via

$$\mathbb{P}(\lambda,\mu) = \frac{W(\lambda,\mu)}{\sum\limits_{\substack{\lambda^{(1)},\ldots,\lambda^{(N)} \\ \mu^{(1)},\ldots,\mu^{(N-1)}}} W(\lambda,\mu)} \quad \text{with } W(\lambda,\mu) = P_{\lambda^{(1)}}(\rho_0^+) P_{\lambda^{(1)}/\mu^{(1)}}(\rho_1^-) P_{\lambda^{(2)}/\mu^{(1)}}(\rho_1^+) \cdots P_{\lambda^{(N)}/\mu^{(N-1)}}(\rho_{N-1}^+) P_{\lambda^{(N)}}(\rho_N^-)$$

Write as

$$\begin{array}{ccccccccccc} & & \lambda^{(1)} & & & \lambda^{(2)} & & & & \lambda^{(N)} & \\ & \overset{\rho_0^+}{\nearrow} & & \overset{\rho_1^-}{\searrow} & \overset{\rho_1^+}{\nearrow} & & \overset{\rho_2^-}{\searrow} & \cdots & \overset{\rho_{N-1}^+}{\nearrow} & & \overset{\rho_N^-}{\searrow} \\ \emptyset & & & \mu^{(1)} & & & \mu^{(2)} & & \mu^{(N-1)} & & \emptyset \end{array}$$

- Normalization calculated by Skew Cauchy and Chapman-Kolmogorov as

$$\sum W(\lambda,\mu) = \prod_{0 \le i < j \le N} \Pi(\rho_i^+ ; \rho_j^-)$$

$$\Pi(\rho;\rho') = \sum_{\lambda} P_\lambda(\rho) Q_\lambda(\rho') = \exp\left\{ \sum_{k \ge 1} \frac{p_k(\rho) p_k(\rho')}{k} \frac{1-t^k}{1-q^k} \right\}$$

We will focus here on a special case in which all $\rho_i^- = 0$ except $\rho_N^- = \rho$ and all $\rho_k^+ = ((a_i); 0; 0)$ [$\in \mathbb{R}_{>0}$]. This implies that the process is supported on seq$^{\underline{s}}$ $\emptyset \preceq \lambda^{(1)} \preceq \lambda^{(2)} \preceq \cdots \preceq \lambda^{(N)}$ of partitions with $\lambda^{(i)}/\lambda^{(i-1)}$ hor. strip and $|\lambda^{(i)}| \le i$

or equiv an interlacing triangular array:

$$\begin{array}{ccccccc} \lambda_N^{(N)} & & \lambda_{N-1}^{(N)} & \cdots & \lambda_2^{(N)} & & \lambda_1^{(N)} \\ & \lambda_{N-1}^{(N-1)} & & \cdots & & \lambda_1^{(N-1)} & \\ & & \ddots & & \therefore & & \\ & & & \lambda_1^{(1)} & & & \end{array} \quad , \ \lambda_j^{(k)} \in \mathbb{Z}_{\ge 0} \text{ interlacing}$$

Def$^{\underline{n}}$: Ascending Macdonald process on $\emptyset \le \lambda^{(1)} \le \cdots \le \lambda^{(N)}$

$$\mathbb{M}_{asc}(a_1, \ldots a_N; \rho)(\lambda^{(1)}, \ldots, \lambda^{(N)}) := \frac{P_{\lambda^{(1)}}(a_1) P_{\lambda^{(2)}/\lambda^{(1)}}(a_2) \cdots P_{\lambda^{(N)}/\lambda^{(N-1)}}(a_N) Q_{\lambda^{(N)}}(\rho)}{\Pi(a_1, \ldots, a_N; \rho)}$$

where (by def^n) $\Pi(a_1, \ldots, a_N; \rho) = \Pi(a_1; \rho) \cdots \Pi(a_N; \rho)$.

- Projection onto level k is simple

Def$^{\underline{n}}$: Macdonald measure on $\lambda^{(k)}$: $\mathbb{MM}(a_1, \ldots a_k; \rho)(\lambda^{(k)}) := \dfrac{P_{\lambda^{(k)}}(a_1, \ldots a_k)\, Q_{\lambda^{(k)}}(\rho)}{\Pi(a_1, \ldots a_k; \rho)}$

Remarks

When $q = t$ these degenerate the Schur case. Other (less trivial) degenerations include that to O'Connell's Whittaker measure/process and a hierarchically slightly higher measure of COSZ. Natural relation to directed polymers via tRSK and recently q-tRSK connection to q-TASEP (more later)

<u>Theorem</u> If $\rho = (0;0;\tau)$ ~~ ~~ and $a_i \equiv 1$ then

$$\mathbb{P}_{MM(a_1,\ldots,a_N;\rho)}\left(\lambda^{(N)}_N - N = x\right) = \mathbb{P}_{\substack{q\text{-TASEP} \\ \text{step initial}}}\left(x_N(\tau) = x\right)$$

- We show via analogous approach as gave rise to TASEP/Schur relation.

<u>Just the highlights</u>

- For ρ, ρ' Mac. pos. spec. define matrices $P^{\uparrow}_{\lambda\to\mu}(\rho;\rho')$, $P^{\downarrow}_{\lambda\to\mu}(\rho;\rho')$ ~~ ~~ $\text{row } \lambda, \text{col } \mu$ as

$$P^{\uparrow}_{\lambda\to\mu}(\rho;\rho') := \frac{P_\mu(\rho)}{P_\lambda(\rho)} \frac{Q_{\mu/\lambda}(\rho')}{\Pi(\rho;\rho')}$$ assume non-zero.

$$P^{\downarrow}_{\lambda\to\mu}(\rho;\rho') := \frac{P_\mu(\rho)}{P_\lambda(\rho,\rho')} P_{\lambda/\mu}(\rho')$$

- $P^{\uparrow}, P^{\downarrow}$ stochastic and act well on Macdonald measures

$$\mathbb{MM}(\rho_1;\rho_2)\, P^{\uparrow}(\rho_2;\rho_3) = \mathbb{MM}(\rho_1,\rho_3;\rho_2)$$

$$\mathbb{MM}(\rho_1;\rho_2,\rho_3)\, P^{\downarrow}(\rho_2;\rho_3) = \mathbb{MM}(\rho_1;\rho_2)$$

Commute $\quad P^{\uparrow}(\rho_1,\rho_2;\rho_3)\, P^{\downarrow}(\rho_1;\rho_2) = P^{\downarrow}(\rho_1;\rho_2)\, P^{\uparrow}(\rho_1,\rho_2;\rho_3)$

- $\mathbb{M}_{asc}(\rho_1,\ldots,\rho_N;\rho)(\lambda^{(1)},\ldots,\lambda^{(N)}) = \mathbb{MM}(\rho_1,\ldots,\rho_N;\rho)(\lambda^{(N)})\, P^{\downarrow}_{\lambda^{(N)}\to\lambda^{(N-1)}}(\rho_1,\ldots,\rho_{N-1};\rho_N)$

$\qquad \cdots\cdots \quad P^{\downarrow}_{\lambda^{(2)}\to\lambda^{(1)}}(\rho_1;\rho_2)$

Thus: We may apply the multivariate Markov Chain construction (similar to Schur case) to construct dynamics on interlacing triangular arrays.

S_k = Young diagrams with at most k rows

$S^{(n)} = \{ (\lambda^{(1)} \preceq \lambda^{(2)} \preceq \cdots \preceq \lambda^{(N)}) \text{ with } \lambda^{(i)} \in S_i \}$

$P_k = \mathbb{P}^{\uparrow}(a_1, \ldots a_k; \rho')$

$\Lambda^k_{k-1} = \mathbb{P}^{\downarrow}(a_1, \ldots, a_{k-1}; a_k)$

The subsequential update rule is that given $\mu \in S_k$ and an update $\lambda \in S_{k-1}$, the ~~meas~~ μ updates to $\nu \in S_k$ with probability equal to $\mathbb{P}(\nu|\lambda,\mu) = \text{const wrt } \nu \cdot Q_{\nu/\mu}(\rho') P_{\nu/\lambda}(a_k)$.

If we apply these dynamics to an ascending Schur process with $(a_1 \ldots a_N; \rho)$ then after the time step ρ has updated to $\rho \cup \rho'$.

If ρ is initially $(0;0;0)$ then the measure is initially supported on $\emptyset \preceq \emptyset \cdots \preceq \emptyset$.

Consider when $\rho' = (0;(c);0)$ then

$$\mathbb{P}(\nu|\lambda,\mu) = \begin{cases} \text{const}\cdot \psi'_{\nu'/\mu'}\psi_{\nu/\lambda}\, c^{|\nu|-|\lambda|} & \nu/\lambda,\ \nu'/\mu' \text{ hor strips} \\ 0 & \text{else.} \end{cases}$$

Take $c=\varepsilon$ and repeat this $\varepsilon^{-1}\tau$ times. Then the

ρ part in the ascending Mac. proc. specialization is

$$\underset{\text{ctns time}}{} (0;\underbrace{(\varepsilon,\ldots\varepsilon)}_{\varepsilon^{-1}\tau \text{ times}};0) \to (0;0;\tau) \quad \text{Plancherel.}$$

The ctns time update rule corresponds to a poisson limit from $O(\varepsilon)$ part of $\mathbb{P}(\nu|\lambda;\mu)$.

- Each coordinate $\lambda_k^{(m)}$ has ind^t exp. clock with rate

$$a_m \cdot \frac{\psi_{(\lambda^{(m)}\cup\square_k)/\lambda^{(m-1)}}}{\psi_{\lambda^{(m)}/\lambda^{(m-1)}}}\,\psi'_{(\lambda^{(m)}\cup\square_k)/\lambda^{(m)}} \qquad \left(\begin{array}{c}\lambda\cup\square_k \text{ means increase} \\ \lambda_k \text{ by } 1.\end{array}\right)$$

- When its clock rings, $\lambda_k^{(m)}$ increases in value by 1 (add box $\square_k$ to $\lambda^{(m)}$)
- By virtue of ψ,ψ', jump rate goes to 0 if jump would violate interlacing with $\lambda^{(m-1)}$, and after a jump of $\lambda_k^{(m)}$ occurs, and interlacing violations at $\lambda^{(m+1)}$ are resolved with rate ∞.
- Unfortunately, ψ,ψ' are still pretty complicated and <u>non-local</u>
- Note: when $q=t$ this reduces to ctns time Schur dynamics from earlier

When $t=0$, Ψ and Ψ' simplify and we find a nice 2^d interacting particle system with local update rule and q-TASEP as a marginal.

__Def__^{n}: The q-deformed 2^d dynamics with rates $a_1, \ldots a_N > 0$ has ~~the~~ each coordinate $\lambda_k^{(m)}$ of an interlacing triangular array jump right by 1 at rate$(\lambda_k^{(m)})$:

$$a_m \cdot \frac{\left(1 - q^{\lambda_{k-1}^{(m-1)} - \lambda_k^{(m)}}\right)\left(1 - q^{\lambda_k^{(m)} - \lambda_{k+1}^{(m)} + 1}\right)}{\left(1 - q^{\lambda_k^{(m)} - \lambda_k^{(m-1)} + 1}\right)}$$

with terms which don't make sense omitted.

$$\begin{tikzcd} \lambda_{k+1}^{(m)} & \lambda_k^{(m)} \arrow[l] & \mathrm{rate}(\lambda_k^{(m)}) \arrow[l, dashed] \\ \lambda_k^{(m-1)} \arrow[ur] & & \lambda_{k-1}^{(m-1)} \arrow[ul] \end{tikzcd}$$

Simulation: $x_k^{(m)} = \lambda_k^{(m)} - k$

- These dynamics have many interesting limits, ~~and~~ properties and explicit formulas
- Set $X_k = \lambda_k^{(k)} - k$ then $X_1 > X_2 > \cdots$ evolves marginally as a Markov proc. which is general rate q-TASEP: Ctns time interacting particle system on $\mathbb{Z}$ ~~with~~ where x_i jumps one to the right at rate $a_i\left(1 - q^{x_{i-1} - x_i - 1}\right)$
- Step initial data $X_n(0) = -n$, $n \geq 1$ corresponds with ρ initially 0.
- Proves Theorem

  If $\rho = (0;0:\tau)$ and $a_i \equiv 1$ then $\mathbb{P}_{\substack{q\text{-TASEP}\\ \text{step}}}(x_N(\tau) = x) = \mathbb{P}_{MM}(\lambda_N^{(N)} - N = x)$

Goal: Compute exact formula for distribution of $x_N(\tau)$ which does not grow in complexity as N, τ, q scale. More generally, develop methods of exact solvability for Macdonald processes / measures allowing us to compute a robust set of expectations for interesting observables.

We will focus on the Macdonald measure on $\lambda^{(N)} = \lambda$

$$\mathbb{P}(\lambda) = \frac{P_\lambda(a_1, \dots a_N)\, Q_\lambda(\rho)}{\Pi(a_1, \dots a_N ; \rho)}, \quad \Pi(a_1, \dots a_N ; \rho) = \sum_{\lambda \in \mathbb{Y}} P_\lambda(a_1 \dots a_N) Q_\lambda(\rho)$$

- Big idea: $\mathcal{D}$ linear operator on Λ_N such that for all $\lambda : \ell(\lambda) \leq N$,

$$\mathcal{D} P_\lambda(x_1, \dots, x_N) = d_\lambda P_\lambda(a_1, \dots, a_N) \quad \text{then}$$

$$\frac{\mathcal{D}^{(a)} \Pi(a_1, \dots, a_N ; \rho)}{\Pi(a_1, \dots, a_N ; \rho)} = \frac{1}{\Pi(a ; \rho)} \cdot \sum_\lambda \mathcal{D}^{(a)} P_\lambda(a) Q_\lambda(\rho)$$

$$= \sum_\lambda d_\lambda \frac{P_\lambda(a) Q_\lambda(\rho)}{\Pi(a ; \rho)} = \mathbb{E}[d_\lambda]$$

Applying products of such operators leads to various expectations.

Example: Ising model $Z(\beta, h) = \sum_\sigma e^{\beta \sum_{x \sim y} \sigma_x \sigma_y + h \sum \sigma_x}$ then

$$\mathbb{E}\left[\sum \sigma_x\right] = \frac{\partial_h Z(\beta, h)}{Z(\beta, h)}, \quad \mathbb{E}\left[\sum_{x \sim y} \sigma_x \sigma_y\right] = \frac{\partial_\beta Z(\beta, h)}{Z(\beta, h)} \quad \text{etc.}$$

$\log Z \to$ free energy contains much of the physics.

There is a whole <u>integrable system</u> of N computing operators diagonalized by P_λ.

<u>Def^n</u>: $(T_{u;x_i} f)(x_1,\ldots,x_N) := f(x_1,\ldots ux_i,\ldots x_N)$

<u>Def^n</u>: Macdonald r^{th} difference operator D_N^r, $1 \le r \le N$

$$D_N^r := \sum_{\substack{I \subset \{1,\ldots,N\} \\ |I| = r}} A_I(x;t) \prod_{i \in I} T_{q;x_i} \quad , \quad A_I(x;t) := t^{\frac{r(r-1)}{2}} \prod_{\substack{i \in I \\ j \notin I}} \frac{tx_i - x_j}{x_i - x_j}$$

- Act taking $\Lambda_N \to \Lambda_N$
- Self adjoint operator wrt $\langle \cdot , \cdot \rangle$ inner product
- All diagonalized by P_λ, $\lambda : \ell(\lambda) \le N$ (i.e. mutually commuting) with

$$D_N^r P_\lambda(x_1,\ldots x_N) = e_r(q^{\lambda_1} t^{N-1}, q^{\lambda_2} t^{N-2}, \ldots, q^{\lambda_N}) P_\lambda(x_1,\ldots x_N)$$

where e_r is r^{th} elementary symmetric ~~function~~ polynomial.

In particular $D_N^1 P_\lambda = (q^{\lambda_1} t^{N-1} + q^{\lambda_2} t^{N-2} + \cdots q^{\lambda_N}) P_\lambda$

and when $t=0$ $D_N^1 P_\lambda = q^{\lambda_N} P_\lambda$.

<u>Remark</u>: When $q=t$ $P_\lambda = s_\lambda$ indep. of q,t. But D_N^r still depend on $q=t$ value. This degeneracy can be used to provide a new route to Schur measure/process correlation function.

Recall that we wish to apply these operators to $\Pi(a_1,\dots,a_N;p) = \Pi(a_1;p)\cdots\Pi(a_N;p)$

so we need only consider the action of the operators on $F(x_1,\dots,x_N) = f(x_1)\cdots f(x_N)$.

<u>Fact</u>: D_N^r action on such F's can be encoded via contour integrals.

Why useful?

- ~~good~~ Like ~~good~~ going to a generating function
- Complex analysis (especially residue calculations) can yield non-trivial combinatorial identities; analytic continuation powerful tool.
- Good for asymptotics and organizing combinatorial information analytically.

Preliminary example $N=r=1$ then $(D_1^1 F)(x) = F(qx)$

Note that
$$f(qx) = \frac{f(x)}{2\pi i} \cdot \int_{C_x} \frac{dz}{z-x} \frac{f(qz)}{f(z)}.$$

Pick residue at $z=x$.

$$N=2, r=1 \quad (D_2^1 F)(x_1,x_2) = \frac{tx_1-x_2}{x_1-x_2} f(qx_1)f(x_2) + \frac{tx_2-x_1}{x_2-x_1} f(x_1)f(qx_2)$$

How is this encoded?

Prop: Assume $F(x_1,\dots x_N) = f(x_1)\cdots f(x_N)$. Consider $a_1,\dots,a_N > 0$ and assume $f(x)$ is holomorphic and non-zero in a complex neighborhood ~~containing~~ around an ~~open~~ interval of $\mathbb{R}$ containing $\{a_j, q a_j\}_{j=1}^N$. Then

$$(D_N^r F)(a_1,\dots,a_N) = F(a_1,\dots a_N)\frac{1}{(2\pi i)^r r!}\oint\cdots\oint \det\left[\frac{1}{tz_k - z_l}\right]_{l,k=1}^r \prod_{j=1}^r \left(\prod_{m=1}^N \frac{tz_j - a_m}{z_j - a_m}\right)\frac{f(qz_j)}{f(z_j)}dz_j$$

Where each of the r integrals is over a pos. oriented contour contain $\{a_m\}_{m=1}^N$ and no other singularities of the integrand. (as long as t small such contour exists)

Proof: (For $r=1$) Note $(D_N^1 F)(a_1,\dots a_N) = F(a_1,\dots a_N)\underbrace{\sum_{i=1}^N \prod_{j\neq i}\frac{t a_i - a_j}{a_i - a_j}\frac{f(qa_i)}{f(a_i)}}_{(*)}$

claim: $(*) = \frac{1}{2\pi i}\int_{C_a}\frac{1}{tz - z}\prod_{m=1}^N \frac{tz - a_m}{z - a_m}\frac{f(qz)}{f(z)}dz$

Assume WLOG all a's diff. Then integral is sum over residues at $z = a_m$ which equals. $\sum_{i=1}^N \frac{ta_i - a_i}{ta_i - a_i}\cdot\prod_{j\neq i}\frac{ta_i - a_j}{a_i - a_j}\cdot\frac{f(qa_i)}{f(a_i)}$ $\square$.

For $r>1$ similar residue calculus plus Cauchy determinant identity yields proof.

Products of D_N^r can be similarly encoded. For example, notice

$$(D_N^1 F)(a_1,\dots a_N) = \frac{1}{2\pi i}\int \frac{1}{tz-z} \prod_{m=1}^{N} \underbrace{\left(f(a_m) \frac{tz-a_m}{z-a_m} \right)}_{\text{call } g(a_m)} \cdot \frac{f(qz)}{f(z)}\, dz$$

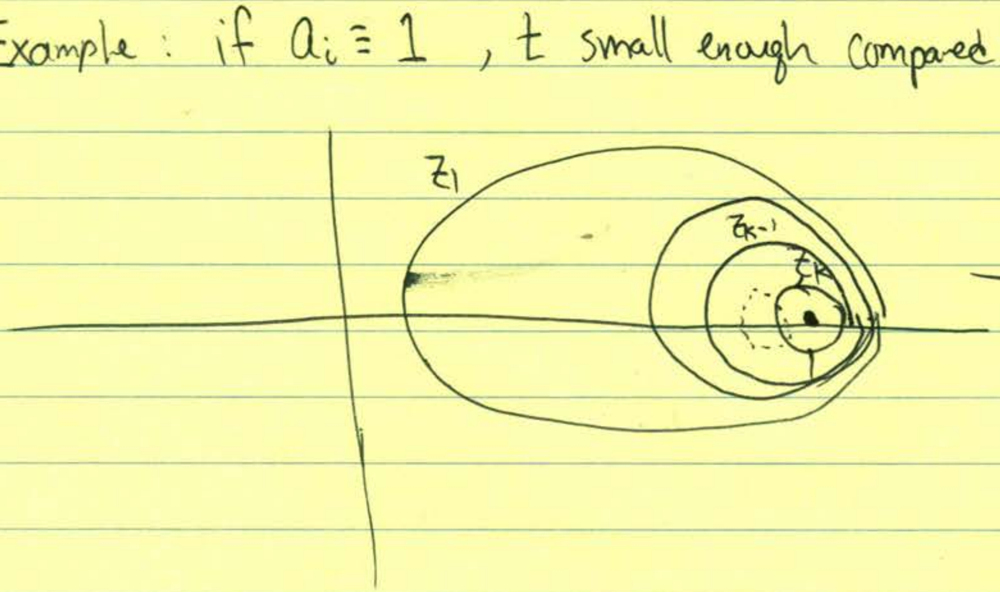
If we apply D_N^1 again, and use linearity of integral we find that we can use the proposition again to get a two fold nested integral.

<u>Prop</u>: Assume F factors, $a_1,\dots,a_N > 0$ and $f(x)$ holomorphic / non zero around $\{ q^j a_m : 0\le j\le k,\ 1\le m\le N \}$. Then

$$((D_N^1)^k F)(a_1,\dots a_N) = \frac{(t-1)^{-k}}{(2\pi i)^k} \oint\cdots\oint \prod_{1\le A<B\le k} \frac{tz_A - qz_B}{z_A - qz_B} \cdot \frac{z_A - z_B}{tz_A - z_B} \prod_{j=1}^{k} \left(\prod_{m=1}^{N} \frac{tz_j - a_m}{z_j - a_m} \right) \frac{f(qz_j)}{f(z_j)} \frac{dz_j}{z_j}$$

where the z_j contour contains $\{ qz_{j+1},\dots qz_k \text{ contours} \}$, $\{a_1,\dots a_N\}$ and no other poles.

Example: if $a_i \equiv 1$, t small enough compared to q then



<u>Nested Contour integral formula</u>

Lets apply this result to q-TASEP (set $t=0$, $a_i \equiv 1$)

Since $D_N^1 P_\lambda = q^{\lambda_N} P_\lambda$ we find that

$$\frac{(D_N^1)^k \,\Pi(a_1,\dots,a_N;(0;0;\tau))}{\Pi(a_1,\dots,a_N;(0;0;\tau))} = \mathbb{E}_{MM(1,\dots,1;(0;0;\tau))}\left[q^{k\lambda_N}\right]$$

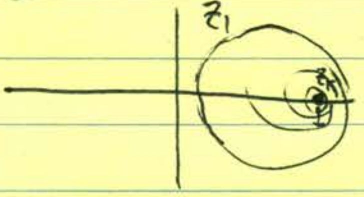
$$= \mathbb{E}_{\substack{q\text{-TASEP}\\ \text{step}}}\left[q^{k(x_N(\tau)+N)}\right]$$

On the other hand,

$$\Pi(a_1,\dots,a_N;(0;0;\tau)) = \Pi(a_1;(0;0;\tau))\cdots\Pi(a_N;(0;0;\tau)), \quad \Pi(x;(0;0;\tau)) = e^{x\tau}.$$

So
$$\mathbb{E}\left[q^{k\lambda_N}\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint\cdots\oint \prod_{1\le A<B\le k} \frac{z_A-z_B}{z_A-qz_B} \prod_{j=1}^{k} \frac{1}{(1-z_j)^N} e^{(q-1)\tau z_j} \frac{dz_j}{z_j}$$

z_A contour contains qz_B, $B>A$ and 1, but not zero.

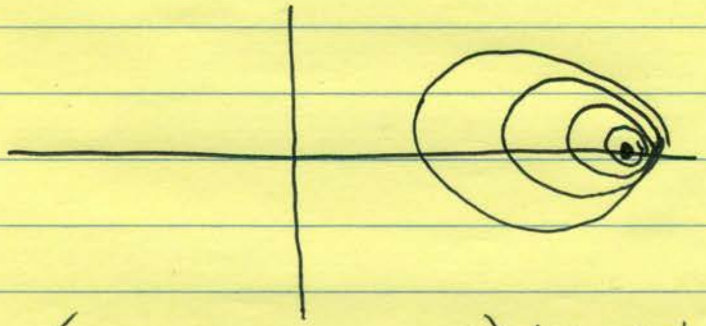


This is one of many nice formulas for expectations wrt Macdonald measure.

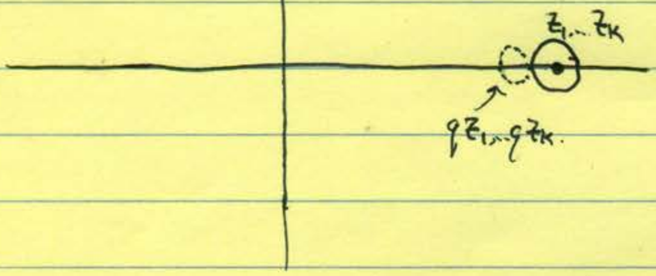
It is also possible to compute joint level expectations of the ascending Macdonald process.

Later we will see how this formula arises for q-TASEP in a different (though related) manner using duality / replica method.

The nesting structure of contours becomes a bit cumbersome as k grows. Can use complex analysis / residue theorem and Cauchy theorem to deform until all contours match. Two choices:

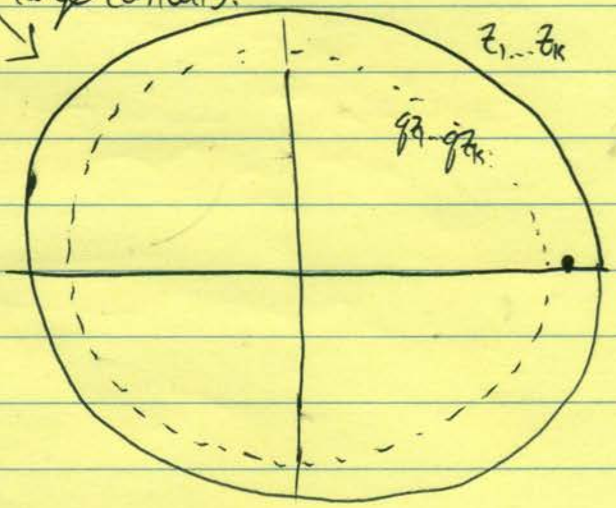


small contours



crosses all poles coming from $\prod_{A<B} \frac{1}{z_A - q z_B}$ term.

large contours.



crosses pole at $z_j = 0$ $\forall j$.

Both yield nice formulas. The large contour formula is easier, but not as useful as the small one, which we now state.

<u>Prop:</u> Assume $f(x)$ is holomorphic and nonzero in a neighborhood of the real interval containing $\{q^i : 0 \le i \le k\}$ then ~~[illegible]~~ for $N_1, \dots, N_k$

$$\frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \underbrace{\oint \cdots \oint}_{\text{Nested}} \prod_{1 \le A < B \le k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{1}{(1-z_j)^{N_j}} \frac{f(qz_j)}{f(z_j)} \frac{dz_j}{z_j}$$

$$= \sum_{\substack{\lambda \vdash k \\ \lambda = 1^{m_1} 2^{m_2} \cdots}} \frac{(1-q)^k}{m_1! m_2! \cdots} \cdot \frac{1}{(2\pi i)^{\ell(\lambda)}} \underbrace{\oint \cdots \oint}_{\substack{\text{small circle} \\ \text{containing } 1}} \det\left[\frac{1}{w_i q^{\lambda_i} - w_j}\right]_{i,j=1}^{\ell(\lambda)} E(w_1, \dots, q^{\lambda_1 - 1} w_1, w_2, \dots, q^{\lambda_2 - 1} w_2, \dots, w_{\ell(\lambda)}, \dots, q^{\lambda_{\ell(\lambda)} - 1} w_{\ell(\lambda)}) \, dw$$

Where $E(z_1, \dots, z_k) = \prod_{j=1}^{k} \frac{f(qz_j)}{f(z_j)} \cdot \sum_{\sigma \in S_k} \prod_{A > B} \frac{z_{\sigma(A)} - q z_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}} \prod_{j=1}^{k} \frac{1}{(1 - z_{\sigma(j)})^{N_j}}$

If all $N_j \equiv N$ then E simplifies to

$$E(z_1, \dots, z_k) = \prod_{j=1}^{k} \frac{f(qz_j)}{f(z_j)} \cdot \frac{1}{(1-z_j)^N} \cdot \underbrace{\sum_{\sigma \in S_k} \prod_{A > B} \frac{z_{\sigma(A)} - q z_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}}}_{C_k}$$

Notice $C_k = \frac{1}{a_\delta(z_1, \dots, z_k)} \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \prod_{A > B} z_{\sigma(A)} - q z_{\sigma(B)}$ must be a constant (by degree considerations)

In fact $C_k = \frac{(1-q)(1-q^2) \cdots (1-q^k)}{(1-q)(1-q) \cdots (1-q)} =: k_q!$

Exercise: As $q \nearrow 1$, $k_q! \to k!$

Hence we conclude that (equiv for q-TASEP $x_N(\tau)+N$)

$$\mathbb{E}\left[q^{k\lambda_N}\right] = k_q! \sum_{\substack{\lambda \vdash k \\ \lambda = 1^{m_1}2^{m_2}\cdots}} \frac{(1-q)^k}{m_1!m_2!\cdots} \cdot \frac{1}{(2\pi i)^{\ell(\lambda)}} \underbrace{\oint \cdots \oint}_{\substack{\text{small} \\ \text{circle} \\ \text{around } 1}} \det\left[\frac{1}{w_i q^{\lambda_i} - w_j}\right]_{i,j=1}^{\ell(\lambda)} \prod_{j=1}^{\ell(\lambda)} \frac{f(q^{\lambda_j} w_j)}{f(w_j)} \left(\frac{1}{(w_j;q)_{\lambda_j}}\right)^N$$

where $f(z) = e^{\tau z}$ and the final terms can from telescoping and $(a;q)_k = (1-a)(1-qa)\cdots(1-q^{k-1}a)$

We will use the above expression to uncover a Fredholm determinant.

But first, let us sketch the <u>proof of the proposition</u>. First consider the possible residues

<u>Example</u>: <u>$k=2$</u> $\qquad \dfrac{z_1 - z_2}{z_1 - qz_2}$, as z_1 shrinks to z_2 contour we cross pole at $z_1 = qz_2$. Can either pick the residue, or the integral.

Hence our integral decomposes into a double integral and a single integral with $z_1 = qz_2$.

<u>$k=3$</u> $\qquad \dfrac{z_1-z_2}{z_1-qz_2}\,\dfrac{z_1-z_3}{z_1-qz_3}\,\dfrac{z_2-z_3}{z_2-qz_3}$ Shrink z_2. Cross pole at $z_2 = qz_3$

If we pick residue then $\dfrac{z_1-qz_3}{z_1-q^2z_3}\,\dfrac{z_1-z_3}{z_1-qz_3}\cdot qz_3-z_3$

The apparent pole at $z_1 = qz_3$ is actually not present due to numerator

Hence we only have pole at $z_1 = q^2 z_3$.

This shows how we get geometric strings of residues.

If (for general k) we shrink z_k, then z_{k-1}, ... then z_1 we find the integral is equal to a sum of integrals with free integration variables $z_{i_{\lambda_1}}, z_{i_{\lambda_2}}, \ldots$ and all other z's fixed according to the following residue subspaces:

For $\lambda \vdash k$ $\quad z_{i_1} = q z_{i_2} = q^2 z_{i_3} = \cdots q^{\lambda_1 - 1} z_{i_{\lambda_1}}$ with $i_1 < i_2 < \cdots < i_{\lambda_1}$

$$z_{j_1} = q z_{j_2} = q^2 z_{j_3} = \cdots q^{\lambda_2 - 1} z_{j_{\lambda_2}} \text{ with } j_1 < j_2 < \cdots < j_{\lambda_2}$$

$\vdots$

etc.

Then we may reverse order and permute the $\{z_j\}$ such that

$$(z_{i_1}, \ldots z_{i_{\lambda_1}}) \longmapsto (y_{\lambda_1}, y_{\lambda_1 - 1}, \ldots, y_1)$$

$$(z_{j_1}, \ldots, z_{j_{\lambda_2}}) \longmapsto (y_{\lambda_1 + \lambda_2}, \ldots, y_{\lambda_1 + 1})$$

$\vdots$ etc.

There is some possible freedom in choice of permutation coming from clusters of residues with same size. If $\lambda = 1^{m_1} 2^{m_2} \cdots$ then there is a total ~~multiplicity~~ of $m_1! m_2! \cdots$ such permutations which suffice. Hence we can write the sum of all residues that ~~for~~ correspond to a given partition λ as follows: $(*)$

$$(*) = \frac{1}{m_1! m_2! \cdots} \cdot \sum_{\sigma \in S_k} \operatorname*{Res}_{\substack{y_{\lambda_1} = q y_{\lambda_1 - 1} = \cdots = q^{\lambda_1 - 1} y_1 \\ y_{\lambda_1+\lambda_2} = \cdots = q^{\lambda_2 - 1} y_{\lambda_1 + 1} \\ \vdots}} \prod_{1 \le A < B \le k} \frac{y_{\sigma(A)} - y_{\sigma(B)}}{y_{\sigma(A)} - q y_{\sigma(B)}} \prod_{j=1}^{k} \frac{f(q y_j)}{f(y_j)} \frac{1}{(1 - y_{\sigma(j)})^{N_j}} \cdot \frac{1}{y_j}$$

Note $$\prod_{1 \le A < B \le k} \frac{y_{\sigma(A)} - y_{\sigma(B)}}{y_{\sigma(A)} - q y_{\sigma(B)}} = \prod_{A \neq B} \frac{y_A - y_B}{y_A - q y_B} \cdot \prod_{A > B} \frac{y_{\sigma(A)} - q y_{\sigma(B)}}{y_{\sigma(A)} - y_{\sigma(B)}}$$

So, introducing $w_j = y_{\lambda_1 + \cdots + \lambda_{j-1} + 1}$ as the remaining integration variables,

$$(*) = \frac{1}{m_1! m_2! \cdots} \operatorname*{Res}_{\substack{y_{\lambda_1} = \cdots q^{\lambda_1 - 1} w_1 \\ y_{\lambda_1 + \lambda_2} = \cdots = q^{\lambda_2 - 1} w_2 \\ \vdots}} \left(\prod_{A \neq B} \frac{y_A - y_B}{y_A - q y_B} \right) \cdot E(w_1, \ldots q^{\lambda_1 - 1} w_1, w_2, \ldots, q^{\lambda_2 - 1} w_2, \ldots w_{\ell(\lambda)}, \ldots, q^{\lambda_{\ell(\lambda)} - 1} w_{\ell(\lambda)}) \cdot q^{k/2} \prod_{j=1}^{\ell(\lambda)} w_j^{-\lambda_j} q^{-\lambda_j^2/2}$$

It remains to prove that

$$\operatorname{Res} \left(\prod_{A \neq B} \frac{y_A - y_B}{y_A - q y_B} \right) = (-1)^k (1-q)^k q^{-k/2} \prod_{j=1}^{\ell(\lambda)} w_j^{\lambda_j} q^{\lambda_j^2/2} \cdot \det \left(\frac{1}{w_i q^{\lambda_i} - w_j} \right)$$

This relies on a careful calculation and the Cauchy det. identity.

Combining this provides the λ term in the proposition and summing over all $\lambda \vdash k$ yields the proposition $\square$.

Now return to $\mathbb{E}[q^{k\lambda_N}]$, notice that we can write (for $k \geq 1$)

$$\mathbb{E}[q^{k\lambda_N}] = k_q! \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \sum_{\lambda_1=1}^{\infty} \cdots \sum_{\lambda_\ell=1}^{\infty} \mathbb{1}\{\lambda_1+\cdots+\lambda_\ell = k\} \oint_{C_1}\frac{dw_1}{2\pi i}\cdots\oint\frac{dw_\ell}{2\pi i} \det\left[\tilde{K}_q(\lambda_i, w_i; \lambda_j, w_j)\right]_{i,j=1}^{\ell}$$

where $\tilde{K}_\zeta(\lambda, w; \lambda', w') = \zeta^\lambda \dfrac{(1-q)^\lambda e^{(q^\lambda-1)\tau w}}{wq^\lambda - w'}(w;q)_\lambda^{-N}$.

This suggests a generating function $G(\zeta)$

$$G(\zeta) := \sum_{k\geq 0} \frac{\mathbb{E}[q^{k\lambda_N}]\zeta^k}{k_q!} = 1 + \sum_{\ell=1}^{\infty}\frac{1}{\ell!}\sum_{\lambda_1=1}^{\infty}\cdots\sum_{\lambda_\ell=1}^{\infty}\oint\cdots\oint \det\left[K_\zeta(\lambda_i, w_i; \lambda_j, w_j)\right]$$

This is the Fredholm expansion of ~~$\det$~~ $I + \tilde{K}_\zeta$ on $L^2(\mathbb{Z}_{>0} \times C_1)$ but since the kernel is independent of λ' we can take the λ summations inside the determinant so that

$$G(\zeta) = \det(I + K_\zeta)_{L^2(C_1)} = 1 + \sum_{\ell=1}^{\infty}\frac{1}{\ell!}\oint\frac{dw_1}{2\pi i}\cdots\oint\frac{dw_\ell}{2\pi i}\det\left[K_\zeta(w_i, w_j)\right]_{i,j=1}^{\ell}$$

with kernel $K_\zeta(w, w') = \displaystyle\sum_{\lambda=1}^{\infty} \frac{\zeta^\lambda (1-q)^\lambda e^{(q^\lambda-1)\tau w}(w;q)_\lambda^{-N}}{wq^\lambda - w'} = \sum_{\lambda=1}^{\infty} g(q^\lambda)[(1-q)\zeta]^\lambda$

with $g(q^\lambda) = \dfrac{e^{(q^\lambda-1)\tau w}\left[\frac{(q^\lambda w;q)_\infty}{(w;q)_\infty}\right]^N}{wq^\lambda - w'}$ (Note $g(\cdot)$ is analytic away from its poles)

We may finally replace the summation by a contour integral via the identity

$$\sum_{\lambda=1}^{\infty} g(q^{\lambda})\xi^{\lambda} = \frac{1}{2\pi i}\int_{\substack{\text{Contains}\\ 1,2,\ldots}} \frac{\pi}{\sin(-\pi s)}(-\xi)^{s} g(q^{s})\,ds$$

"Mellin-Barnes" type representation

which makes analytic sense for suitable g and choices of contours.

<u>Thm</u>

$$G(\xi) = \sum_{k=0}^{\infty} \frac{\mathbb{E}[q^{k\lambda_N}]\xi^k}{k_q!} = \det(I + K_{\xi})_{L^2(C_1)}$$

$$K_{\xi}(w,w') = \frac{1}{2\pi i}\int_{-i\infty+1/2}^{i\infty+1/2} \frac{\pi}{\sin(-\pi s)}(-(1-q)\xi)^{s} \frac{\left(\frac{(q^s w;q)_\infty}{(w;q)_\infty}\right)^N e^{\tau w(q^s-1)}}{q^s w - w'}\,ds$$

<u>Question</u>: How does this help us compute the distribution of λ_N (eg. $x_{N(t)+N}$)

Note that since $q \in (0,1)$ and $\lambda_N \geq 0$, $q^{\lambda_N} \leq 1$, hence all its moments finite

This means that (by Carleman's condition) the moments of q^{λ_N} determine its distribution, and hence since $G(\xi)$ determines these moments, it does too the dist.

The ~~question~~ remaining task is to find an inverse of the transform from distribution to $G(\xi)$.

In 1949 Hahn introduced two q-deformed exponential functions

$$e_q(x) := \frac{1}{((1-q)x;q)_\infty} \quad , \quad E_q(x) := (-(1-q)x;q)_\infty$$

Exercise: Show pointwise convergence of $e_q(x), E_q(x) \to e^x$ as $q \to 1$.

Focusing on e_q, there is a "Taylor series" which is consequence of q-Binomial thm

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{k_q!}$$

This, along with the fact that $q^{\lambda_N} \leq 1$, implies that for ζ small enough

$$\mathbb{E}\left[e_q(\zeta q^{\lambda_N}) \right] = \sum_{k=0}^{\infty} \frac{\mathbb{E}[q^{k\lambda_N}]\zeta^k}{k_q!} = \det\left[I + K_\zeta\right]_{L^2(C_1)}$$

The left and right are analytic in ζ away from poles, hence identity $\forall \zeta$.

This is a q-Laplace transform wrt spectral variable ζ.

It is completely surprising that this is ~~such~~ a ~~simple~~ Fred. det.

In the Schur/RMT/TW limits this recovers the fact that

the dist. of the top (or bottom) eigenvalue is given by such a Fred. det.

That was a consequence of det. p.p. structure – what has

taken that place?

<u>Def^{n}</u>: For a function $f \in \ell^1(\mathbb{Z}_{\geq 0})$ define for $z \in \mathbb{C}/\{q^{-m}\}_{m\geq 0}$

$$\hat{f}^q(z) := \sum_{n=0}^{\infty} \frac{f(n)}{(zq^n;q)_\infty}$$

<u>Prop</u>: Can recover f from $\hat{f}^q$ via $f(n) = -q^n \frac{1}{2\pi i}\int (q^{n+1}z;q)_\infty \hat{f}^q(z)\,dz$

with z contour containing only $z = q^{-m}$ $0 \leq m \leq n$ poles.

This is the q-deformed Laplace transform and has many nice properties such as linearity; scaling; shift; transformation under q-derivative/integral q-product / convolution (see Gaspard Bangerezako manuscript for uses in solving q-Difference equations)

Hence we have found expression for distribution of λ_N (eq. $x_N(\tau)+N$) in which complexity does not grow with (N, τ, q). In practice we will generally deal with (q)-Laplace transforms since convergence of such transform suffices for weak convergence of distribution, and ultimately this transform converges to a probability (in a certain scaling)

Final note: Had we pursued here the "large contour" residue formula we are also led to a simpler (but less useful) Fred. det.

<u>Thm</u> $$\mathbb{E}\left[e_q(\zeta q^{\lambda_N})\right] = \frac{1}{((1-q)\zeta;q)_\infty} \det(I+\tilde{K}_\zeta)_{L^2(\oplus)}$$

$$\tilde{K}_\zeta(w,w') = \zeta \frac{(1-q)\,(1-w)^{-N} e^{(q-1)\tau w}}{qw'-w}$$

Not as good for asymptotics because separately the det and prefactor ~~do not have~~ has (in the scalings we will take) no clear limit, but rather there must be some intract cancelation which is a priori not obvious.