Equivariant K-theory of Laumon Spaces and Dual Verma Module of $U_q(\mathfrak{gl}_n)$

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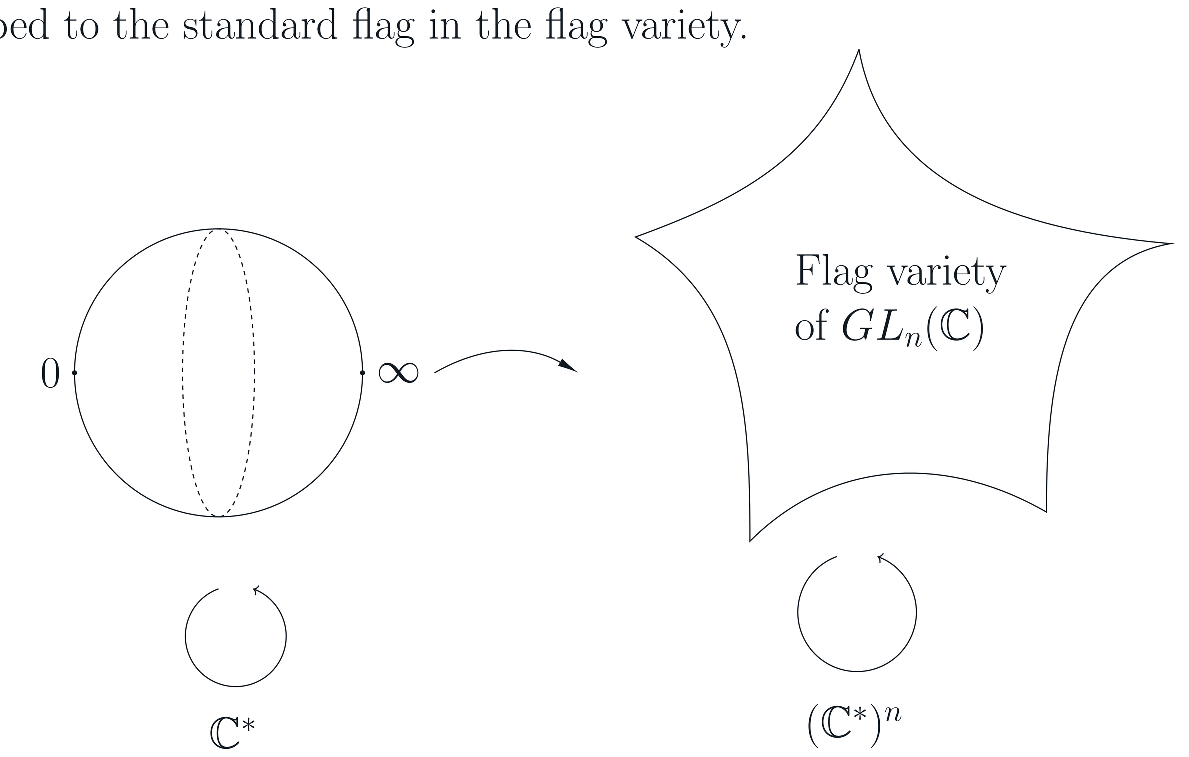
Background

Given $\mathbf{d} = (d_1, ..., d_{n-1})$, the Laumon space $\mathcal{M}_{\mathbf{d}}$ (also known as based quasimaps to the flag variety) is a partial compactification of the moduli space of degree $\mathbf{d}$ maps from $\mathbb{P}^1$ to the flag variety of $GL_n(\mathbb{C})$ such that $\infty \in \mathbb{P}^1$ is mapped to the standard flag in the flag variety.



The torus $T = \mathbb{C}^* \times (\mathbb{C}^*)^n$ acts on $\mathcal{M}_{\mathbf{d}}$, so we can consider the *equivariant* cohomology or K-theory of it. This was extensively studied in [2, 1, 8, 4, 5]. An action of $U_q(\mathfrak{gl}_n)$ on $\bigoplus_{\mathbf{d}} K_T(\mathcal{M}_{\mathbf{d}})$ can be constructed geometrically. It was first observed in [1] that, for a generic highest weight $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_{n-1})$,

$$\bigoplus_{\mathbf{d}} K_T(\mathcal{M}_{\mathbf{d}})\Big|_{a_i = q^{\lambda_i}} \simeq \mathrm{Verma}(\boldsymbol{\lambda} - \rho)$$

where ρ is half sum of positive roots.
The data on the geometric side and on the representation theory side match nicely. For example,

$$\begin{aligned} \text{equivariant parameters} &\longleftrightarrow \text{highest weight} \\ \text{degree } \mathbf{d} &\longleftrightarrow \text{weight space} \\ \text{Euler characteristic pairing on } K_T(\mathcal{M}_{\mathbf{d}}) &\longleftrightarrow \text{Shapovalov form} \\ \text{Fixed point basis of } K_T(\mathcal{M}_{\mathbf{d}}) &\longleftrightarrow \text{Gelfand-Tsetlin basis} \end{aligned}$$

The goal of this paper is to study what happens when the equivariant parameters a_i's are specialized to integral highest weight $(\lambda_1, ..., \lambda_n)$.

Laumon Spaces $\mathcal{M}_{\mathbf{d}}$

Given $\mathbf{d} = (d_1, ..., d_{n-1}), d_i \in \mathbb{Z}_{\geq 0}$, let $\mathcal{M}_{\mathbf{d}}$ denote the moduli of flag of sheaves on $\mathbb{P}^1$:

$$\mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_n \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$$

such that

- $\mathcal{F}_i$ is locally free of rank i and degree d_i
- $\mathcal{F}_i \to \mathcal{F}_{i+1}$ is injective as a map of sheaves.
- Restricting to $\infty \in \mathbb{P}^1$, the flag
$$\mathcal{F}_1|_\infty \to \mathcal{F}_2|_\infty \to \cdots \to \mathcal{F}_n|_\infty \simeq \mathbb{C}^{\oplus n}$$
is the standard flag in $\mathbb{C}^n$.

It is known that for each $\mathbf{d}$, $\mathcal{M}_{\mathbf{d}}$ is a smooth algebraic variety of dimension $2(d_1 + ... + d_{n-1})$.

Equivariant K-theory and Correspondences

$K_T(\mathcal{M}_{\mathbf{d}})$ is a module over

$$K_T(pt) \simeq \mathbb{Z}[q^\pm, a_1^\pm, ..., a_n^\pm]$$

The a_i's and q are called equivariant parameters.
The key technical tool is the localization formula in equivariant K-theory. For any $\alpha \in K_T(\mathcal{M}_{\mathbf{d}})$

$$\alpha = \sum_{F \in \mathcal{M}_{\mathbf{d}}^T} \frac{\alpha|_F}{\Lambda^\bullet N_{F/\mathcal{M}_{\mathbf{d}}}^\vee}$$

where $N_{F/\mathcal{M}_{\mathbf{d}}}$ denotes the normal bundle to F, and $\Lambda^\bullet$ is the signed sum of exterior powers.
The action of $U_q(\mathfrak{gl}_n)$ is defined using *correspondences*. Let $\mathcal{C}_{\mathbf{d},i}$ be the moduli space of

$$\mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \mathcal{F}_i' \to \mathcal{F}_i \to \cdots \to \mathcal{F}_n \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$$

satisfying similar conditions as $\mathcal{M}_{\mathbf{d}}$, but with the additional constraint that

- $\mathcal{F}_i/\mathcal{F}_i' = \mathcal{O}_0$, the structure sheaf supported at $0 \in \mathbb{P}^1$.

There are two projection maps

$$\begin{tikzcd} & \mathcal{C}_{\mathbf{d},i} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{M}_{\mathbf{d}} & & \mathcal{M}_{\mathbf{d}+i} \end{tikzcd}$$

Up to invertable constants from $K_T(pt)$, the generators e_i, f_i of $U_q(\mathfrak{gl}_n)$ are given by

$$e_i := q_* p^*(\cdot) : K_T(\mathcal{M}_{\mathbf{d}}) \to K_T(\mathcal{M}_{\mathbf{d}+i})$$
$$f_i := p_*(q^*(\cdot) \otimes \mathcal{L}_i) : K_T(\mathcal{M}_{\mathbf{d}}) \to K_T(\mathcal{M}_{\mathbf{d}-i})$$

where $\mathcal{L}_i$ denotes the tautological line bundle defined by $\mathcal{F}_i/\mathcal{F}_i'$.

Example: $n = 2$

In this case, $\mathcal{M}_{\mathbf{d}} \simeq \mathbb{C}^{2d}$, with equivariant weights

$$q,\ q^2,\ ...,\ q^d,\ \frac{a_2}{a_1}q,\ \frac{a_2}{a_1}q^2,\ ...,\ \frac{a_2}{a_1}q^d$$

and $\mathcal{C}_d \simeq \mathcal{M}_{\mathbf{d}}$. For each d, $K_T(\mathcal{M}_{\mathbf{d}}) \simeq \mathbb{Z}$ and is generated by the structure sheaf.

$$\begin{tikzcd} \mathcal{O}_{\mathcal{M}_0} \arrow[loop above, "\Psi_1, \Psi_2"] \arrow[r, bend right, "E"'] & \mathcal{O}_{\mathcal{M}_1} \arrow[loop above, "\Psi_1, \Psi_2"] \arrow[l, bend right, "F"'] \arrow[r, bend right, "E"'] & \mathcal{O}_{\mathcal{M}_2} \arrow[loop above, "\Psi_1, \Psi_2"] \arrow[l, bend right, "F"'] \arrow[r, bend right, "E"'] & \cdots \arrow[l, bend right, "F"'] & \cdots \end{tikzcd}$$

The coefficients of the $U_q(\mathfrak{gl}_2)$ action are given by

$$E[\mathcal{O}_{\mathcal{M}_{\mathbf{d}}}] = (1 - q^{-d})(1 - \frac{a_1}{a_2}q^{-d})[\mathcal{O}_{\mathcal{M}_{\mathbf{d}+1}}], \quad F[\mathcal{O}_{\mathcal{M}_{\mathbf{d}}}] = \frac{1}{a_1 q^{-d}}[\mathcal{O}_{\mathcal{M}_{\mathbf{d}-1}}]$$

$$\Psi_1[\mathcal{O}_{\mathcal{M}_{\mathbf{d}}}] = \frac{q^d}{q a_1}[\mathcal{O}_{\mathcal{M}_{\mathbf{d}}}], \quad \Psi_2[\mathcal{O}_{\mathcal{M}_{\mathbf{d}}}] = \frac{q^{-d}}{q^2 a_2}[\mathcal{O}_{\mathcal{M}_{\mathbf{d}}}]$$

A technical remark: The generators E, F, Ψ_1, Ψ_2 we use here differ from the usual generators e, f, ψ_1, ψ_2 by

$$E = \psi_2 e, \quad F = f\psi_1, \quad \Psi_i = \psi_i^2 \text{ for } i = 1, 2$$

This choice fits better with the geometric picture.

Main Results

Theorem 1. *The module*

$$\bigoplus_{\mathbf{d}} K_T(\mathcal{M}_{\mathbf{d}})\Big|_{a_i = q^{\lambda_i}}$$

is isomorphic to the dual Verma module of $U_q(\mathfrak{gl}_n)$ of highest weight $\boldsymbol{\lambda} - \rho$, where ρ is half sum of positive roots of $\mathfrak{gl}_n$.

This can be easily verified in the $n = 2$ example. The action of F is never 0, but the action of E vanishes when $\frac{a_1}{a_2} = q^k$ for a positive integer k. In this case, the dual Verma module has a k-dimensional submodule.

Verma module has a natural basis – the *PBW basis* fullwoing the Poincaré-Birkhoff-Witt theorem. Their dual basis form a basis of the dual Verma module. The dual PBW basis can be realized geometrically. To this end, we will need the notion of *stable envelopes*, which is a map

$$K_T(\mathcal{M}_{\mathbf{d}}^T) \to K_T(\mathcal{M}_{\mathbf{d}})$$

Stable envelopes were initially defined in [3, 7] for Nakajima quiver varieties. In a later paper [6], the construction was generalized to varieties that satisfy the milder condition that "attracting directions can be globalized". It is this construction that can be applied to $\mathcal{M}_{\mathbf{d}}$.

Theorem 2. *The stable envelopes of $\mathcal{O}_F$ for $F \in \mathcal{M}_{\mathbf{d}}^T$ coincide with the dual PBW basis under the isomorphism in Theorem 1.*

Generalization to Affine Case

Theorem 1 can be generalized to the affine case. More precisely, we consider the *affine* Laumon space $\mathcal{M}_{\mathbf{d}}^{\mathrm{aff}}$, which is the moduli space of parabolic sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ framed at the divisor $\infty \times \mathbb{P}^1 \cup \mathbb{P}^1 \times \infty$.
On the representation theory side, one considers the dual Verma module of $U_q(\widehat{\mathfrak{gl}}_n)$.

Theorem 3. *For any highest weight that is not at the critical level, the module $\bigoplus_{\mathbf{d}} K_T(\mathcal{M}_{\mathbf{d}}^{\mathrm{aff}})$ with equivariant parameters specialized according to the highest weight is isomorphic to the dual Verma module of $U_q(\widehat{\mathfrak{gl}}_n)$.*

References

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