Integrable probability:
Macdonald processes, quantum integrable systems and the Kardar-Parisi-Zhang universality class

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A physicist's guide to solving the Kardar-Parisi-Zhang equation

KPZ: $\frac{\partial h}{\partial t} = \frac{1}{2}\frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x}\right)^2 + \dot{W}$ SHE: $\frac{\partial Z}{\partial t} = \frac{1}{2}\frac{\partial^2 Z}{\partial x^2} + \dot{W}\cdot Z$

1. Think of the Cole-Hopf transform instead: $Z = e^h$ solves the SHE

2. Look at the moments $\langle Z(t,x_1)\cdots Z(t,x_k)\rangle$. They are solutions of the quantum delta Bose gas evolution [Kardar '87], [Molchanov '87].

$$\frac{\partial}{\partial t}\langle Z(t,x_1)\cdots Z(t,x_k)\rangle = \frac{1}{2}\left(\sum_{i=1}^{k}\frac{\partial^2}{\partial x_i^2} + \sum_{i\neq j}\delta(x_i - x_j)\right)\langle Z(t,x_1)\cdots Z(t,x_k)\rangle$$

3. Use Bethe ansatz to solve it [Bethe '31], [Lieb-Liniger '63], [McGuire '64], [Yang '67], [Oxford '79] [Heckman-Opdam '97]

4. Reconstruct solution using the known moments: The replica trick. [Calabrese-Le Doussal-Rosso '10+], [Dotsenko '10+]

Possible mathematician's interpretation. Be wise - discretize!

1. Start with a good discrete system that converges to KPZ.

2. Find 'moments' that would solve an integrable autonomous system of equations.

3. Reduce to direct sum of 1-dim equations and 2-body boundary conditions and use Bethe ansatz to solve, for arbitrary initial data

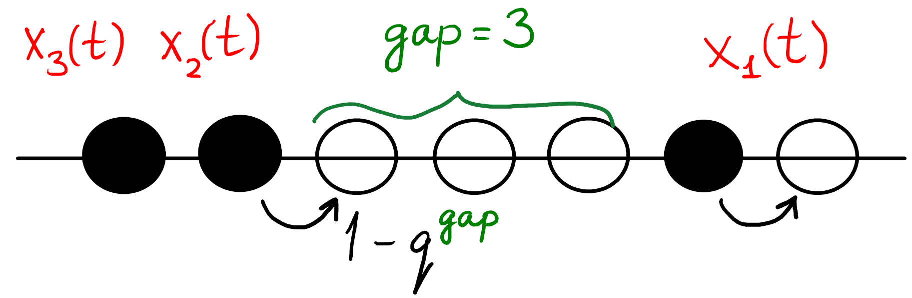
4. Reconstruct the solution using the known 'moments' and take the limit to KPZ/SHE. A mathematically rigorous replica trick.

We can do 1-3 for a few systems: q-TASEP, ASEP, q-Hahn TASEP, higher-spin vertex models

So far we can do 4 only for very special initial conditions.

q-TASEP [Borodin-C '11]

Particles jump right by one according to exponential clocks of rate $1-q^{gap}$.

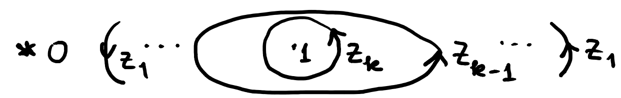


Theorem [B-C '11], [B-C-Sasamoto '12], [B-C-Gorin-Shakirov '13]

For the q-TASEP with step initial data $\{x_n(0)=-n\}_{n\geq 1}$

$$\mathbb{E}\left[q^{(x_{n_1}(t)+n_1)+\dots+(x_{n_k}(t)+n_k)}\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k}\oint\cdots\oint \prod_{A<B}\frac{z_A-z_B}{z_A-qz_B}\prod_{j=1}^{k} \frac{e^{(q-1)tz_j}}{(1-z_j)^{n_j}}\frac{dz_j}{z_j}$$

$(n_1\geq n_2\geq \dots \geq n_k)$



Eventually yields one-point Fredholm determinant and KPZ asymptotics [B-C '11], [B-C-Ferrari '12], [Ferrari-Veto '13].

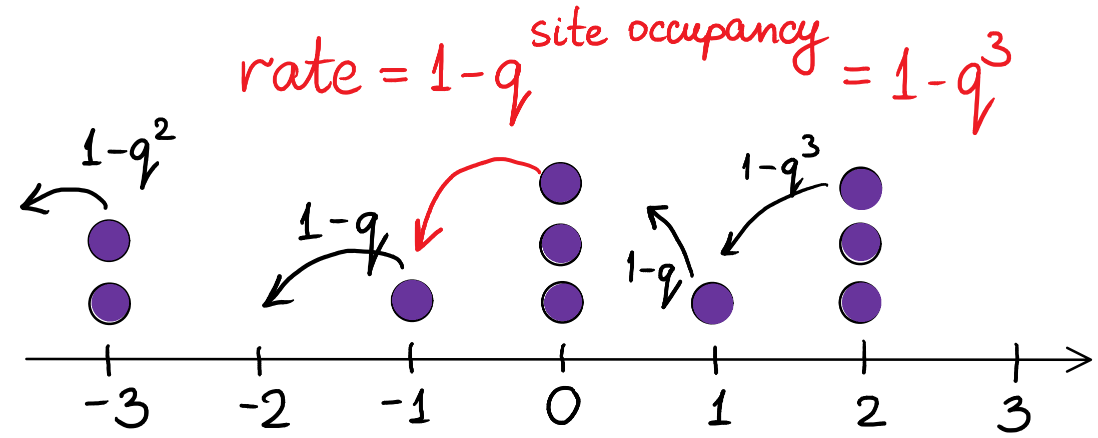
The original proof involved Macdonald processes. A simpler one?

q-Boson process [Sasamoto-Wadati '98]

Top particles at each location jump to the left by one indep. with rates $1-q^{\#\text{ of particles at the site}}$.

The generator is $\left(\vec{n}_j^- = (\cdots, n_j - 1, \cdots)\right)$

$$(Hf)(\vec{n}) = \sum_{\text{clusters } i} \left(1-q^{c_i}\right)\left(f\left(\vec{n}^-_{c_1+\dots+c_i}\right) - f(\vec{n})\right)$$



particles at $\vec{n} = (2,2,2,1,0,0,0,-1,-3,-3)$

clusters $\vec{c} = (3,1,3,1,2)$

Proposition [Borodin-C-Sasamoto '12] For q-TASEP with finitely many particles on the right, $f(t,\vec{n}) = \mathbb{E}\left[\prod_{j=1}^{k} q^{x_{n_j}(t)+n_j}\right]$ is the unique solution of

$$\frac{d}{dt} f(t,\vec{n}) = (Hf)(t,\vec{n}), \qquad f(0,\vec{n}) = \mathbb{E}\left[\prod_{j=1}^{k} q^{x_{n_j}(0)+n_j}\right].$$

q-TASEP and q-Boson particle system are dual (as Markov processes).

The q-Boson system is a discretization of the delta Bose gas.

Coordinate integrability of the q-Boson system

The generator of k free (distant) particles is

$$(\mathcal{L}u)(\vec{n}) = (1-q)\sum_{i=1}^{k}(\nabla_i u)(\vec{n}),$$

∇_i is $(\nabla f)(x) = f(x-1) - f(x)$ acting in n_i

Define the boundary conditions as

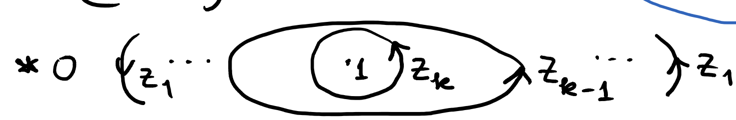
$$(\nabla_i - q\nabla_{i+1})u\Big|_{n_i = n_{i+1}} = 0 \qquad \text{for all } 1 \le i \le k-1$$

Proposition [Borodin-C-Sasamoto '12] If $u : \mathbb{Z}^k \times \mathbb{R}_{\ge 0} \to \mathbb{C}$ satisfies the free evolution equation $\frac{d}{dt}u = \mathcal{L}u$ and boundary conditions, then its restriction to $\{n_1 \geq \ldots \geq n_k\}$ satisfies the q-Boson process evolution equation $\frac{d}{dt}u = Hu$.

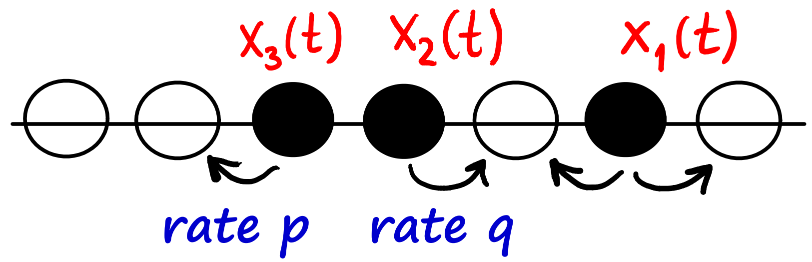
This suffices to re-prove the nested integral formula

$$\mathbb{E}\left[q^{(x_{n_1}(t)+n_1)+\ldots+(x_{n_k}(t)+n_k)}\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k}\oint\cdots\oint \prod_{A<B}\frac{z_A - z_B}{z_A - q z_B}\prod_{j=1}^{k}\frac{e^{(q-1)tz_j}}{(1-z_j)^{n_j}}\frac{dz_j}{z_j}$$

free evolution (the factor $\frac{e^{(q-1)tz_j}}{(1-z_j)^{n_j}}$); initial data (the factor $\frac{1}{z_j}$ in $\frac{dz_j}{z_j}$); boundary conditions (the factor $\frac{z_A - z_B}{z_A - q z_B}$)



The ASEP story (briefly)



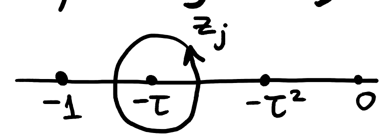
$0<p<q<1, \quad p+q=1$

Set $\tau = p/q < 1, \quad n_y(t) = \#\{m \geq 1 : x_m(t) \geq y\}, \quad Q_y = \dfrac{\tau^{n_y} - \tau^{n_{y-1}}}{\tau - 1}.$

Theorem [B-C-Sasamoto, '12] For ASEP with step initial data $\{x_n(0) = -n\}_{n\geq 1}$

$$\mathbb{E}\left[Q_{y_1}(t)\cdots Q_{y_k}(t)\right] = \frac{\tau^{k(k-1)/2}}{(2\pi i)^k} \cdot \oint \cdots \oint \prod_{A<B} \frac{z_A - z_B}{z_A - \tau z_B} \prod_{j=1}^{k} e^{-\frac{z_j (p-q)^2 t}{(1+z_j)(p+q z_j)}} \left(\frac{1+z_j/\tau}{1+z_j}\right)^{y_j+1} \frac{dz_j}{\tau + z_j}$$

$(y_1 > y_2 > \cdots > y_k)$



ASEP is self-dual [Schutz '97], and integrable in coordinate and algebraic sense.

[Tracy-Widom '08+] used Bethe ansatz to study ASEP's transition probabilities and prove Fredholm determinants and KPZ asymptotics.

q-Hahn distribution

$$\varphi_{q,\mu,\nu}(j|m) := \mu^j \frac{(\nu/\mu;q)_j(\mu;q)_{m-j}}{(\nu;q)_m} \binom{m}{j}_q \mathbb{1}_{j\in\{0,\dots,m\}}$$

$q\in(0,1)$, $0\le\nu\le\mu\le1$, $m>0$

$(a;q)_n := \prod_{i=0}^{n-1}(1-aq^i)$, $\binom{m}{j}_q := \frac{(q;q)_m}{(q;q)_{m-j}(q;q)_j}$

Related to weight for q-Hahn orthogonal polynomials.

At q->1 becomes binomial distribution (other interesting limits).

Limit of Macdonald polynomial binomial formula [Okounkov '97]

Lemma [Povolotsky '13] Let A,B satisfy $AB = \alpha AA + \beta BA + \gamma BB$ then,

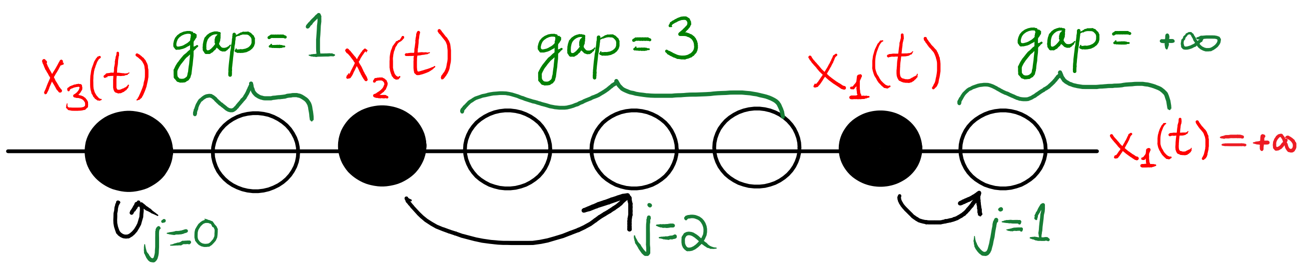
for $p = \frac{\mu-\nu}{1-\nu}$, $\alpha = \frac{\nu(1-q)}{1-q\nu}$, $\beta = \frac{q-\nu}{1-q\nu}$, $\gamma = \frac{1-q}{1-q\nu}$

$$(pA + (1-p)B)^m = \sum_{j=0}^{m} \varphi_{q,\mu,\nu}(j|m) A^j B^{m-j}$$

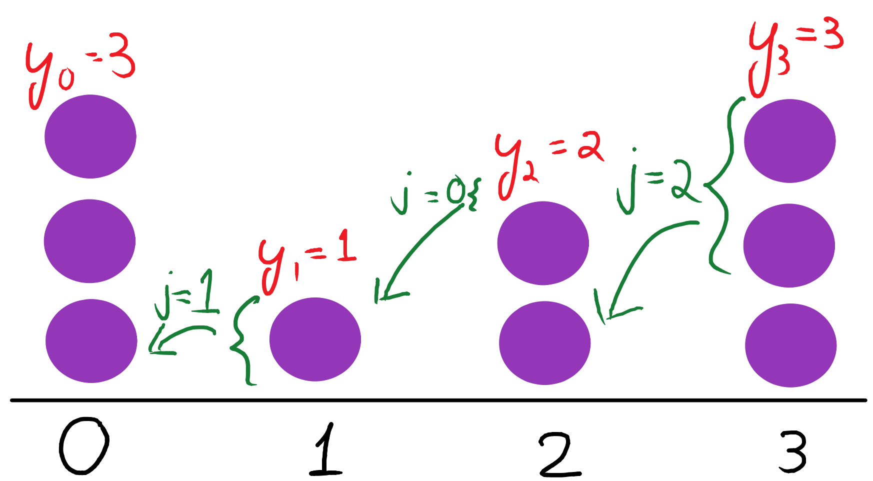
q-Hahn TASEP [C '14] and Boson process [Povolotsky '13]

q-Hahn distribution: $\varphi_{q,\mu,\nu}(j|m) := \mu^j \frac{(\nu/\mu;q)_j (\mu;q)_{m-j}}{(\nu;q)_m} \binom{m}{j}_q \mathbb{1}_{j\in\{0,\dots,m\}}$

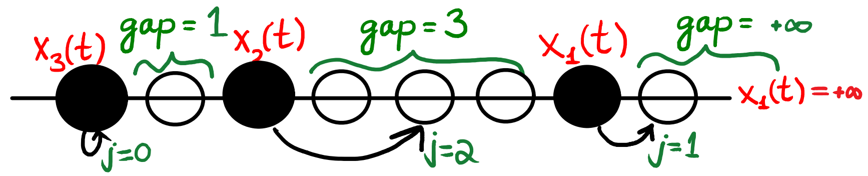
q-Hahn TASEP: In parallel jump by j according to $\varphi_{q,\mu,\nu}(j|gap)$.



q-Hahn Boson process: In parallel move j particles according to $\varphi_{q,\mu,\nu}(j|y)$.



q-Hahn TASEP degenerations



- $\nu = 0$ -> discrete time geometric q-TASEP [Borodin-C '13].
- $\nu = 0$, $\mu = \varepsilon$, $t = \varepsilon^{-1}\tau$, $\varepsilon \searrow 0$ -> continuous time q-TASEP.
- $\nu = \frac{q-\varepsilon}{1-\varepsilon}$, $\mu = q$, $t = \varepsilon^{-1}\tau$, $\varepsilon \searrow 0$ -> multiparticle hopping asymmetric diffusion process [Sasamoto-Wadati '98], [Lee '12], [Barraquand
- -C '15]; jump right distance $j \in \{1, \ldots, gap\}$ at rate $= \frac{(q^{-1}-1)}{(q^{-j}-1)}$.

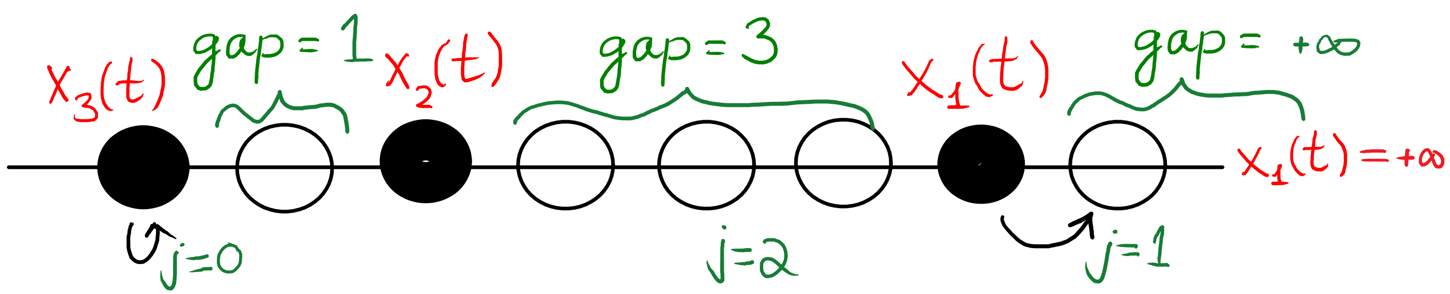
Taking $q=1$, particles jump to any $j \in \{1, \ldots, gap\}$ at rate $1/j$.

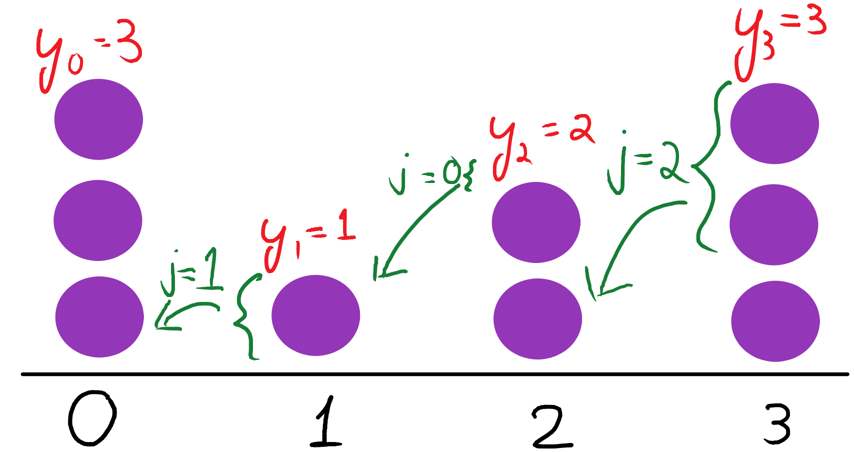
Taking $q=\infty$, particles jump to any $j \in \{1, \ldots, gap\}$ at rate 1.

Many other degenerations remain to be studied...

q-Hahn TASEP 'moments' and Fredholm determinant

Theorem [C '14] $f(t,\vec{y}) := \mathbb{E}\Big[\prod_{i=0}^{N} q^{(x_i(t)+i)y_i}\Big]$ is the unique solution to $f(t+1,\vec{y}) = P^{\mathrm{Boson}} f(t,\vec{y})$ subject to initial data $f(0,\vec{y}) = \mathbb{E}\Big[\prod_{i=0}^{N} q^{(x_i(0)+i)y_i}\Big]$.





The q-Hahn-Boson process transition matrix is integrable.

Theorem [C '14] For q-Hahn TASEP with step initial data $\{x_n(0)=-n\}_{n\geq 1}$

$$\mathbb{E}\Big[q^{(x_{n_1}(t)+n_1)+\cdots+(x_{n_k}(t)+n_k)}\Big] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint\cdots\oint \prod_{A<B} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \left(\frac{1-\nu z_j}{1-z_j}\right)^{n_j} \left(\frac{1-\mu z_j}{1-\nu z_j}\right)^{t} \frac{dz_j}{z_j}$$

$(n_1 \geq n_2 \geq \cdots \geq n_k)$

$*0 \; (z_1 \cdots (\cdot 1) z_k) z_{k-1} \cdots) z_1 \;\cdot\; \nu^{-1}$

q-Hahn Boson spectral theory

Two motivations to look at q-Hahn Boson process spectral theory:

- Develop an umbrella theory containing q-TASEP and ASEP (i.e., Tracy-Widom's ASEP transition formulas and 'magical identities'.)
- Solve for other q-Hahn TASEP/Boson initial data.

Starting point was our previous work on q-Boson spectral theory in [Borodin-C-Petrov-Sasamoto '13].

At the q-Hahn level, the proofs necessitated new methods.

I will explain the q-Hahn spectral theory (magenta= 'Hahn' terms)

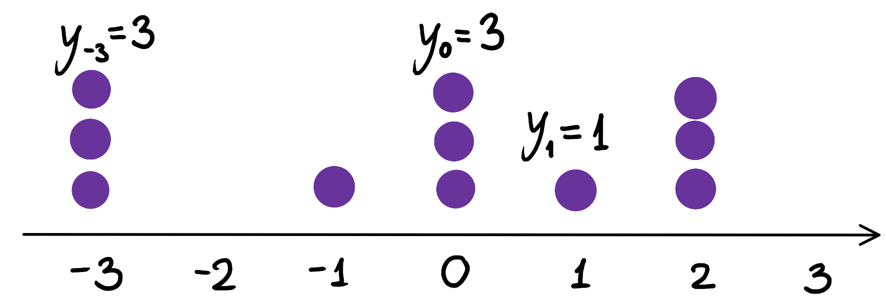
PT-invariance

To solve q-Hahn Boson system (thus q-Hahn TASEP) for general initial conditions, we want to diagonalize its transition matrix.

It is not self-adjoint, but PT-invariance (under joint space reflection and time inversion) effectively replaces self-adjointness:

Let m be an invariant product measure

$$m(dy) = \bigotimes_{n\in\mathbb{Z}} m_0(dy_n), \quad m_0(k) = \text{const}\cdot\begin{cases} \alpha^k \frac{(\nu;q)_k}{(q;q)_k}, & k \geq 0, \\ 0, & k<0 \end{cases}$$



Then $P^{Boson} = R\,(P^{Boson})^* R^{-1}$ in $L^2(\{y_n\}_{n\in\mathbb{Z}}, m)$

with $(Rf)(\{y_n\}_{n\in\mathbb{Z}}) = f(\{y_{-n}\}_{n\in\mathbb{Z}})$ (parity transformation).

Coordinate Bethe ansatz [Bethe '31]

(Algebraic) eigenfunctions for a (direct) sum of 1d operators

$$(\mathcal{L}\Psi)(\vec{x}) = \sum_{i=1}^{k} (L_{x_i}\Psi)(\vec{x}), \qquad \vec{x} = (x_1, \dots, x_k) \in \mathfrak{X}^k,$$

that satisfy boundary conditions

$$B_{x_i x_{i+1}} \Psi \Big|_{x_i = x_{i+1}} = 0, \qquad 1 \le i \le k-1, \qquad B : \{\text{functions on } \mathfrak{X}^2\} \circlearrowleft$$

Example: $B(g(x,y)) = (\nabla_x - q\nabla_y) g(x,y)$

can be found via

1. Diagonalizing 1d operator $L\psi_z = \lambda_z \psi_z, \quad \psi_z : \mathfrak{X} \to \mathbb{C}$
2. Taking linear combinations $\Psi_{\vec{z}}(\vec{x}) = \sum_{\sigma \in S(k)} A_\sigma(\vec{z})\, \psi_{z_{\sigma(1)}}(x_1) \cdots \psi_{z_{\sigma(k)}}(x_k)$
3. Evaluating $A_\sigma(\vec{z}) = \mathrm{sgn}(\sigma) \prod_{a>b} \dfrac{S(z_{\sigma(a)}, z_{\sigma(b)})}{S(z_a, z_b)}, \qquad S(z_1, z_2) = \dfrac{B(\psi_{z_1} \otimes \psi_{z_2})(x,x)}{\psi_{z_1}(x)\, \psi_{z_2}(x)}$

No quantization of spectrum (Bethe equations) in infinite volume.

q-Hahn Boson integrability

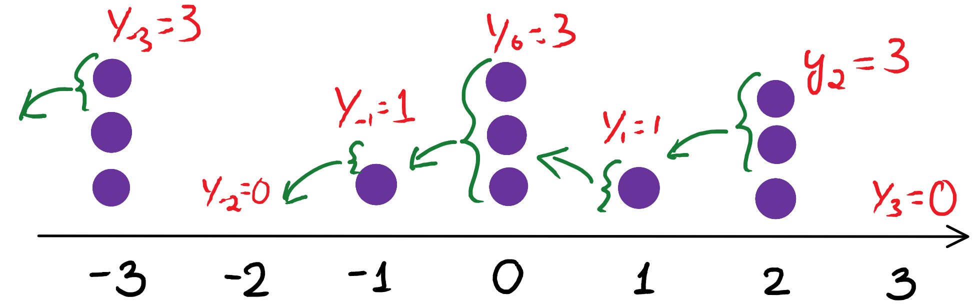
Transition matrix for k free particles:

$$(P^{\text{free}}u)(\vec{n}) := \prod_{i=1}^{k} [\nabla_{\mu,\nu}]_i \, u(\vec{n})$$

with $(\nabla_{\mu,\nu} f)(n) := \frac{\mu-\nu}{1-\nu} f(n-1) + \frac{1-\mu}{1-\nu} f(n)$

particles at $\vec{n} = (2,2,2,1,0,0,0,-1,-3,-3)$

heights $\vec{y}(\vec{n})$, $k = 11$ particles



In parallel move j particles according to $\varphi_{q,\mu,\nu}(j|y)$.

Two-body boundary conditions: for $1 \le i \le k-1$ and all $\vec{n}$: $n_i = n_{i+1}$

$$\alpha\, u(\ldots, n_i - 1, n_i - 1, \ldots) + \beta\, u(\ldots, n_i, n_i - 1, \ldots) + \gamma\, u(\ldots) - u(\ldots, n_i - 1, n_i, \ldots) = 0$$

where $\alpha = \frac{\nu(1-q)}{1-q\nu}$, $\beta = \frac{q-\nu}{1-q\nu}$, $\gamma = \frac{1-q}{1-q\nu}$.

Proposition [Povolotsky '13] If $u : \mathbb{Z}^k \to \mathbb{C}$ satisfies the two-body boundary conditions then $(P^{\text{free}}u)(\vec{n}) = (P^{\text{Boson}}u)(\vec{y}(\vec{n}))$ restricted to $\vec{n} : n_1 \ge \cdots \ge n_k$.

q-Hahn Boson eigenfunctions

Bethe ansatz and PT-invariance yields $(z_1,\dots,z_k \in \mathbb{C}\setminus\{1,\nu^{-1}\})$

$$\Psi^{\ell}_{\vec z}(\vec n) = \sum_{\sigma\in S(k)} \prod_{a>b} \frac{z_{\sigma(a)} - q z_{\sigma(b)}}{z_{\sigma(a)} - z_{\sigma(b)}} \prod_{j=1}^{k} \left(\frac{1-\nu z_{\sigma(j)}}{1-z_{\sigma(j)}}\right)^{n_j}$$

$$\Psi^{r}_{\vec z}(\vec n) = \frac{1}{C_q(\vec n)} \sum_{\sigma\in S(k)} \prod_{a>b} \frac{z_{\sigma(a)} - q^{-1} z_{\sigma(b)}}{z_{\sigma(a)} - z_{\sigma(b)}} \prod_{j=1}^{k} \left(\frac{1-\nu z_{\sigma(j)}}{1-z_{\sigma(j)}}\right)^{-n_j}$$

with $C_q(\vec n) = (-1)^k q^{-k(k-1)/2} \frac{(q;q)_{c_1}}{(1-q)^{c_1}(\nu;q)_{c_1}} \frac{(q;q)_{c_2}}{(1-q)^{c_2}(\nu;q)_{c_2}} \cdots$, and eigenvalues

$$P^{\mathrm{Boson}} \Psi^{\ell}_{\vec z} = \prod_{j=1}^{k} \frac{1-\mu z_j}{1-\nu z_j} \Psi^{\ell}_{\vec z}, \qquad (P^{\mathrm{Boson}})^T \Psi^{r}_{\vec z} = \prod_{j=1}^{k} \frac{1-\mu z_j}{1-\nu z_j} \Psi^{r}_{\vec z}.$$

Eigenfunctions only contain ν, so P^{Boson} commute for different μ.

Direct and inverse Fourier type transforms

Let $\mathcal{W}^k = \{ f : \{ n_1 \geq \ldots \geq n_k \mid n_j \in \mathbb{Z} \} \to \mathbb{C}$ of compact support $\}$

$\mathcal{C}^k = \mathbb{C}\left[\left(\frac{1-\nu z_1}{1-z_1}\right)^{\pm 1}, \ldots, \left(\frac{1-\nu z_k}{1-z_k}\right)^{\pm 1} \right]^{S(k)}$ = symmetric Laurent poly's in $\left(\frac{1-\nu z_j}{1-z_j}\right)$, $1 \leq j \leq k$.

Direct tranform: $\mathcal{F} : \mathcal{W}^k \to \mathcal{C}^k$

$$\mathcal{F} : f \longmapsto \sum_{n_1 \geq \ldots \geq n_k} f(\vec{n}) \cdot \Psi^r_{\vec{z}}(\vec{n}) =: \langle f, \Psi^r_{\vec{z}} \rangle_{\mathcal{W}}$$

Inverse transform: $\mathcal{J} : \mathcal{C}^k \to \mathcal{W}^k$

$$\mathcal{J} : G \longmapsto (q-1)^k q^{-\frac{k(k-1)}{2}} \frac{1}{(2\pi i)^k k!} \oint_{\substack{|w_j| = R \in (1, \nu^{-1}) \\ j=1,\ldots,k}} \cdots \oint \det\left[\frac{1}{q w_i - w_j}\right]_{i,j=1}^{k} \prod_{j=1}^{k} \frac{w_j}{(1-w_j)(1-\nu w_j)} \Psi^{\ell}_{\vec{w}}(\vec{n}) G(\vec{w}) \, d\vec{w}$$

$$=: \langle \Psi^{\ell}(\vec{n}), G \rangle_{\mathcal{C}}$$

Contour deformations

Inverse transform: $\mathcal{J} : \mathcal{C}^k \to \mathcal{W}^k$

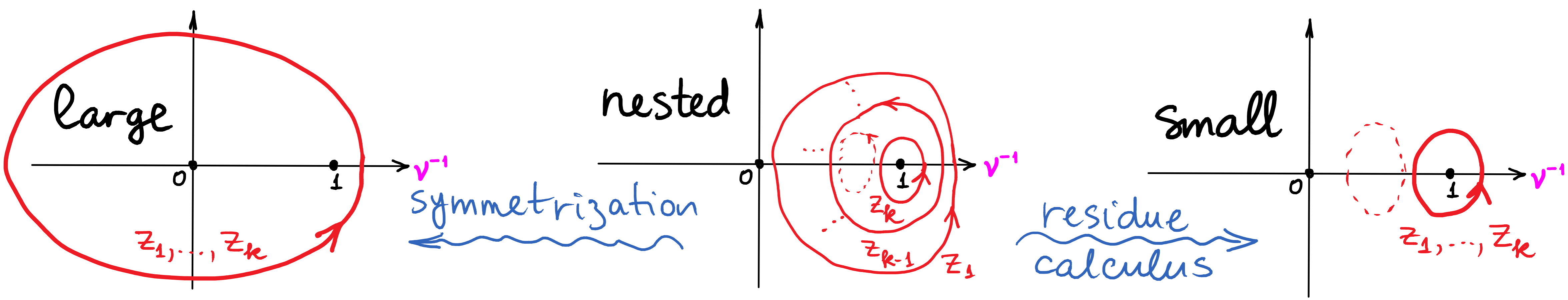
$$G \mapsto (q-1)^k q^{-\frac{k(k-1)}{2}} \frac{1}{(2\pi i)^k k!} \underset{\text{large}}{\oint \cdots \oint} \det\left[\frac{1}{q w_i - w_j}\right]_{i,j=1}^{k} \prod_{j=1}^{k} \frac{w_j}{(1-w_j)(1-\nu w_j)} \Psi^{\ell}_{\vec{w}}(\vec{n}) G(\vec{w}) \, d\vec{w}$$

$$= \frac{1}{(2\pi i)^k} \underset{\text{nested}}{\oint \cdots \oint} \prod_{a<b} \frac{z_a - z_b}{z_a - q z_b} \prod_{j=1}^{k} \left(\frac{1-\nu z_j}{1-z_j}\right)^{n_j} \frac{1}{(1-z_j)(1-\nu z_j)} G(\vec{z}) \, d\vec{z} =$$

$$= \sum_{\substack{\lambda = (\lambda_1 \geq \ldots \geq \lambda_\ell \geq 0) \\ \lambda_1 + \ldots + \lambda_\ell = k \quad \lambda = 1^{m_1} 2^{m_2} \ldots}} \frac{(q-1)^k q^{-k^2/2}}{m_1! \, m_2! \cdots} \frac{1}{(2\pi i)^\ell} \underset{\text{small}}{\oint \cdots \oint} \det\left[\frac{1}{q^{\lambda_i} w_i - w_j}\right]_{i,j=1}^{\ell} \prod_{j=1}^{\ell} \frac{w_j \, q^{\lambda_j^2/2}}{(w_j;q)_{\lambda_j} (w_j \nu; q)_{\lambda_j}} \Psi^{\ell}_{\vec{w} \circ \lambda}(\vec{n}) \, G(\vec{w} \circ \lambda) \, dw_1 \cdots dw_\ell$$

different sets of Bethe states

$$\vec{w} \circ \lambda = (w_1, q w_1, \ldots, q^{\lambda_1 - 1} w_1, w_2, q w_2, \ldots, q^{\lambda_2 - 1} w_2, \ldots, w_\ell, q w_\ell, \ldots, q^{\lambda_\ell - 1} w_\ell)$$



Plancherel isomorphism theorem

Theorem [Borodin-C-Petrov-Sasamoto '14] On spaces $\mathcal{W}^k$ and $\mathcal{C}^k$, operators $\mathcal{F}$ and $\mathcal{J}$ are mutual inverses of each other.

Isometry:

$$\langle f, g\rangle_{\mathcal{W}} = \langle \mathcal{F}f, \mathcal{F}g\rangle_{\mathcal{C}} \quad \text{for } f,g\in\mathcal{W}^k$$

$$\langle F, G\rangle_{\mathcal{C}} = \langle \mathcal{J}F, \mathcal{J}G\rangle_{\mathcal{W}} \quad \text{for } F,G\in\mathcal{C}^k$$

Biorthogonality:

$$\langle \Psi^{\ell}_{\bullet}(\vec{m}), \Psi^{r}_{\bullet}(\vec{n})\rangle_{\mathcal{C}} = \delta_{\vec{m},\vec{n}}$$

in a certain weak sense

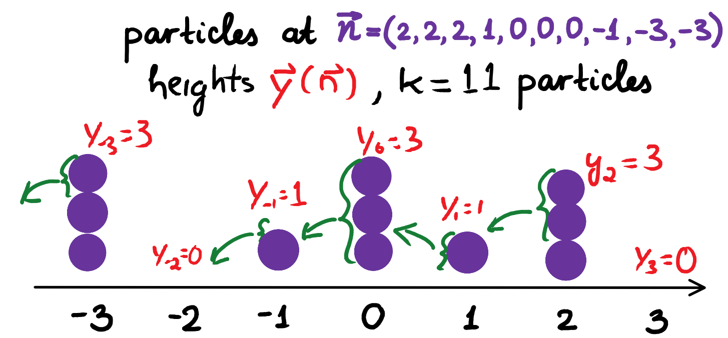
$$\langle \Psi^{\ell}_{\vec{z}}(\cdot), \Psi^{r}_{\vec{w}}(\cdot)\rangle_{\mathcal{W}} = \prod_{a\neq b}\frac{z_a - qz_b}{z_a - z_b}\prod_{j=1}^{k}(1-z_j)(1-\nu z_j)\det\left[\delta(z_i - w_j)\right]_{i,j=1}^{k}$$

Proof of $\mathcal{J}\mathcal{F} = \mathrm{Id}$ uses residue calculus in nested contour version of $\mathcal{J}$, while proof of $\mathcal{F}\mathcal{J} = \mathrm{Id}$ uses existence of simultaneously diagonalized family of matrices

Back to the q-Hahn Boson particle system

Corollary The (unique) solution of the q-Hahn Boson evolution equation

$f(t+1,\vec{y}) = P^{Boson} f(t,\vec{y})$ with $f(0,\vec{y}) = f_0(\vec{y})$ is



$$f(t,\vec{y}(\vec{n})) = \mathcal{J}\left(\left(\prod_{j=1}^{k}\frac{1-\mu z_j}{1-\nu z_j}\right)^t \mathcal{F} f_0\right)$$

eigenvalue of H corr. to $\Psi_{\vec{z}}$

$$= \frac{1}{(2\pi i)^k}\underset{\text{nested}}{\oint\cdots\oint}\prod_{a<b}\frac{z_a-z_b}{z_a-qz_b}\prod_{j=1}^{k}\left(\frac{1-\nu z_j}{1-z_j}\right)^{n_j}\frac{1}{(1-z_j)(1-\nu z_j)}\left(\prod_{j=1}^{k}\frac{1-\mu z_j}{1-\nu z_j}\right)^t \langle f_0, \Psi^r_{\vec{z}}\rangle_{\mathcal{W}}\, d\vec{z}$$

The computation of $\mathcal{F} f_0$ can still be difficult. It is, however, automatic if $f_0 = \mathcal{J} G \Rightarrow \mathcal{F} f_0 = \mathcal{F}\mathcal{J} G = G$.

Eg: q-TASEP step initial data $f_0(\vec{n}) = \mathbb{1}_{\{n_i \geq 1,\, 1\leq i\leq k\}}$, $G(\vec{z}) = q^{k(k-1)/2}\prod_{j=1}^{k}\frac{z_j-1}{z_j}$.

ASEP/XXZ degeneration of the Plancherel theorem

Specializing $\nu = q^{-1}$ (require some care since now $\nu > 1$) yields ASEP eigenfunctions:

- $\mathcal{I}\mathcal{F} = \mathrm{Id}$ becomes equivalent to the time zero version of [Tracy-Widom '08] k-particle ASEP transition probability result

$$P_{\vec{y}}(t;\vec{x}) = \sum_{\sigma \in S(k)} \oint \cdots \oint A_\sigma(\xi) \prod_{j=1}^{k} \xi_{\sigma(j)}^{x_j - y_{\sigma(j)} - 1} e^{(p\xi_j^{-1} + q\xi_j - 1)t} d\xi_j \,, \qquad A_\sigma(\xi) = \prod_{\substack{\{\alpha,\beta\} \\ \text{inversions of } \sigma}} - \frac{p + q\xi_\alpha\xi_\beta - \xi_\alpha}{p + q\xi_\beta\xi_\alpha - \xi_\beta}$$

- $\mathcal{F}\mathcal{I}G = G$ applied to a certain G yields TW 'magical-identity'.

XXZ in k-magnon sector is similarity transform of ASEP, so we recover results of [Babbitt-Gutkin '90] (the proof of which seems to be lost in the literature).

Further limit to XXX in k-magnon sector [Babbitt-Thomas '77]

Degenerations of wave functions at $\nu = 0$

$$\sum_{\sigma \in S(k)} \sigma \left[\prod_{a>b} \frac{z_a - q z_b}{z_a - z_b} \prod_{j=1}^{k} \frac{1}{(1-z_j)^{n_j}} \right]$$

q-Boson particle system

$z_j \gg 1$ (towards Hall-Littlewood polynomials)

$q = e^{-\varepsilon} \to 1$

$1 - z_j = O(\varepsilon)$ (towards semi-discrete Brownian polymer)

$$\sum_{\sigma \in S(k)} \sigma \left[\prod_{a>b} \frac{z_a - q z_b}{z_a - z_b} \prod_{j=1}^{k} z_j^{-n_j} \right]$$

Hall-Littlewood polynomials

$$\sum_{\sigma \in S(k)} \sigma \left[\prod_{a>b} \frac{z_a - z_b - 1}{z_a - z_b} \prod_{j=1}^{k} z_j^{-n_j} \right]$$

semi-discrete Brownian polymer

$q = e^{-\varepsilon} \to 1$

$1 - z_j = O(\varepsilon)$ (from Hall-Littlewood polynomials)

rescale near crit. pt. of e^{tz}/z^n with $t \sim \sqrt{n} \to \infty$ (time is involved) (from semi-discrete Brownian polymer)

$$\sum_{\sigma \in S(k)} \sigma \left[\prod_{a>b} \frac{z_a - z_b - 1}{z_a - z_b} \prod_{j=1}^{k} e^{x_j z_j} \right]$$

continuous delta Bose gas / KPZ

Some further directions

- Asymptotic analysis of these systems and associated formulas [Veto '14], [Barraquand-C '15]...
- Put into general context of higher spin vertex models and unite processes in terms of generators [Borodin-C-Gorin '14], [C-Petrov '15]
- Properties of eigenfunctions as symmetric functions [Borodin '14] (biorthogonal symmetric polynomials generalizing the Hall-Littlewood polynomials)
- Develop Macdonald process lifting related to q-Hahn processes ($\nu = 0$ related to Macdonald difference operator commutation relations [Borodin-C '13])
- Analyze discrete time TAZRP's with product invariant measure (see [Evans-Majumdar-Zia '04], [Balazs-Komjathy-Seppalainen '12])

Summary

- Using duality we computed formulas for 'moment' of q-Hahn TASEP which identify the distribution through a q-Laplace transform yielding rigorous replica method type computations.
- q-Hahn Boson process spectral theory provides an umbrella for q-TASEP, ASEP/XXZ, and various delta Bose gases;
  also enables study of general initial data q-Hahn TASEP/Boson.
- Work needed to turn algebraic advances into new analytic results.
- After break we will pick up on the higher spin vertex models which unite all known KPZ class integrable models.