Determinant point processes

The following discussion will lead to asymptotics.

Schur measure can be mapped onto a point process which has a special property of being determinantal.

- Let $\mathfrak{X}$ be a one particle state space. We will focus on discrete spaces $\mathbb{Z}$ or $\mathbb{N}$ (though with minor modifications $\mathbb{R}$)
- Let X be a (locally finite) collection of points in $\mathfrak{X}$ (i.e. element of $\{0,1\}^{\mathfrak{X}}$).
- A point process is a prob. measure $\mathbb{P}$ on $\{0,1\}^{\mathfrak{X}}$.

<u>Def</u>

Def^n For $A \subseteq \mathfrak{X}$ finite, the <u>correlation function</u> for X is

$$\rho(A) = \mathbb{P}(A \subseteq X).$$

For $|A| = n$, $A = \{x_1, \ldots, x_n\}$, $\rho_n(x_1, \ldots, x_n) := \rho(A)$.

Exercises

(1) For $n \geq 1$ and a cpctly supported bdd. Borel function f on $\mathcal{X}^n$ one has.

$$\int_{\mathcal{X}^n} f \rho_n = \mathbb{E}\left(\sum_{x_{i_1},\dots,x_{i_n} \in X} f(x_{i_1},\dots,x_{i_n}) \right)$$

2) Show that a point process on a discrete set X is uniquely determined by its correlation functions

3) For $\mathcal{X} = \mathbb{R}$ the correlation f^n's ρ_n are measured relative to Lebesgue. Show that a poisson point process with intensity 1 has $\rho_n \equiv 1$ for all n.

Def^n: A point process on a discrete space $\mathcal{X}$ is determinantal if there exists $K: \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ such that $\rho_n(x_1, \dots, x_n) = \det\left[K(x_i, x_j)\right]_{i,j=1}^{n}$ for all $n \geq 1$ and $x_i \in \mathcal{X}$. The function K is called the correlation kernel.

— A substantial reduction in the ammount of data to describe it —

- History (briefly)

↳ Early 1960's Dyson in RMT

↳ 1975 Macchi considered $K(x,y) = \overline{K(y,x)}$ "Fermionic process"

"Repel": $\rho_2(x,y) = \rho_1(x)\rho_1(y) - |K(x,y)|^2 \leq \rho_1(x)\rho_1(y)$

↳ Det. point process terminology Borodin-Olshanski '00

↳ Many sources known
- RMT / Schur processes
- Dimers on bipartite graphs.
- Unif span tree
- Non-intersecting paths
- Zeros of GAF

Schur measure is determinantal

Define $X(\lambda)_i = \lambda_i - i + 1/2 \subseteq \mathbb{Z} + 1/2$.

Theorem (Okounkov '01)

Suppose $\lambda \in \mathbb{Y}$ distributed as $S_{\rho_1;\rho_2}$ (ρ_1, ρ_2 Schur pos.)

Then $X(\lambda)$ is a determinant point process on $\mathbb{Z}+1/2$

with correlation kernel $K(i,j)$ defined via generating series

$$\sum_{i,j \in \mathbb{Z}+1/2} K(i,j) v^i w^{-j} = \frac{H(\rho_1;v)\,H(\rho_2;w^{-1})}{H(\rho_2;v^{-1})\,H(\rho_1;w)} \sum_{k=1/2,3/2,\dots} \left(\frac{w}{v}\right)^k .$$

Where recall for $\rho = (\alpha;\beta;\gamma)$: $H(\rho;z) = e^{\gamma z} \prod_{i \geq 1} \frac{1+\beta_i z}{1-\alpha_i z}$

1) Proof in Borodin-Gorin notes. based on fact that $s_\lambda = \frac{\det(\,)}{\det(\,)}$.

2) Earlier work: Johansson, Borodin-Olshanski-Okounkov.

3) Above is a formal power series identity

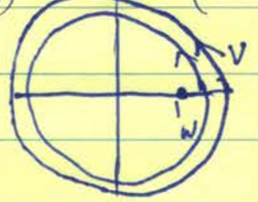
4) Under suitable conditions can treat as analytic identity and invert via contour integral to get $K(i,j)$.

Let us apply to the case related to TASEP where

$$\rho_1 = ((\overbrace{1,\dots,1}^{N});0;0), \quad \rho_2 = (0;0;\tau), \quad \tau \geq 0.$$

then $H(\rho_1;z) = \left(\frac{1}{1-z}\right)^N$ $\quad H(\rho_2;z) = e^{\tau z}$ so

$$K(i,j) = \frac{1}{(2\pi i)^2} \oint\oint \left(\frac{1-w}{1-v}\right)^N \frac{e^{\tau w^{-1}}}{e^{\tau v^{-1}}} \frac{\sqrt{vw}}{v-w} \frac{dv\,dw}{v^{i+1} w^{-j+1}}$$



- From K how can we extract distribution of λ_N (or equivalently $X_N(\tau)$)?

Note: for $m \in \mathbb{Z}$, $\mathbb{P}(X_N(\lambda) > m) = \mathbb{P}(X_N(\lambda) \cap [-\infty, m] = \emptyset)$

(see Borodin's RMT book chapter, page 3)

From inclusion/exclusion, for $I \subseteq \mathcal{X}$

$$\mathbb{P}(X \cap I = \emptyset) = \det(I - K_I)_{L^2(\mathcal{X})}$$

where $K_I = \chi_I K \chi_I$ is K restricted to $L^2(I)$ and

$$\det(I-K)_{L^2(\mathcal{X}, d\mu)} := 1 + \sum_{n \geq 1} \frac{(-1)^n}{n!} \int_{\mathcal{X}} d\mu(x_1) \cdots \int_{\mathcal{X}} d\mu(x_n) \det[K(x_i, x_j)]_{i,j=1}^{n}$$

"Fredholm determinant"

From such a formula, we can take asymptotics to prove the claimed TASEP limit theorem. Instead, we will first degenerate to the GUE_n measure and take asymptotics there. Same in spirit but a little easier.

Lets consider the $GUE_N(1/2)$ ensemble with measure $\frac{1}{Z}\prod_{i<j}^{N}|x_i-x_j|^2\prod_{i=1}^{N}e^{-x_i^2}dx_i$. By ~~these~~ scaling we want to show largest/smallest eigenvalue near $\pm\sqrt{2N}$ with $N^{-1/6}$ fluctuations

We will, infact, study the limiting behavior of the correlation function $K(x,y): \mathbb{R}\times\mathbb{R}\to\mathbb{C}$ in three scaling regimes

By asymptotics of $K(i,j)$ (or similar derivation)

$$K(x,y)=\frac{e^{x^2-y^2}}{2(\pi i)^2}\oint_{\Gamma_0}\int_{-i\infty}^{i\infty}\left(\frac{w}{v}\right)^N\frac{e^{w^2-2wy}}{e^{v^2-2vx}}\frac{dv\,dw}{w-v}$$

technical, to make trace class kernel. Doesn't change kernel.

Idea of steepest descent

$$\int e^{Nf(z)}\,dz$$

- deform contour so $\mathrm{Im}\, f(z)$ is constant.
- by C-R-eqs $\mathrm{Re}\, f(z)$ will be changing most rapidly along this curve
- Find critical point (i.e. maximum) along path and then the equation localizes its value to $e^{Nf(z_{max})}$.
- Higher order corrections depend on order of the critical point
- In practice one need not always find exact steepest descent contours, just need to be able to bound integrand away from critical point.

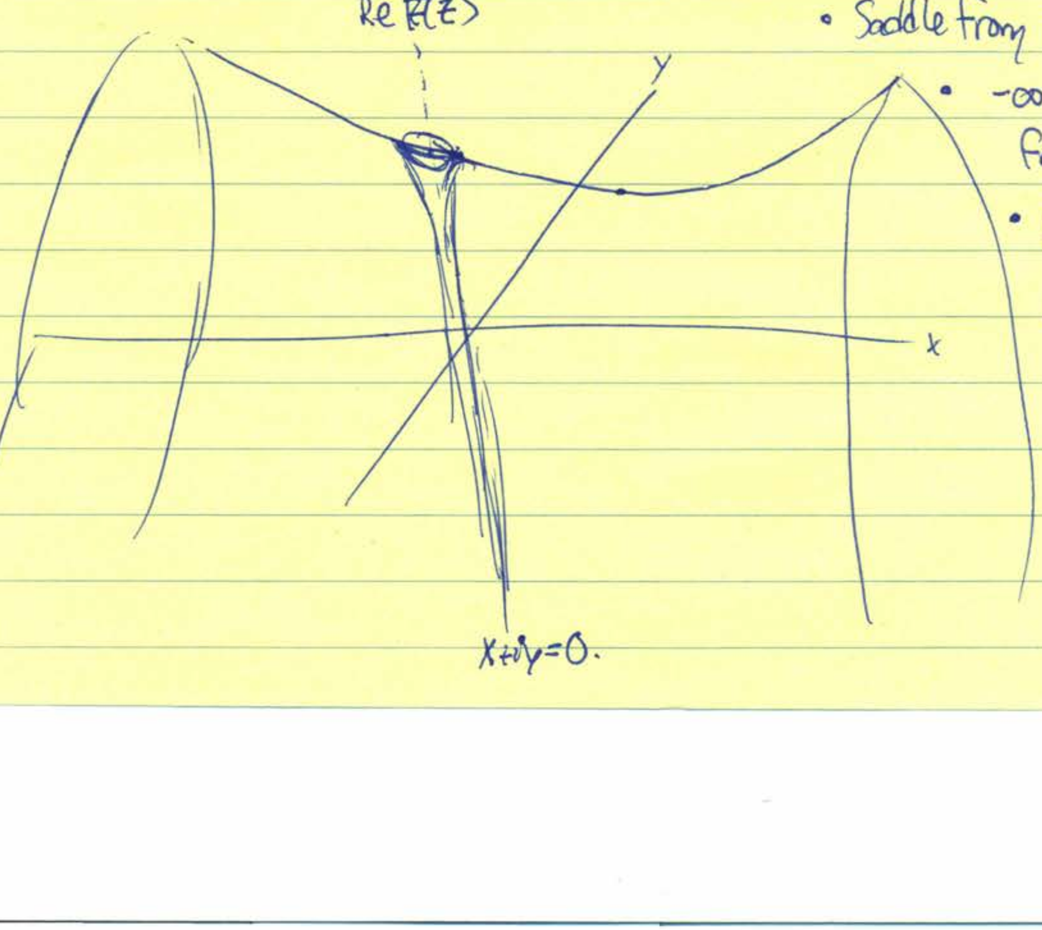
$N \to \infty$ asymptotics.

Consider $N \log w + w^2 - 2wy$. We are interested in $y \sim \sqrt{N}\, Y$. So set $w = \sqrt{N}\,\tilde{w}$, $v = \sqrt{N}\,\tilde{v}$ and $\cancel{Y} = \sqrt{N}\,Y - \tilde{y}$, $x = \sqrt{N}\cancel{X} - \tilde{x}$. Then the kernel becomes

$$\frac{\sqrt{N}}{2(\pi i)^2} \oint_{\Gamma_0} \int_{-i\infty}^{i\infty} \frac{e^{NF(\tilde{w})}}{e^{NF(\tilde{v})}} \frac{e^{2\sqrt{N}\,\tilde{w}\tilde{y}}}{e^{2\sqrt{N}\,\tilde{v}\tilde{x}}} \frac{d\tilde{v}\,d\tilde{w}}{\tilde{w}-\tilde{v}}$$

with $F(z) = \log z + z^2 - 2zY$. We want to find a set of contours along which $F(\tilde{w})$ and $F(\tilde{v})$ are not oscillatory, and on which $\mathrm{Re}\,F(\tilde{w})$ is minimized and $\mathrm{Re}\,F(\tilde{v})$ maximized.

Plot of $\mathrm{Re}\,F(z)$ $(z = x+iy)$ looks like



- Saddle from z^2
- $-\infty$ at $z=0$ from $\log z$
- Relative position from Y.

Saddle point is where $\frac{\partial \operatorname{Re} F}{\partial \operatorname{Re} z} = \frac{\partial \operatorname{Re} F}{\partial \operatorname{Im} z} = 0$

which by Cauchy-Riemann is when $F'(z) = \frac{1}{z} + 2z - 2Y = 0$.

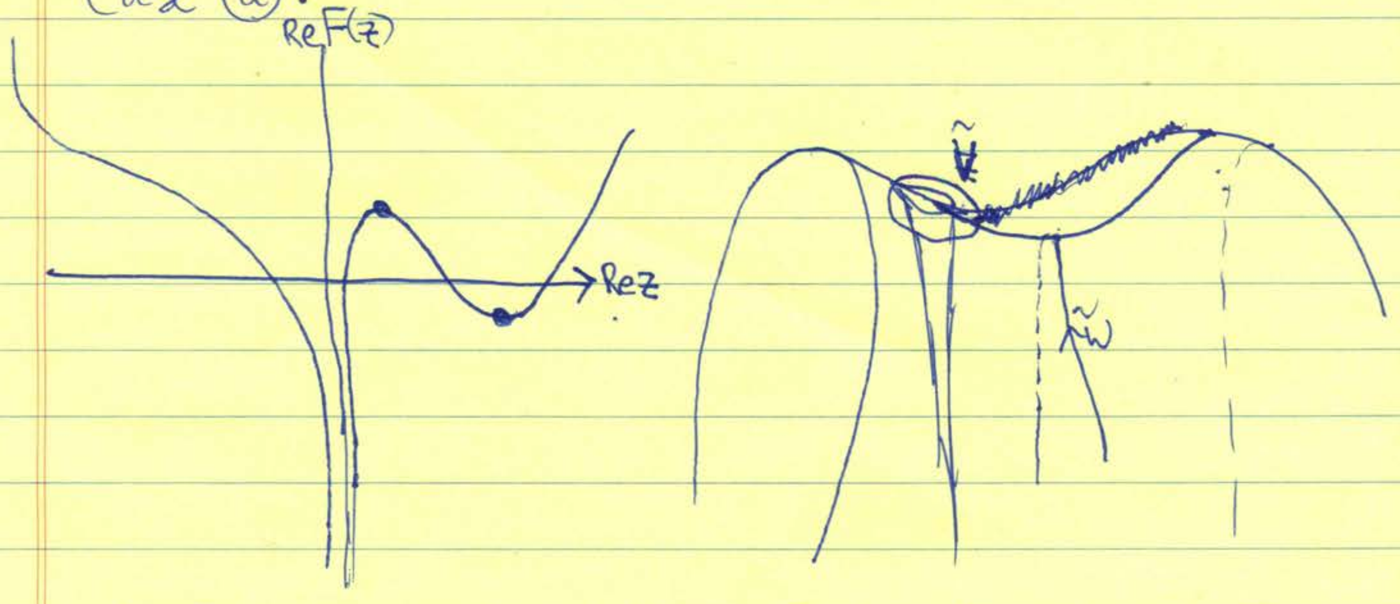
i.e. $z = \frac{2Y \pm \sqrt{4(Y^2-2)}}{4}$

(a) If $|Y| > \sqrt{2}$ then two real, positive roots

(b) If $|Y| = \sqrt{2}$ double roots at $\frac{1}{\sqrt{2}}$.

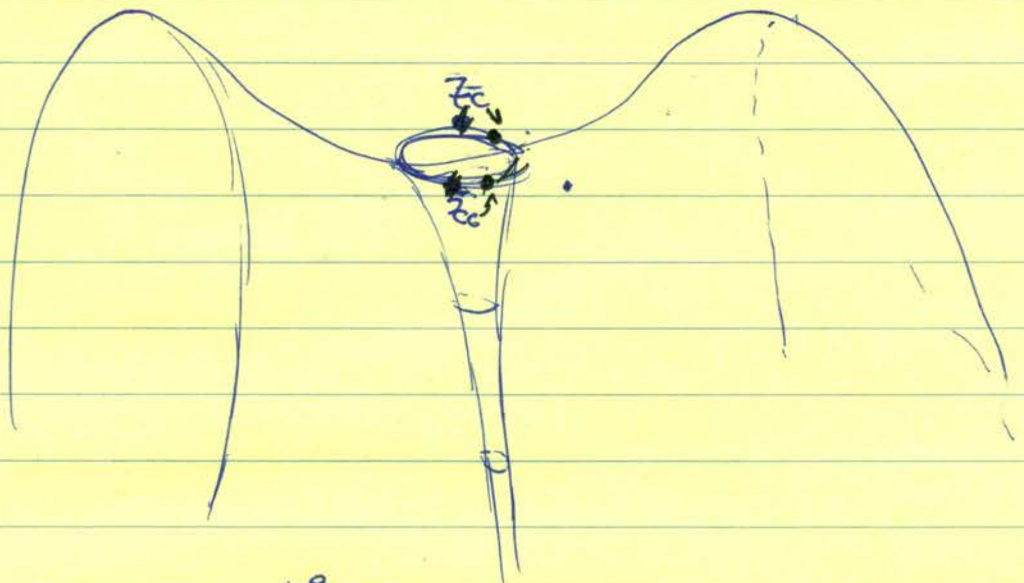
(c) If $|Y| < \sqrt{2}$ Complex conjugate roots.

Case (a):

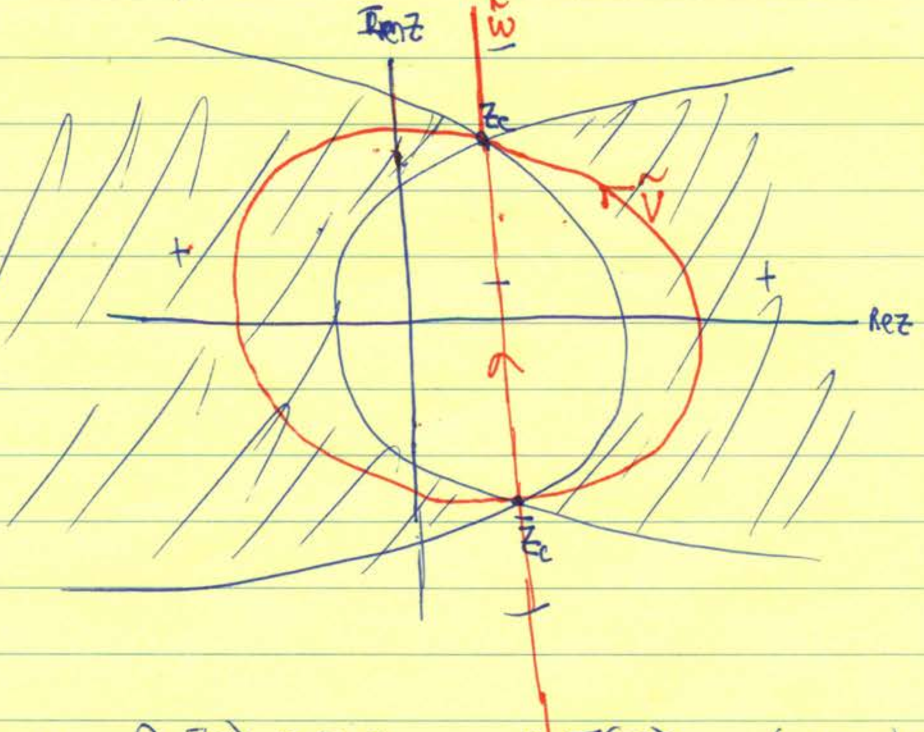


Can choose $\tilde{v}$ and $\tilde{w}$ contours so that $\operatorname{Re}(\tilde{v}) > \operatorname{Re} \tilde{w}$ on whole contour. Hence integral goes to zero as $N \to \infty$ exponentially fast!

Case (c): Critical pts $z_c, \bar{z}_c = \dfrac{\gamma \pm \sqrt{\gamma^2 - 2}}{2}$.



The curves where $\mathrm{Re}\,F = \mathrm{Re}\,F(z_c)$ have form.



Since $\mathrm{Re}\,F(\tilde{v}) \geq \mathrm{Re}\,F(z_c) \geq \mathrm{Re}\,F(\tilde{w})$ (with strict inequality away from $z_c, \bar{z}_c$) the integral over these contours goes to zero.

But, in deforming from original contours to these, we crossed poles at $\hat{w}-\bar{v}=0$ and picked up a residue equal to

$$\frac{1}{\pi i}\int_{\bar{z}_c}^{z_c} e^{2z(x-y)\sqrt{N}}\,dz = \frac{\sin\left(2\,\mathrm{Im}\,z_c\,(x-y)\sqrt{N}\right)}{\pi(x-y)\sqrt{N}}.$$

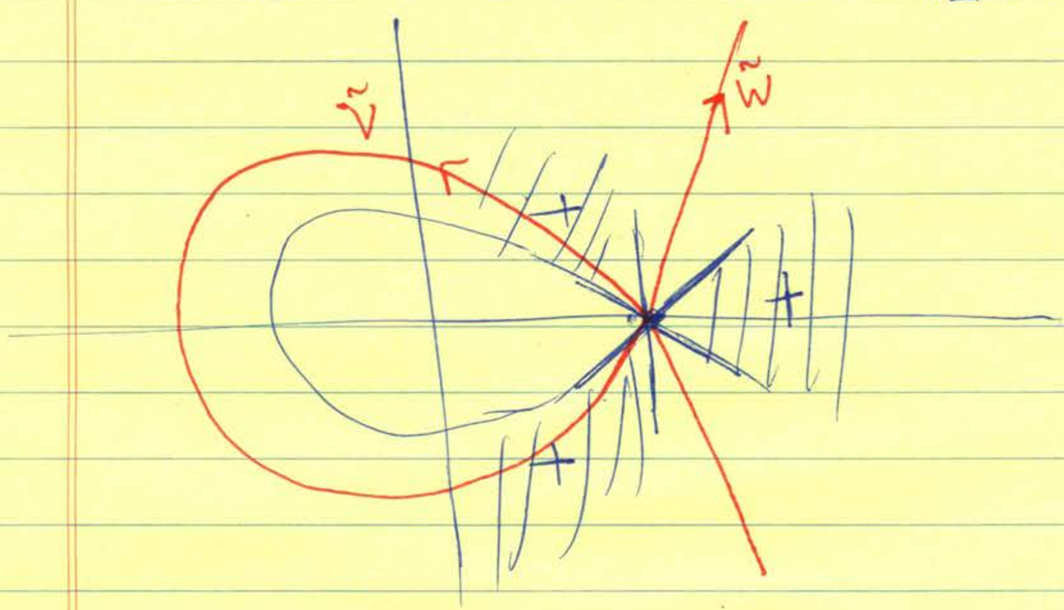
Thus

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}K_N\left(\sqrt{N}Y+\frac{\tilde{x}}{\sqrt{N}},\sqrt{N}Y+\frac{\tilde{y}}{\sqrt{N}}\right) = \frac{\sin\left((2\,\mathrm{Im}\,z_c)(\tilde{x}-\tilde{y})\right)}{\pi(\tilde{x}-\tilde{y})}.$$

Where $2\,\mathrm{Im}(z_c) = \sqrt{2-Y^2}$.

This is Dyson's Sine kernel and $\sqrt{2-Y^2}$ is the Wigner semi-circle density.

Case (b): $Y = \sqrt{2}$ then $z_c = \bar{z}_c = \frac{1}{\sqrt{2}}$. Consider $\operatorname{Re} F(z)$ near z_c



so near z_c $\quad F(z) = \frac{2\sqrt{2}}{3}(z - z_c)^3 + \cdots$

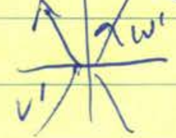
Note away from z_c, $\operatorname{Re} F(\tilde{v}) > \operatorname{Re} F(z_c) > \operatorname{Re} F(\tilde{w})$

Localize integral near z_c. Scale $\tilde{w} = z_c + N^{-1/3}\frac{w'}{\sqrt{2}}$

$\tilde{v} = z_c + N^{-1/3}\frac{v'}{\sqrt{2}}$

and $\tilde{x} = N^{-1/6}\frac{x'}{\sqrt{2}}$, $\tilde{y} = N^{-1/6}\frac{y'}{\sqrt{2}}$ then (up to guage transform)

$$K_N(x,y) \approx N^{1/6}\sqrt{2}\,\frac{1}{(2\pi i)^2}\iint \frac{e^{\frac{(w')^3}{3} - w'y'}}{e^{\frac{(v')^3}{3} - v'x'}}\,\frac{dv'\,dw'}{w'-v'}$$



So

$$\lim_{N\to\infty} \frac{K_N\left(\sqrt{2N}+\frac{x}{\sqrt{2}N^{1/6}},\ \sqrt{2N}+\frac{y}{\sqrt{2}N^{1/6}}\right)}{\sqrt{2}N^{1/6}} = \mathbb{A}(x,y)$$

where

$$\mathbb{A}(x,y) = \left(\frac{1}{2\pi i}\right)^2 \iint \frac{e^{w^3/3 - wy}}{e^{v^3/3 - vx}} \frac{dv\,dw}{(w-v)}$$

or equiv

$$= \frac{Ai(x)\,Ai'(y) - Ai'(x)\,Ai(y)}{x-y}$$

with $Ai(s) = \frac{1}{2\pi i}\int e^{z^3/3 - zs}\,dz$ the Airy function

(equiv has $Ai''(s) = s\,Ai(s)$, $Ai(s) \to 0$ exp fast as $s\to\infty$)

or equiv.

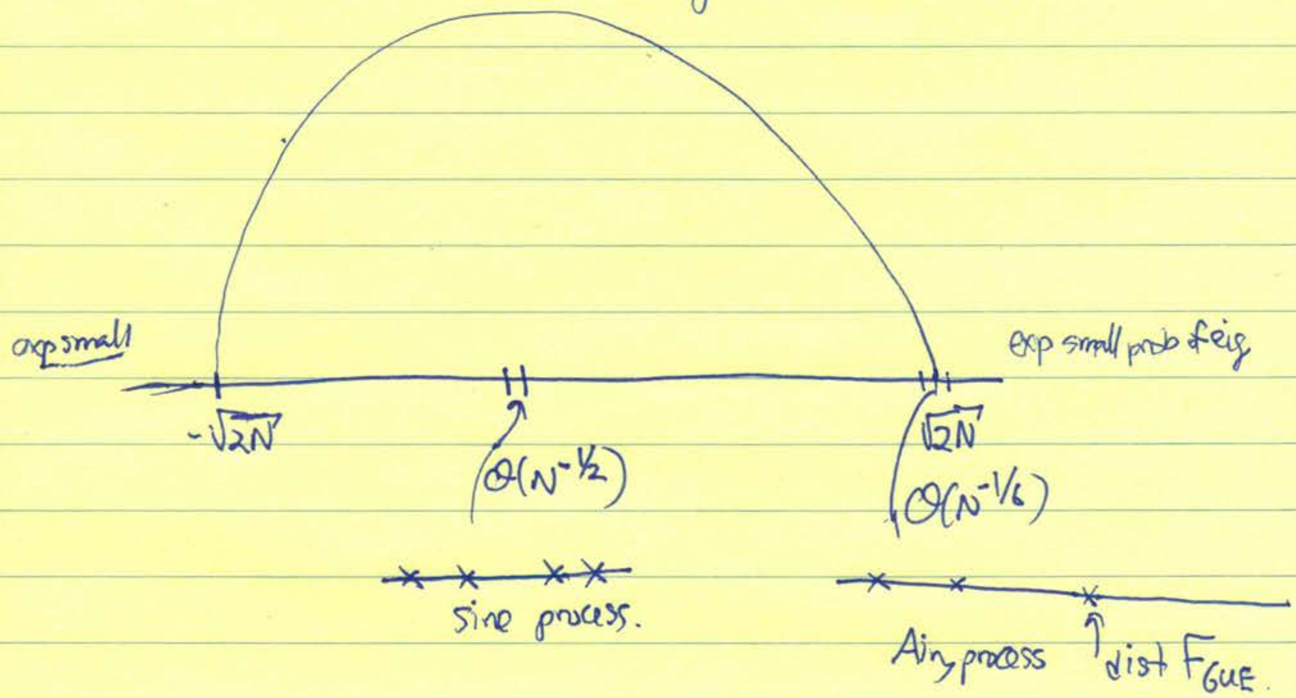
$$= \int_{-\infty}^{0} Ai(x-r)\,Ai(y-r)\,dr$$

Exercise: Show equivalence of them all.

Note that

$$F_{GUE}(s) = \det(I - K_{\mathbb{A}})_{L^2(s,\infty)} \quad \square.$$

To summarize: Plot eigenvalues of $GUE_N(1/2)$



Schur process

Okounkov - Reshetikhin '01

<u>Def</u> Schur process of rank N is a probability measure on seq of Young diagrams $\lambda^{(1)}, \mu^{(1)}, \lambda^{(2)}, \mu^{(2)}, \dots, \mu^{(N-1)}, \lambda^{(N)}$ parameterized by $2N$ Schur-positive specializations $\rho_0^+, \rho_1^-, \rho_1^+, \rho_2^-, \rho_2^+, \dots, \rho_N^-$ and given by

$$\mathbb{P}\left(\lambda^{(1)}, \mu^{(1)}, \lambda^{(2)}, \dots, \lambda^{(N)}\right) = \frac{1}{Z} s_{\lambda^{(1)}}(\rho_0^+)\, s_{\lambda^{(1)}/\mu^{(1)}}(\rho_0^-)\, s_{\lambda^{(2)}/\mu^{(1)}}(\rho_1^+)$$

$$s_{\lambda^{(2)}/\mu^{(2)}}(\rho_1^-) \cdots\cdots s_{\lambda^{(N)}/\mu^{(N-1)}}(\rho_N^+)\, s_{\lambda^{(N)}}(\rho_N^-)$$

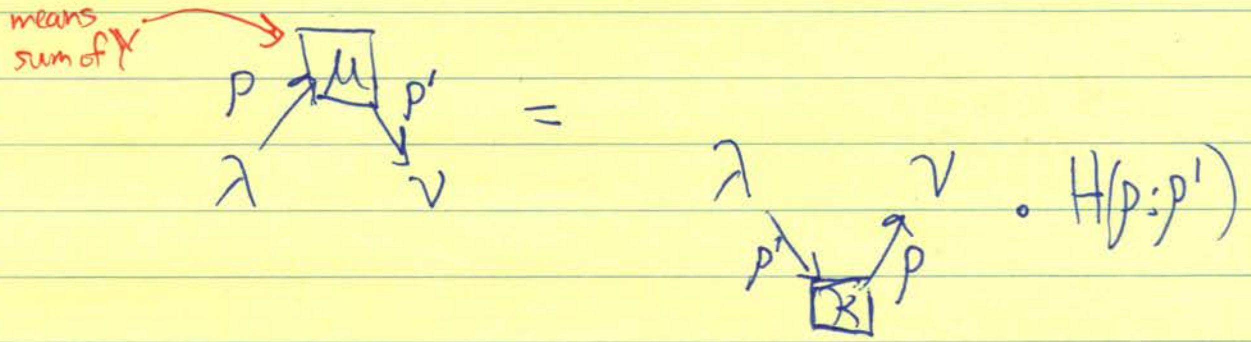
Definition implies $\lambda^{(1)} \supset \mu^{(1)} \subset \lambda^{(2)} \supset \mu^{(2)} \subset \cdots \supset \mu^{(N-1)} \subset \lambda^{(N)}$

We pictorally illustrate this as

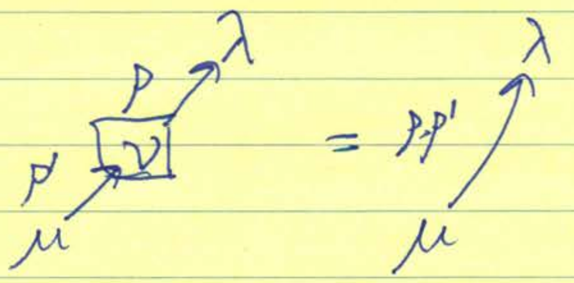
$$\begin{tikzcd}[column sep=small]
 & \lambda^{(1)} \arrow[dr, "\rho_1^-"] & & \lambda^{(2)} \arrow[dr, "\rho_2^-"] & \cdots\cdots & & \lambda^{(N)} \arrow[dr, "\rho_N^-"] & \\
\emptyset \arrow[ur, "\rho_0^+"] & & \mu^{(1)} \arrow[ur, "\rho_1^+"] & & \mu^{(2)} & \mu^{(N-1)} \arrow[ur, "\rho_{N-1}^+"] & & \emptyset
\end{tikzcd}$$

We will use Skew Cauchy, Chapmn-Kolmogorov at length.

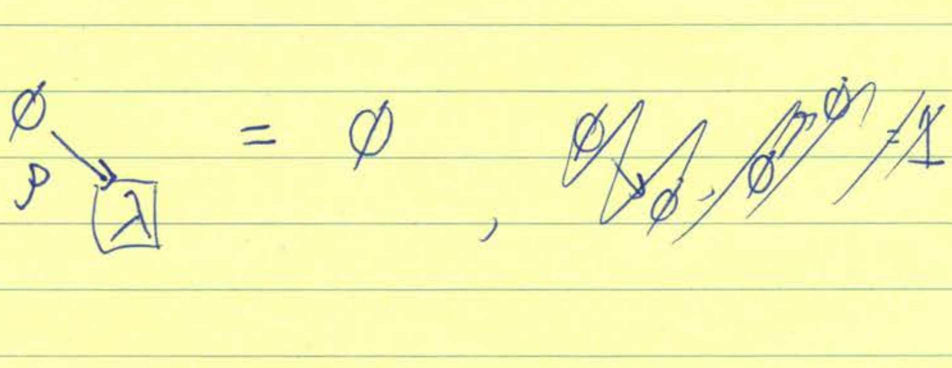
<u>Skew Cauchy</u> $\sum_{\mu\in\mathbb{Y}} S_{\mu/\lambda}(\rho)\, S_{\mu/\nu}(\rho') = H(\rho;\rho') \sum_{\kappa} S_{\lambda/\kappa}(\rho')\, S_{\nu/\kappa}(\rho)$



<u>Chapmn-Kolmogorov:</u> $\sum_{\nu\in\mathbb{Y}} S_{\lambda/\nu}(\rho)\, S_{\nu/\mu}(\rho') = S_{\lambda/\mu}(\rho,\rho')$



<u>Zero rule.</u> $S_{\emptyset/\lambda} = \begin{cases} 1 & \lambda=\emptyset \\ 0 & \text{else} \end{cases}$



Claim: The normalizing constant Z is Schur process is

$$Z = \prod_{i<j} H(\rho_i^+ ; \rho_j^-)$$

Pf:

$$\begin{tikzcd}[column sep=small] & \boxed{\lambda^{(1)}} \arrow[dr, "\rho_1^-"] & & \lambda^{(2)} \arrow[dr, "\rho_2^-"] & \\ \emptyset \arrow[ur, "\rho_0^+"] & & \mu^{(1)} \arrow[ur, "\rho_1^+"] & & \emptyset \end{tikzcd}$$

$$= \begin{tikzcd}[column sep=small] & & & \lambda^{(2)} \arrow[dr, "\rho_2^-"] & \\ \emptyset \arrow[dr, "\rho_1^-"'] & & \mu^{(1)} \arrow[ur, "\rho_1^+"] & & \emptyset \\ & \lambda^{(1)} \arrow[ur, "\rho_0^+"'] & & & \end{tikzcd} \cdot H(\rho_0^+ ; \rho_1^-)$$

$$= \begin{tikzcd}[column sep=small] & & \lambda^{(2)} \arrow[dr, "\rho_2^-"] & \\ & \boxed{\mu^{(1)}} \arrow[ur, "\rho_1^+"] & & \emptyset \\ \emptyset \arrow[ur, "\rho_0^+"'] & & & \end{tikzcd} \cdot H(\rho_0^+ ; \rho_1^-)$$

$$= \begin{tikzcd}[column sep=small] & \boxed{\lambda^{(2)}} \arrow[dr, "\rho_2^-"] & \\ \emptyset \arrow[ur, "{\rho_0^+, \rho_1^+}"] & & \emptyset \end{tikzcd} \cdot H(\rho_0^+ ; \rho_1^-)$$

$$= \begin{tikzcd}[column sep=small] \emptyset \arrow[dr, "\rho_2^-"'] & & \emptyset \\ & \boxed{\lambda^{(2)}} \arrow[ur, "{\rho_0^+, \rho_1^+}"'] & \end{tikzcd} \cdot H(\rho_0^+ ; \rho_1^-) \cdot H(\rho_0^+, \rho_1^+ ; \rho_2^-)$$

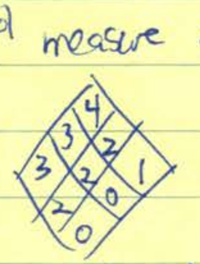
$$= \emptyset \to \emptyset = 1 \cdot H(\rho_0^+ ; \rho_1^-)\, H(\rho_0^+ ; \rho_2^-)\, H(\rho_1^+ ; \rho_2^-)$$

$\square$.

Note from defn: $H(\rho_1, \ldots \rho_k ; \rho_1', \ldots \rho_m') = \prod_{i=1}^{k} \prod_{j=1}^{m} H(\rho_i ; \rho_j')$

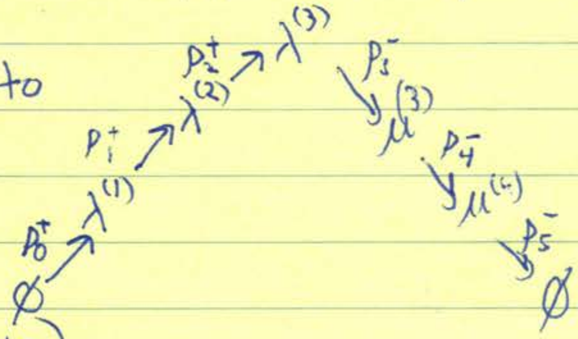
- In a similar way, one shows that the projection of Schur process to $\lambda^{(k)}$ is Schur measure $\mathbb{S}_{\rho_1;\rho_2}$ with $\rho_1=(\rho_0^+,\rho_1^+,\dots,\rho_{k-1}^+)$, $\rho_2=(\rho_k^-,\rho_{k+1}^-,\dots,\rho_N^-)$
- Just like Schur measure, Schur process gives rise to a determinant point process. ~~The~~

<u>Exercise</u> (Boxed) Plane partitions with q^{vol} measure are Schur processes.

A boxed plane partition $\Pi =$  such that 

$\mathrm{vol}\,\Pi = \Sigma$ entries. , $0<q<1$. Project this measure onto a seq^n of partitions $\emptyset, (3), (3,2), (4,2,0), (2,0), (1), \emptyset$.

<u>Claim</u>: This is distributed according to



Where all missing spec. are 0 and.

$\rho_j^+ = ((q^{-j});0,0)$; $\rho_j^- = ((q^j);0;0)$

Hint: recall that for $\alpha_1=c$ and all else in ρ, 0, $s_{\lambda/\mu}=\begin{cases} c^{|\lambda|-|\mu|} & \text{hor. strip} \\ 0 & 0 \end{cases}$

<u>Def$^{\underline{n}}$</u> : Ascending Schur process has form.

$$\emptyset \overset{\rho_0^+}{\nearrow} \lambda^{(1)} \overset{0}{\searrow} \lambda^{(1)} \overset{\rho_1^+}{\nearrow} \lambda^{(2)} \overset{0}{\searrow} \lambda^{(2)} \cdots\cdots \lambda^{(N-1)} \overset{\rho_{N-1}^+}{\nearrow} \lambda^{(N)} \overset{\rho_N^-}{\searrow} \emptyset \qquad \text{or} \qquad \emptyset \overset{\rho_0^+}{\nearrow} \lambda^{(1)} \overset{\rho_1^+}{\nearrow} \lambda^{(2)} \cdots \lambda^{(N)} \overset{\rho_N^{-1}}{\downarrow} \emptyset$$

Can write as $0 \subseteq \lambda^{(1)} \subseteq \lambda^{(2)} \subseteq \cdots \subseteq \lambda^{(N)}$,

and is specified by $\rho_0^+, \rho_1^+, \ldots, \rho_{N-1}^+, \rho_N^-$ according to the measure

$$\frac{S_{\lambda^{(1)}}(\rho_0^+)\, S_{\lambda^{(2)}/\lambda^{(1)}}(\rho_1^+) \cdots S_{\lambda^{(N)}/\lambda^{(N-1)}}(\rho_{N-1}^+)\, S_{\lambda^{(N)}}(\rho_N^-)}{H(\rho_0^+, \rho_1^+, \ldots, \rho_{N-1}^+ ; \rho_N^-)}$$

It is not hard to come up with Markov operators (stocastic matrices) that act well on Schur measure and ultimately which act well on Schur process

Def^{n}: For ρ, ρ' Schur pos. specializations with $H(\rho;\rho')<\infty$ define matrices $P^{\uparrow}_{\lambda\to\mu}(\rho;\rho')$, $P^{\downarrow}_{\lambda\to\mu}(\rho;\rho')$ with rows and column indexed by Young diagrams λ and μ as follows

- $$P^{\uparrow}_{\lambda\to\mu}(\rho;\rho') := \frac{S_\mu(\rho)}{S_\lambda(\rho)} \cdot \frac{S_{\mu/\lambda}(\rho')}{H(\rho;\rho')}$$

- $$P^{\downarrow}_{\lambda\to\mu}(\rho;\rho') := \frac{S_\mu(\rho)}{S_\lambda(\rho,\rho')} S_{\lambda/\mu}(\rho')$$

Prop $P^{\uparrow}_{\lambda\to\mu}$ and $P^{\downarrow}_{\lambda\to\mu}$ are stochastic (i.e. non-neg and for all $\lambda\in\mathbb{Y}$ (1) $\sum_\mu P^{\uparrow}_{\lambda\to\mu}(\rho;\rho') = 1$

(2) $\sum_\mu P^{\downarrow}_{\lambda\to\mu}(\rho;\rho') = 1$)

Pf: Non-neg follows from Schur-pos. spec.
(1) from skew Cauchy (2) from Chap-Kol.

Thus $P^{\uparrow}_{\lambda\to\mu}$, $P^{\downarrow}_{\lambda\to\mu}$ can be viewed as transition prob. for Markov Chains.

where $P^{\uparrow}_{\lambda\to\mu} = 0$ unless $\mu\supset\lambda$ "increases"

$P^{\downarrow}_{\lambda\to\mu} = 0$ unless $\mu\subset\lambda$ "decreases"

These transition matrices preserve class of Schur measures.

Prop For any $\mu \in \mathbb{Y}$

(1) $\sum_{\lambda \in \mathbb{Y}} \mathbb{S}_{\rho_1;\rho_2}(\lambda)\, p^{\uparrow}_{\lambda\to\mu}(\rho_2;\rho_3) = \mathbb{S}_{\rho_1,\rho_3;\rho_2}(\mu)$

(2) $\sum_{\lambda \in \mathbb{Y}} \mathbb{S}_{\rho_1;\rho_2,\rho_3}(\lambda)\, p^{\downarrow}_{\lambda\to\mu}(\rho_2;\rho_3) = \mathbb{S}_{\rho_1;\rho_2}(\mu)$

Pf (1) By Chap. Kol (2) By Skew Cauchy.

Example In the limit discussed earlier to GUE_N here is what happens.

- $p^{\uparrow}_{\lambda\to\mu}$ becomes $\frac{V(y)}{V(x)}\Delta$ where Δ dirichlet Laplacian on $x_1 > x_2 > \dots > x_N$ and $V(x) = \prod_{i<j}(x_i - x_j)$

  This is the generator of a stoch. process called Dyson BM which maps $GUE_N(t) \to GUE_N(t+dt)$

- $p^{\downarrow}_{\lambda\to\mu}$ becomes indicator function that $x_1 > y_1 > x_2 > y_2 \cdots > y_{N-1} > x_N$ and maps $GUE_N(t) \to GUE_{N-1}(t)$.

- The ascending Schur process becomes the GUE minor process which is measure of eigenvalues of $N\times N$, $N-1\times N-1, \dots, 1\times 1$ principle minors.

In general a Schur process can be written as a trajectory of a Markov Chain involving $P^\uparrow$, $P^\downarrow$ with particular specializations.

Ascending Schur process $\emptyset \subset \lambda^{(1)} \subset \cdots \subseteq \lambda^{(N)}$ with $\rho_0^+, \ldots, \rho_{N-1}^+, \rho_N^-$ spec.

equals $\mathbb{S}_{\rho_0^+, \ldots, \rho_{N-1}^+; \rho_N^-}(\lambda^{(N)}) P^\downarrow_{\lambda^{(N)} \to \lambda^{(N-1)}}(\rho_0^+, \ldots, \rho_{N-2}^+; \rho_{N-1}^+) \cdots P^\downarrow_{\lambda^{(2)} \to \lambda^{(1)}}(\rho_0^+; \rho_1^+)$

[**Exercise**: Figure out how this works in general.

Prop: $P^\uparrow(\rho_1, \rho_2; \rho_3) P^\downarrow(\rho_1; \rho_2) = P^\downarrow(\rho_1; \rho_2) P^\uparrow(\rho_1; \rho_3)$

This means that on Schur processes

$$\mathbb{S}_{\rho_4; \rho_1, \rho_2} P^\uparrow(\rho_1, \rho_2; \rho_3) = \mathbb{S}_{\rho_3, \rho_4; \rho_1, \rho_2}$$

$$\mathbb{S}_{\rho_3, \rho_4; \rho_1, \rho_2} P^\downarrow(\rho_1; \rho_2) = \mathbb{S}_{\rho_3, \rho_4; \rho_1}$$

and

$$\mathbb{S}_{\rho_4; \rho_1, \rho_2} P^\downarrow(\rho_1; \rho_2) = \mathbb{S}_{\rho_4; \rho_1}$$

$$\mathbb{S}_{\rho_4; \rho_1} P^\uparrow(\rho_1; \rho_3) = \mathbb{S}_{\rho_3, \rho_4; \rho_1}$$

We will use this commutation relation to build dynamics preserving Schur processes.

A general construction of multivariate Markov Chains

(or an exact sampling method for certain types of Gibbs measures)

- Let $(S_1, \dots, S_n)$ n-tuple of discrete countable sets
- $P_k : S_k \times S_k \to [0,1]$ s.t. $\sum_{y \in S_k} P_k(x,y) = 1 \quad \forall x \in S_k$
- $\Lambda^k_{k-1} : S_k \times S_{k-1} \to [0,1]$ s.t. $\sum_{y \in S_{k-1}} \Lambda^k_{k-1}(x,y) = 1 \quad \forall x \in S_k$.

Assume $\Delta^k_{k-1} := \Lambda^k_{k-1} P_{k-1} = P_k \Lambda^k_{k-1}$

$$\begin{tikzcd} S_k \arrow[r, "P_k"] \arrow[d, "\Lambda^k_{k-1}"'] \arrow[dr, phantom, "\circlearrowleft"] & S_k \arrow[d, "\Lambda^k_{k-1}"] \\ S_{k-1} \arrow[r, "P_{k-1}"'] & S_{k-1} \end{tikzcd}$$

Then we can define a multivariate Markov chain with transition matrix $P^{(n)}$

and state space $S^{(n)} = \{ (x_1, \dots, x_n) \in S_1 \times \dots S_n : \prod_{k=2}^{n} \Lambda^k_{k-1}(x_k, x_{k-1}) \neq 0 \}$

via $(\underline{X} = (x_1, \dots x_n), \underline{Y} = (y_1, \dots, y_n))$

$$P^{(n)}(\underline{X}, \underline{Y}) = P_1(x_1, y_1) \prod_{k=2}^{n} \frac{P_k(x_k, y_k) \Lambda^k_{k-1}(y_k, y_{k-1})}{\Delta^k_{k-1}(x_k, y_{k-1})}$$

if $\prod_{k=2}^{n} \Delta^k_{k-1}(x_k, y_{k-1}) > 0$ and 0 otherwise.

This means that we sequentially update $\underline{X}$ starting at x_1.

We choose y_1 according to $P_1(x_1, y_1)$, then

choose y_2 according to $\dfrac{P_2(x_2,y_2)\,\Lambda_1^2(y_2,y_1)}{\Delta_1^2(x_2,y_1)}$

which is conditional dist. of application of P_2 the Λ_1^2 conditioned

on starting at x_2 and finishing at y_1., and repeat.

$$\begin{tikzcd} x_2 \arrow[r, dashed] \arrow[dr] & y_2 \arrow[d] \\ x_1 \arrow[r] & y_1 \end{tikzcd}$$

Prop Let m_n be prob measure on S_n and define

$$m^{(n)}(\underline{X}) = m_n(x_n)\,\Lambda_{n-1}^{n}(x_n, x_{n-1}) \cdots \Lambda_1^2(x_2, x_1)$$

Set $\tilde{m}_n = m_n P_n$ and

$$\tilde{m}^{(n)}(\underline{X}) = \tilde{m}_n(x_n)\,\Lambda_{n-1}^{n}(x_n, x_{n-1}) \cdots \Lambda_1^2(x_2, x_1)$$

Then $m^{(n)} P^{(n)} = \tilde{m}^{(n)}$.

Pf: Repeated application of Compatibility relation.

Remark: The marginal of these dynamics on level k

is P_k (i.e. all of the P_k chains are linked together)

Let us apply this construction with

$$S_i = \mathbb{Y}, \qquad P_k(\lambda,\mu) = p^{\uparrow}_{\lambda\to\mu}(\rho_1^+,\dots,\rho_k^+;\rho^-)$$

$$\Lambda^k_{k-1}(\lambda,\mu) = p^{\downarrow}_{\lambda\to\mu}(\rho_1^+,\dots,\rho_{k-1}^+;\rho_k^+)$$

$$S^{(n)} = \{(\lambda^{(1)}\subset\lambda^{(2)}\subset\cdots\subset\lambda^{(n)}) \mid \prod_{k=2}^{n}\Lambda^k_{k-1}(\lambda^{(k)},\lambda^{(k-1)}) \neq 0\}.$$

and $m_n(\lambda^{(n)}) = \mathbb{S}_{\rho_1^+,\dots,\rho_N^+;\rho^-}(\lambda^{(n)})$.

Then as we saw before $m^{(n)}(\lambda^{(1)}\subset\cdots\subset\lambda^{(n)})$ = Schur process $\emptyset \xrightarrow{\rho_1^+} \lambda^{(1)} \cdots \xrightarrow{\rho_n^+} \lambda^{(n)} \xrightarrow{\rho^-} \emptyset$

The previous construction implies that the Markov dynamics on $\lambda^{(1)}\subset\cdots\subset\lambda^{(n)}$ takes the Schur process to $\emptyset \xrightarrow{\rho_1^+} \lambda^{(1)} \cdots \xrightarrow{\rho_n^+} \lambda^{(n)} \xrightarrow{\rho^-,\rho} \emptyset$.

↳ Hence, if we start with an ascending Schur process, we will have (marginally) another Schur process after any number of steps, but with new parameters in the specialization ρ^-.

There are several natural/simple examples of such dynamics. We consider 1

- Take $\rho_k^+ = ((1);0;0)$ $\forall k$ and $\rho' = (0;(b);0)$
- Consider the discrete time Markov chain $\lambda(t) = (\lambda^{(1)}(t) \subset \lambda^{(2)}(t) \subset \cdots \subset \lambda^{(n)}(t))$

  with ρ^- initially 0, hence $\lambda(0) = (\emptyset \subset \cdots \subset \emptyset)$.
- After time t, the marginal on $\lambda^{(k)}(t)$ is $\mathbb{S}_{\underbrace{\rho^+_{1}\cdots\rho^+_{k}}_{((1,\dots,1);0;0)} ; (0;(\overbrace{b,\dots,b}^{t});0)}$,

  and hence supported on diagrams of at most k non-empty rows.
- The condition $\Lambda^k_{k-1}(\lambda^{(k)},\lambda^{(k-1)}) > 0$ ~~iff~~ implies $S_{\lambda^{(k)}/\lambda^{(k-1)}}((1);0;0) > 0$

  which is only true if $\lambda^{(k)} = \lambda^{(k-1)}$ + horizontal strip. This means

$$\lambda^{(k)}_k \le \lambda^{(k-1)}_{k-1} \le \lambda^{(k)}_{k-1} \le \lambda^{(k-1)}_{k-2} \le \cdots \quad \lambda^{(k)}_2 \le \lambda^{(k-1)}_1 \le \lambda^{(k)}_1$$

and is written as $\lambda^{(k-1)} \preceq \lambda^{(k)}$ meaning they interlace.

- ~~$\lambda^{(1)}$ has a single row, hence is one number (as a partition).~~

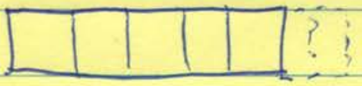
$$\cancel{P_{\lambda^{(1)}\to\mu^{(1)}}(\rho_i;\rho') = H(\rho_i;\rho')\cdot\begin{cases} b & \mu=\lambda+1 \\ 1 & \mu=\lambda \end{cases}}$$

• Consider evolution of $\lambda^{(1)}(t)$, which is one rowed, hence a single number.

$$P^{\uparrow}_{\lambda^{(1)}\to\mu^{(1)}}(p_1^+; p') = \frac{S_{\mu^{(1)}}((1);0;0)}{S_{\lambda^{(1)}}((1);0;0)} \cdot \frac{S_{\mu^{(1)}/\lambda^{(1)}}(0;(b);0)}{H(((1);0;0);(0;(b);0))}$$

$S_{\mu/\lambda}(0;(b);0) > 0$ if μ/λ vertical strip

$$= \frac{1}{1+b} \cdot \begin{cases} b & \mu^{(1)}=\lambda^{(1)}+1 \\ 1 & \mu^{(1)}=\lambda^{(1)} \end{cases}$$



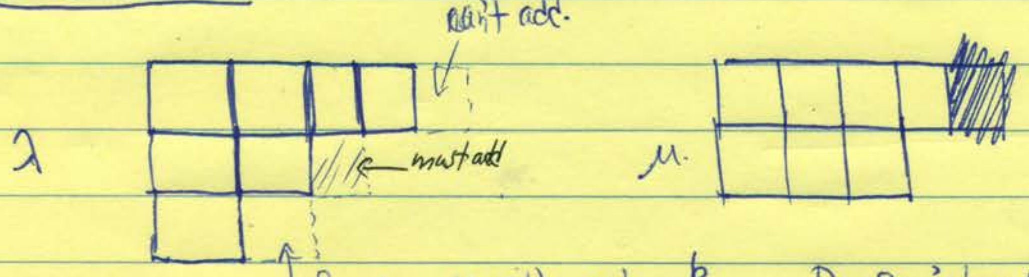
So $\lambda^{(1)}(t)$ ~~performs~~ ~~random~~ increases by 1 with prob $\frac{b}{1+b}$, and otherwise stays same.

• How does $\lambda^{(k)}(t+1)$ evolve given $\lambda^{(k)}(t)$ and $\lambda^{(k-1)}(t+1)$

Call $\lambda^{(k)}(t+1)=\nu$, $\lambda^{(k)}(t)=\lambda$, $\lambda^{(k-1)}(t+1)=\mu$, then

$$\mathbb{P}(\nu \mid \lambda,\mu) = \frac{S_{\nu/\lambda}(0;b;0)\, S_{\nu/\mu}(1;0;0)}{\sum_{\eta} S_{\eta/\lambda}(0;b;0)\, S_{\eta/\mu}(1;0;0)}$$

Three cases



free to add with prob $\frac{b}{1+b}$. , Do so independently.

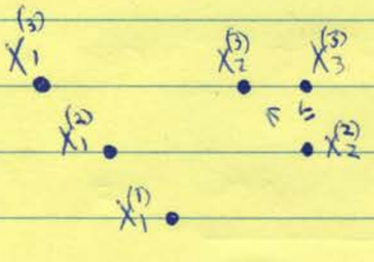
must have $\mu \preceq \nu$ and ν/λ vertical strip.

In order to better visualize these dynamics define

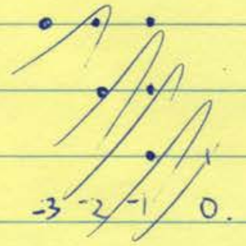
$$X_i^{(j)} = \lambda_{j+1-i}^{(j)} - N + i \qquad \text{(i.e. reorder coordinates and make strictly increasing)}$$

The interlacing rule becomes $X_{i-1}^{(j)} < X_{i-1}^{(j-1)} \le X_i^{(j)}$

Gelfand - Tsetlin Pattern



initial ~~data is~~



Dynamics • Start at level 1 update

$$X_1^{(1)}(t+1) = X_1^{(1)}(t) + \begin{cases} 1 & \text{prob } \frac{b}{1+b} \\ 0 & \text{prob } \frac{1}{1+b} \end{cases}$$

• For level k, if $X_i^{(k)}(t) = X_{i-1}^{(k-1)}(t+1) - 1$ then X_i^k is pushed by $X_{i-1}^{(k-1)}$ so that $X_i^{(k)}(t+1) = X_i^{(k)}(t) + 1$

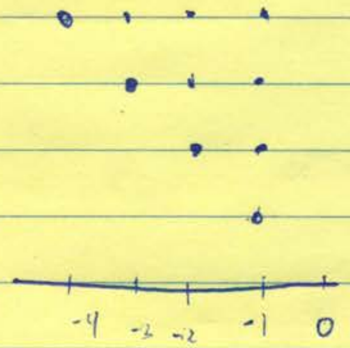
• For level k, if $X_i^{(k-1)}(t+1) = X_i^k(t)+1$, particle $X_i^{(k)}$ is blocked by $X_i^{(k-1)}$ and $X_i^{(k)}(t+1) = X_i^{(k)}(t)$

• Otherwise $X_i^{(k)}(t+1) = X_i^{(k)}(t) + \begin{cases} 1 & \text{prob } \frac{b}{1+b} \\ 0 & \text{prob } \frac{1}{1+b} \end{cases}$

Show simulation

If we start with $\rho^- = 0$ then the measure is delta on $\emptyset \subset \emptyset \subset \cdots \subset \emptyset$

or



Running one step is like taking

$$\rho^- \mapsto \left(\rho^-, (0;(b);0)\right)$$

So that after time t

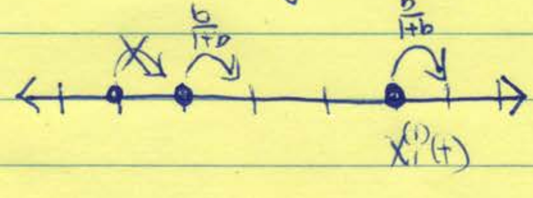
$$\lambda(t) = \left(\lambda^{(1)} \preceq \lambda^{(2)} \preceq \cdots \preceq \lambda^{(n)}\right)$$ is distributed

as a Schur process with $\rho_1^+ = \cdots = \rho_n^+ = 1$ and $\rho^-_{\emptyset} = (0;\underbrace{(b,\ldots,b)}_{t};0)$

and $\lambda^{(n)}$ is marginally distributed as

$$S_{(\underbrace{1,\ldots,1}_{n};0;0);\,(0;\underbrace{(b,\ldots,b)}_{t};0)}\left(\lambda^{(n)}\right)$$

• Observe that the left edge $x_1^{(1)}(t), x_1^{(2)}(t), \ldots, x_1^{(n)}(t)$ is marginally a Markov process with right to left subsequential update



Close to TASEP, but discrete time.

We can take a continuous time (Poisson) limit of these dynamics

- Let $b = \varepsilon$, $t = \varepsilon^{-1}\tau$ then the edge dynamics becomes TASEP
- The limit of $\rho' = (0; \underbrace{(\varepsilon, \dots, \varepsilon)}_{\varepsilon^{-1}\tau}; 0)$ is $(0; 0; \tau)$
- This proves that if $\lambda^{(n)}$ distributed according to $\mathbb{S}_{(\underbrace{1,\dots,1}_{n};0;0);(0;0;\tau)}$ then $\lambda^{(n)}_n - n$ is distributed as then n^{th} particle left in TASEP with $x_1^{(k)}(0) = -k$ (step initial data)

  Can then apply det. point process technology to get marginal distributions and take limit (analog of GUE limit)
- Full 2^d dynamics have limit in which each particle ~~has~~ at ind't rate 1 jumps right unless blocked from below, and particles push those above it

  (higher particles ~~are~~ defer to lower particles)

Finally, can take a diffusive scaling limit.

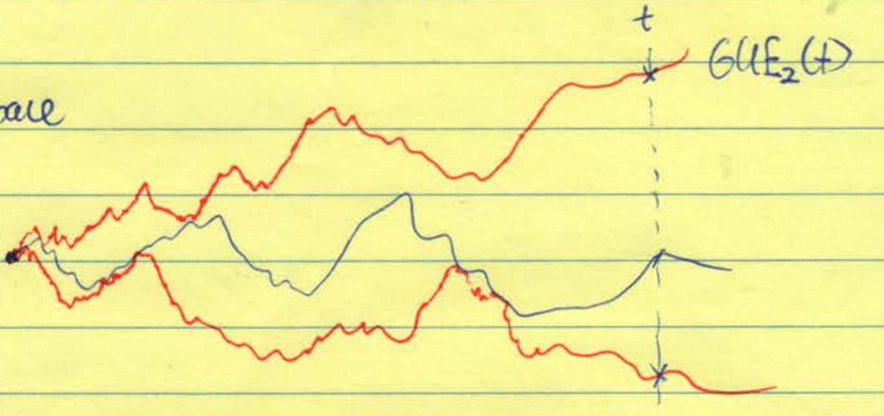
$$y_j^{(k)}(t) = \lim_{\varepsilon \to 0} \varepsilon^{1/2}\left(x_j^{(k)}(\varepsilon t) - \varepsilon^{-1} t \right)$$

We saw before that $y^{(k)}$ distributed as $GUE_k(t)$. The dynamics become the following:

$y_1^{(1)}(t)$ performs a Brownian motion.

$y_j^{(k)}(t)$ performs a Brownian motion reflected off $y_j^{(k-1)}(t)$, $y_{j-1}^{(k-1)}(t)$.

- The ensemble at time t is GUE minor distributed.
- The $\{y_1^{(k)}(t)\}_{k=1}^{n}$ is a ctns space TASEP with BM's reflected off the one to the right.



This is called Warren's process.

When we move onto Macdonald processes we will go through a similar degeneration, though with additional parameters which give access to many new, and interesting prob. models.