Laumon spaces and Representations of $\mathfrak{gl}_n$

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Eisenstein Series over Function Field

- C: smooth projective curve over $k = \bar{k}$
- G: reductive group (with $[G, G]$ simply-connected)
- $T \subset B \subset G$ maximal torus and Borel subgroup
- Bun_G (resp. Bun_T, Bun_B): stacks of G (resp. T, B)-bundles on C

$$\mathrm{Bun}_T \xleftarrow{\pi_T} \mathrm{Bun}_B \xrightarrow{\pi_G} \mathrm{Bun}_G$$

Remark:
If $k = \overline{\mathbb{F}}_q$, then $\pi_{G,!}\pi_T^* : \mathrm{Shv}(\mathrm{Bun}_T) \to \mathrm{Shv}(\mathrm{Bun}_G)$ recovers the unramified principal Eisenstein series over the function field $\mathbb{F}_q(C_{\mathbb{F}_q})$ by taking trace of Frobenius action.

Compatifications of Bun_B

For $G = GL(n)$,Bun_B parametrizes flags

$$V_1 \subset V_2 \subset \cdots \subset V_n$$

where

- V_i: locally free sheaf on C of rank i
- $V_i \subset V_{i+1}$: subbundle (pointwise injective)

Two compactifications:

- $\overline{\mathrm{Bun}}_B^{\mathrm{Lau}} := \{ V_1 \to V_2 \to \cdots \to V_n \}$ where $V_i \to V_{i+1}$ is injective as coherent sheaves. (Only defined for $GL(n)$.) [Laumon '88]
- $\overline{\mathrm{Bun}}_B^{\mathrm{Dri}}$: can be defined for any type. [Braverman-Gaitsgory '02] following a suggestion of Drinfeld.

For $GL(n)$, there is a map $\overline{\mathrm{Bun}}_B^{\mathrm{Lau}} \to \overline{\mathrm{Bun}}_B^{\mathrm{Dri}}$ which is a small resolution.

Laumon Space and Zastava Space

Let $k = \mathbb{C}$ and $C = \mathbb{P}^1$. Define the Laumon space $\mathcal{M}$ to be the moduli space of flags

$$V_1 \to V_2 \to \cdots \to V_n$$

where

- $V_n = \mathcal{O}^{\oplus n}$ the trivial bundle (i.e. we fix an isomorphism $V_n \simeq \mathcal{O}^{\oplus n}$)
- The restriction at $\infty \in \mathbb{P}^1$ is the standard flag.

For $\mathbf{d} = (d_1, ..., d_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$, denote by $\mathcal{M}_{\mathbf{d}}$ the connected component of $\mathcal{M}$ with $\deg V_i = -d_i$.

Fact: For each $\mathbf{d}$, $\mathcal{M}_{\mathbf{d}}$ is a smooth scheme of dimension $2|\mathbf{d}|$.

A similar definition can be made for $\overline{\mathrm{Bun}}_B^{\mathrm{Dri}}$. The resulting spaces are called Zastava spaces $\mathcal{Z}_{\mathbf{d}}$. They are singular in general.

Action on Cohomology

For each $\mathbf{d}$, $\mathbb{T} = T \times \mathbb{C}^*$ acts on $\mathcal{M}_{\mathbf{d}}$ where T scales $\mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ and $\mathbb{C}^*$ scales $\mathbb{P}^1$.

One can construct an action of $U(\check{\mathfrak{gl}}_n)$ on the equivariant cohomology $\bigoplus_d H^\bullet_{T\times\mathbb{C}^*}(\mathcal{M}_d, \mathbb{C})$ by "Hecke correspondences".

$$\mathcal{C}_{d,i} = \left\{ \begin{array}{l} V_1 \to \cdots \to V_{i-1} \to V_i' \to V_i \to \cdots \to V_n = \mathcal{O}_{\mathbb{P}^1}^{\oplus n} \\ \text{where } V_i/V_i' = \mathcal{O}_0 \text{ (skyscraper sheaf at } 0 \in \mathbb{P}^1) \end{array} \right\}$$

$$\begin{array}{ccccc} & & \mathcal{C}_{d,i} & & \\ & \swarrow^{p} & & \searrow^{q} & \\ \mathcal{M}_{\mathbf{d}} & & & & \mathcal{M}_{\mathbf{d}+\delta_i} \end{array}$$

$H^\bullet_{\mathbb{T}}(\mathcal{M}_{\mathbf{d}})$ is a module over $H^\bullet_{\mathbb{T}}(\mathrm{pt}) = \mathbb{C}[a_1, \ldots, a_n, \epsilon]$.

Action on Cohomology (cont.)

Let

$$E_i := -q_* p^* : H^\bullet_{\mathbb{T}}(\mathcal{M}_{\mathbf{d}}) \to H^\bullet_{\mathbb{T}}(\mathcal{M}_{\mathbf{d}+\delta_i})$$
$$F_i := p_* q^* : H^\bullet_{\mathbb{T}}(\mathcal{M}_{\mathbf{d}+\delta_i}) \to H^\bullet_{\mathbb{T}}(\mathcal{M}_{\mathbf{d}}), \quad i = 1, \dots, n-1$$
$$H_j := -a_j + (d_j - d_{j-1} - j)\epsilon, \quad j = 1, \dots, n$$

Theorem (Feigin–Finkelberg–Frenkel–Rybnikov '08)

- *These operators form an action of* $U(\mathfrak{gl}_n)$ *on* $\bigoplus_{\mathbf{d}} H^\bullet_{\mathbb{T}}(\mathcal{M}_{\mathbf{d}})$.
- *The module* $\bigoplus_{\mathbf{d}} H^\bullet_{\mathbb{T}}(\mathcal{M}_{\mathbf{d}})$ *is isomorphic to the universal dual Verma module of* $U(\mathfrak{gl}_n)$ *with lowest weight* $(-a_1 - \epsilon, -a_2 - 2\epsilon, \dots, -a_n - n\epsilon)$.

More precisely, the algebra should be $U_\epsilon(\mathfrak{gl}_n)$, the Rees algebra of $U(\mathfrak{gl}_n)$, which is an algebra over $\mathbb{C}[\epsilon]$ and relations homogenized using ϵ.

Example: Laumon Space for $GL(2)$

When $G = GL(2)$, for each $d \in \mathbb{Z}_{\geq 0}$, $\mathcal{M}_d$ parametrizes maps

$$\mathcal{O}_{\mathbb{P}^1}(-d) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$$

satisfying certain conditions at $\infty \in \mathbb{P}^1$.

We have $\mathcal{M}_d \simeq \mathbb{C}^{2d}$. The $\mathbb{T}$-weights are

$$(\epsilon, \dots, d\epsilon,\ a_2 - a_1 + \epsilon, \dots, a_2 - a_1 + d\epsilon)$$

Let $1_d \in H^\bullet_{\mathbb{T}}(\mathcal{M}_d)$ be the generator, then

$$\begin{aligned} E(1_{d-1}) &= -d\epsilon(a_2 - a_1 + d\epsilon)1_d \\ F(1_d) &= 1_{d-1} \\ H(1_d) &= a_2 - a_1 + (2d+1)\epsilon \end{aligned}$$

In the $GL(2)$ case, one checks that

- $[E, F] = \epsilon H$, etc. (Rees algebra of $U(\mathfrak{gl}_2)$)
- If we set $a_2 - a_1 = -\lambda\epsilon$, $\lambda \in \mathbb{Z}_{\geq 0}$, then E kills 1_λ.

$$\bigoplus_d H^\bullet_{\mathbb{T}}(\mathcal{M}_d)\big|_{a_2 - a_1 = -\lambda\epsilon} \simeq V^\vee(-\lambda + 1)$$

Remarks:

- For general n and lowest weight $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, setting $a_i \mapsto \lambda_i\epsilon$, we get $V^\vee(-\lambda - \rho)$ where $\rho = (1, \ldots, n)$ is the half-sum of positive roots up to overall shift.
- If the evaluation at $\infty \in \mathbb{P}^1$ is not the standard flag x_0 but another point $w(x_0)$ for some $w \in S_n$ (Weyl group of $GL(n)$), then we get $V^\vee(w(-\lambda) - \rho)$ under the above specialization.

More Remarks

1. Generalizations:
   - K-theory: $U_q(\mathfrak{gl}_n)$ [Braverman–Finkelberg '05]
   - Tautological bundles: $Y(\mathfrak{gl}_n)$, $U_q(\widehat{\mathfrak{gl}}_n)$
   - Affine Laumon space [FFNR '08, Tsymbaliuk '10, Negut '13]
   - Partial flag varieties [BFFR '11, Nakajima '11, Finkelberg–Tsymbaliuk '19]
2. $U(\check{\mathfrak{g}})$ acts on $\bigoplus_{\mathbf{d}} IH^\bullet_{\mathbb{T}}(\mathcal{Z}_{\mathbf{d}})$ for the Zastava space $\mathcal{Z}_{\mathbf{d}}$ of G using derived geometric Satake.
   [BFN '20]; see also [FFKM '99], [Braverman–Gaitsgory '06], [Gaitsgory], [Campbell-Hayash]

Relative Quasimaps

We want to compactify $\mathcal{M}_{\mathbf{d}}$'s while keeping a well-defined evaluation map at $\infty \in \mathbb{P}^1$.

One way to do this is to let the domain $\mathbb{P}^1$ degenerate into nodal curves. This uses the quasimaps moduli space constructed in [Ciocan-Fontanine–Kim–Maulik '11].

We use their construction in the special case where the domain is a *parametrized* $\mathbb{P}^1$ with one marked point and the target is $W//G$ where

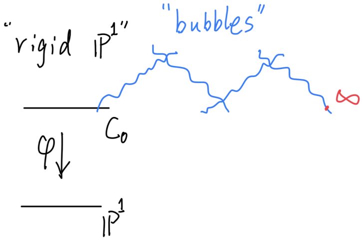
$$W = \bigoplus_{i=1}^{n-1} \mathrm{Hom}(\mathbb{C}^i, \mathbb{C}^{i+1}), \qquad G = \prod_{i=1}^{n-1} GL(i)$$

Denote this moduli space by QM_{rel}

Definitions

Closed points in QM_{rel} corresponds to the following data:

- Domain curve $C = \mathbb{P}^1$ or a chain of $\mathbb{P}^1$'s, together with a map $\varphi\colon C \to \mathbb{P}^1$ and a marked point ∞, such that $\varphi|_{C_0}$ is an isomorphism and $\varphi|_{C \setminus C_0}$ maps to a point and ∞ is on the component on the opposite end.



- A flag of locally free sheaves on C

$$V_1 \to V_2 \to \cdots \to V_n = \mathcal{O}_C^{\oplus n} \quad (1)$$

Definitions (cont.)

Subject to the stability condition:

- If p is a node or ∞ in C, the maps in the flag (1) at p are injective
- For each $\mathbb{P}^1$ in the bubble, at least one V_i has non-zero degree when restricted to this $\mathbb{P}^1$.

$QM^{\mathbf{d}}_{rel}$: degree $\mathbf{d}$ connected component;

$QM^{\mathbf{d}}_{rel,w(x_0)}$: with $ev(\infty) = w(x_0)$.

Fact([CKM]): For any $\mathbf{d}$, $QM^{\mathbf{d}}_{rel}$ is a smooth, proper Deligne-Mumford stack.

$\mathcal{M}_{\mathbf{d}} \subset QM^{\mathbf{d}}_{rel,x_0}$ is a dense open substack.

$U(\mathfrak{gl}_n)$ Action on $H_{\mathbb{T}}(QM_{rel})$

$\mathcal{C}_{d,i,rel}$: similar to before, modification at 0. $\rightsquigarrow$ operators E_i, F_i, $i = 1, 2, ..., n-1$.

$$H_j = -c_1(0^*\mathcal{V}_j) + c_1(0^*\mathcal{V}_{j-1}) - j\epsilon, \quad j = 1, 2, ..., n$$

where 0 and ∞ are sections of the universal curve $\mathfrak{C}$ over QM_{rel} and $\mathcal{V}_i$'s are tautological bundles.

$$\begin{array}{c} \mathfrak{C} \\ {\scriptstyle 0,\infty}\uparrow\downarrow{\scriptstyle \pi} \\ QM_{rel} \end{array}$$

Theorem

The operators $E_i, F_i, i = 1, ..., n-1$ and $H_j, j = 1, ..., n$ form an action of $U(\mathfrak{gl}_n)$ on $\bigoplus_{\mathbf{d}} H^\bullet_{\mathbb{T}}(QM^{\mathbf{d}}_{rel})$.

$U(\mathfrak{gl}_n)$ Action on $H^\bullet_{\mathbb{T}}(QM_{rel})$ (cont.)

Theorem

$H^\bullet_{\mathbb{T}}(QM_{rel})$ is self dual as a $U(\mathfrak{gl}_n)$ module.

This follows from the properness of QM_{rel}. The Poincare pairing on $H^\bullet_{\mathbb{T}}$ is a contravariant pairing with respect to $U(\mathfrak{gl}_n)$ action.

Theorem

$H^\bullet_{\mathbb{T}}(QM_{rel})$ has an infinite filtration by the universal dual Verma module $V^\vee(w(-a)-\rho)$ for various $w \in S_n$.

This is because QM_{rel} has a filtration by $(\bigsqcup_{\mathbf{d}} \mathcal{M}_{\mathbf{d},w(x_0)}) \times$ (affine space) where $\mathcal{M}_{\mathbf{d},w(x_0)}$ is defined in the same way as $\mathcal{M}_{\mathbf{d}}$ but with $ev(\infty) = w(x_0)$.

A Finitely Generated Submodule

These properties resembles those of tilting modules in category $\mathcal{O}$. But note that $H^\bullet_{\mathbb{T}}(QM_{rel})$ is not finitely generated.

There is a natural way to make it smaller. Fix $\lambda = (\lambda_1 > \cdots > \lambda_n)$ dominant regular, $\epsilon_0 \in \mathbb{C}$ generic,

$$H^\bullet_{\mathbb{T}}(QM_{rel,\, w(x_0)})\Big|_{a_i \mapsto \lambda_i \epsilon, \epsilon \mapsto \epsilon_0} \simeq H^\bullet(QM^{\mathbb{C}^\times_\lambda}_{rel,\, w(x_0)})$$

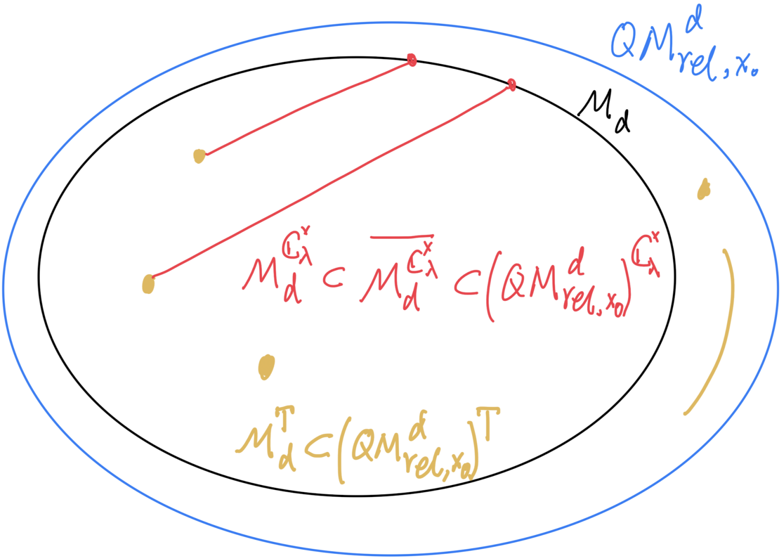
where $\mathbb{C}^\times_\lambda \subset \mathbb{T}$ is a subtorus determined by λ.

Define:

$$H_{\lambda,w} := H^\bullet\left(\overline{\mathcal{M}^{\mathbb{C}^\times_\lambda}_{\mathbf{d},w(x_0)}}\right)$$

(closure inside $QM^{\mathbb{C}^\times_\lambda}_{rel,\, w(x_0)}$).

Closure of $\mathbb{C}^\times_\lambda$ Fixed Points



Category $\mathcal{O}'$ and $H_{\lambda,w}$

Category $\mathcal{O}'$: Cartan action is *not* semisimple; center action *is* semisimple.

For λ regular, there is an equivalence $\Psi : \mathcal{O}'_\lambda \simeq \mathcal{O}_\lambda$

Theorem

- *$H_{\lambda,w}$ is a direct summand of* $H^\bullet_{\mathbb{T}}(QM_{rel,\, w(x_0)})\Big|_{a_i \mapsto \lambda_i \epsilon, \epsilon \mapsto \epsilon_0}$
- *The image of $H_{\lambda,w}$ under Ψ is a tilting module.*
- $[H_{\lambda,w} : V(u(\lambda) - \rho)] = \#\{\textit{Bruhat paths } w \to u\}$

(In particular, $H_{\lambda,w}$'s are not indecomposible in general.)

Example: $n = 2$

Set $a_2 - a_1 = -\lambda\epsilon$, $\lambda \in \mathbb{Z}$.
We have seen that $\mathcal{M}_d \simeq \mathbb{C}^{2d}$ with $\mathbb{T}$-weights $(\epsilon, \ldots, d\epsilon,\ a_2 - a_1 + \epsilon, \ldots, a_2 - a_1 + d\epsilon)$.
If $\lambda \leq 0$,

$$\mathcal{M}_{\mathbf{d}}^{\mathbb{C}^\times_\lambda} = pt \text{ for all } d$$

Antidominant Verma module is a tilting module.

If $\lambda > 0$,

$$\mathcal{M}_d^{\mathbb{C}^\times_\lambda} = \begin{cases} \mathrm{pt} & d < \lambda \\ \mathbb{C} & d \geq \lambda \end{cases}$$

$$\overline{\mathcal{M}_d^{\mathbb{C}^\times_\lambda}} = \begin{cases} \mathrm{pt} & d < \lambda \\ \mathbb{P}(1, \lambda) & d \geq \lambda \end{cases}$$

$T(-\lambda + 1)$ is an extension of $V(\lambda + 1)$ by $V(-\lambda + 1)$.

Thank you!