1. MONODROMY PERMUTATION REPRESENTATION

In the previous lecture, we saw how $\mathrm{Aut}(Y/X)$ acts on a covering $h : Y \to X$. Restricting to a fiber $h^{-1}(x)$, this gives us an action $\mathrm{Aut}(Y/X)$ on $h^{-1}(x)$. One can also define an action of $\pi_1(X,x)$ on a fiber $h^{-1}(x)$ in terms of liftings of maps, which is closely related to the former action when the covering $h : Y \to X$ is Galois. For this, we need the following proposition.

Proposition 1.1 (Homotopy lifting property). *Given a covering space $h : Y \to X$, a homotopy $f_t : Z \to X$ ($t \in [0,1]$) and a map $\widetilde{f_0} : Z \to Y$ lifting f_0, there exists a unique homotopy $\widetilde{f_t} : Z \to Y$ of $\widetilde{f_0}$ lifting f_t.*

In particular, taking Z to be a point, we obtain the *path lifting property* of a covering space $h : Y \to X$: for any path $f : [0,1] \to X$ and any lift y_0 of the starting point $f(0) = x_0$, there exists a unique path $\tilde{f} : [0,1] \to Y$ lifting f starting at y_0. Furthermore, for any homotopy f_t of f, there exists a unique lift $\widetilde{f_t}$ of f_t such that $\widetilde{f_t}$ is a homotopy of $\tilde{f}$.

The path lifting property now enables us to define an action of $\pi_1(X,x)$ on a fiber $h^{-1}(x)$ as follows. For a loop $[l] \in \pi_1(X,x)$ and a point $y \in h^{-1}(x)$, we define $y \cdot [l]$ to be the ending point $\tilde{l}(1)$, where $\tilde{l}$ is the lift of l with starting point $\tilde{l}(0) = y$. The induced representation $\rho_x : \pi_1(X,x) \to \mathrm{Aut}(h^{-1}(x))$ is called the *monodromy permutation representation* of $\pi_1(X,x)$. Moreover, one can show that if $h : Y \to X$ is a Galois covering, then the composite

$$\pi_1(X,x) \twoheadrightarrow h_*(\pi_1(Y,y))\backslash\pi_1(X,x) \cong \mathrm{Gal}(Y/X) \xrightarrow{\text{restrict to a fiber } h^{-1}(x)} \mathrm{Aut}(h^{-1}(x))$$

is precisely the monodromy permutation representation. Conversely, from the monodromy permutation representation, one can recover the covering $h : Y \to X$ up to isomorphism.

Example 1.2. Suppose our base space X is the circle S^1. If $h : Y \to X$ is one of the finite cyclic coverings $h_n : S^1 \to S^1$ as in Example 1.2 of Lecture 3, then $\mathrm{Gal}(Y/X)$ is a finite cyclic group $\mathbb{Z}/n\mathbb{Z}$, $h^{-1}(x)$ consists of n points, $\mathrm{Aut}(h^{-1}(x))$ is the symmetric group S_n and the image of the inclusion $\mathrm{Gal}(Y/X) \hookrightarrow S_n$ is the subgroup generated by the cyclic permutation $(1\ 2\ \cdots\ n)$. If $h : Y \to X$ is the universal covering $h_\infty : \mathbb{R}^1 \to S^1$, then $\mathrm{Gal}(\mathbb{R}^1/S^1)$ is the infinite cyclic group $\mathbb{Z}$, $\mathrm{Aut}(h^{-1}(x))$ is the infinite symmetric group S_∞ and the image of the inclusion $\mathrm{Gal}(\mathbb{R}^1/S^1) \hookrightarrow S_\infty$ is the subgroup generated by the shift $m \mapsto m+1$ ($m \in \mathbb{Z}$).

2. LINKING NUMBERS AND LEGENDRE SYMBOLS

To get a better feeling for working not only with knots, but also with links, and not only at a single prime, but with multiple primes, the first analogy between knot theory and number theory that we shall study is that between linking numbers and Legendre symbols. The linking number and Legendre symbol are the first invariants that come to mind when one considers a 2-component link and a pair of primes respectively, and surprisingly, there is an analogy between them.

2.1. Linking numbers.

Definition 2.1. Let K and L be disjoint oriented simple closed curves K and L in S^3 (i.e. a 2-component link). The *linking number of K and L*, denoted by $\mathrm{lk}(L,K)$, is defined as follows. Let Σ_L be a Seifert surface of L; by perturbing Σ_L suitably, we may assume that K intersects Σ_L transversely. Let $P_1, \ldots, P_m$ be the set of intersection points of K and Σ_L. According as the tangent vector of K at P_i has the same or opposite direction as the normal vector of Σ_L at P_i, assign a number $\varepsilon(P_i) := 1$ or -1 to each P_i, as in Figure 1. The linking number $\mathrm{lk}(L,K)$ is defined by

$$\mathrm{lk}(L,K) := \sum_{i=1}^{m} \varepsilon(P_i).$$

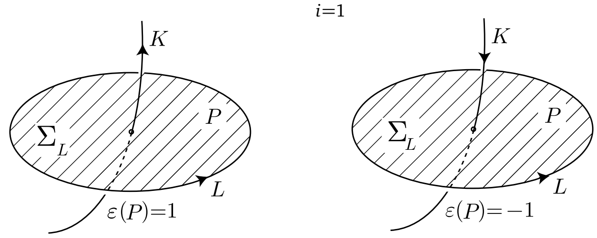

FIGURE 1. Calculation of the linking number.

Remark 2.2. The linking number can also be computed from a link diagram by the formula

$$\mathrm{lk}(L,K) = \frac{1}{2}(\#\text{ positive crossings} - \#\text{ negative crossings}),$$

from which we see that $\mathrm{lk}(L,K)$ is symmetric:

$$\mathrm{lk}(L,K) = \mathrm{lk}(K,L).$$

Example 2.3.

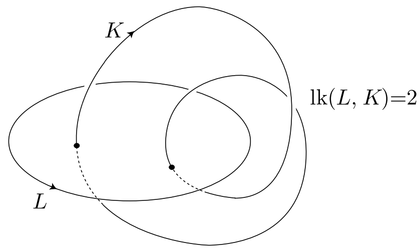

FIGURE 2. 2-component link with linking number 2.

We have seen that covering spaces provide a link between topological and arithmetic fundamental groups. Thus, it is natural to try to formulate the linking number in terms of covering spaces as a first step in establishing the analogy between linking numbers and Legendre symbols. We use the notation in Example 1.9 of Lecture 3: for a meridian α of L, let $\psi_\infty : G_L \to \mathbb{Z}$ be the surjective homomorphism sending α to 1, let X_∞ be the infinite cyclic covering of X_L corresponding to $\ker(\psi_\infty)$ and let τ denote the generator of $\mathrm{Gal}(X_\infty/X_L)$ corresponding to $1 \in \mathbb{Z}$. Let $\rho_\infty : G_L \twoheadrightarrow \mathrm{Gal}(X_\infty/X_L)$ be the natural surjection. We shall want to think of this as a monodromy permutation reprsentation.

Proposition 2.4. $\rho_\infty([K]) = \tau^{\mathrm{lk}(L,K)}$.

Proof. Recall (Example 1.9 of Lecture 3) that X_∞ is constructed by gluing together copies Y_i $(i \in \mathbb{Z})$ of the space Y obtained by cutting X_L along the Seifert surface Σ_L of L, as in Figure 3.

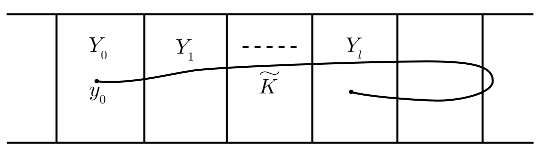

FIGURE 3. Lift $\tilde{K}$ of K to X_∞.

Let $\tilde{K}$ be a lift of K to X_∞. Then, when K crosses Σ_L with intersection number $\varepsilon = 1$ (respectively -1), $\tilde{K}$ crosses from Y_i to Y_{i+1} (respectively from Y_{i+1} to Y_i) for some i since Σ_i^+ is identified with Σ_{i+1}^- in X_∞. Therefore, if the starting point y_0 of $\tilde{K}$ is in Y_0, then its ending point lies in Y_l, $l = \mathrm{lk}(L,K)$, that is, $\rho_\infty([K])(y_0) \in Y_l$. Since τ maps Y_i onto Y_{i+1}, it follows that $\rho_\infty([K] = \tau^{\mathrm{lk}(L,K)}$. □

Let $\psi_2 : G_L \to \mathbb{Z}/2\mathbb{Z}$ be the composite of ψ_∞ with the surjection $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, and let $h_2 : X_2 \to X_L$ be the double covering corresponding to $\ker(\psi_2)$. Let $\rho_2 : G_L \twoheadrightarrow \mathrm{Gal}(X_2/X_L)$ be the natural surjection, then by Proposition 2.4, the image of $[K]$ in $\mathrm{Gal}(X_2/X_L) \cong \mathbb{Z}/2\mathbb{Z}$ under ρ_2 is given by $\mathrm{lk}(L,K) \bmod 2$. A similar argument to that in the proof of Proposition 2.4 tells us that

$$\rho_2([K])(y) = \text{ending point of a lift of } K \text{ with starting point } y.$$

We conclude that

$$\rho_2([K]) = \mathrm{id}_{X_2} \iff h_2^{-1}(K) = K_1 \cup K_2 \quad \text{(2-component link)};$$
$$\rho_2([K]) = \tau \iff h_2^{-1}(K) = \mathfrak{K} \quad \text{(knot in } X_2\text{)}.$$

We thus obtain the following result:

Proposition 2.5.

$$h_2^{-1}(K) = \begin{cases} K_1 \cup K_2 & \textit{if } \mathrm{lk}(L,K) \equiv 0 \pmod 2, \\ \mathfrak{K} & \textit{if } \mathrm{lk}(L,K) \equiv 1 \pmod 2. \end{cases}$$

Note that this result has the same form as the decomposition of prime numbers in the ring of Gaussian integers $\mathbb{Z}[i]$ that was established in the previous lecture: a prime $p \in \mathbb{Z}$ decomposes in $\mathbb{Z}[i]$ as

$$p = \begin{cases} \alpha\bar{\alpha} & \text{if } \left(\frac{-1}{p}\right) = 1, \\ p & \text{if } \left(\frac{-1}{p}\right) = -1. \end{cases}$$

We shall see that this result can be extended more generally to the ring of integers $\mathcal{O}_k$ of any quadratic field $\mathbb{Q}[\sqrt{q}]$.

Example 2.6.

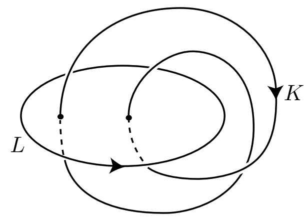

FIGURE 4. 2-component link $K \cup L$ with $\mathrm{lk}(L,K) = 2$.

Let $K \cup L$ be the two-component link in Figure 4. Since $\mathrm{lk}(L,K) = 2$, K decomposes in the two-sheeted cover X_2 of X_L as $h^{-1}(K) = K_1 \cup K_2$. We can see this pictorially as follows. The knot complement X_L is homeomorphic to a solid torus. (Imagine expanding the two linked solid tori in Figure 5 via a homeomorphism to fill up the ambient space.) The two-sheeted cover X_2 is obtained by slicing X_L along the disc bounded by L and gluing together two sliced copies of X_L, hence it is also a solid torus, with each of the copies of X_L "stretched out" to form half of it, as in Figure 6. (As a guide, the left intersection point of K with Σ_L in Figure 4 lifts to the left intersection point along the gluing boundary on the left in Figure 6, and the right intersection point along the gluing boundary on the right.) Thus, $h^{-1}(K)$ is a Hopf link in X_2.

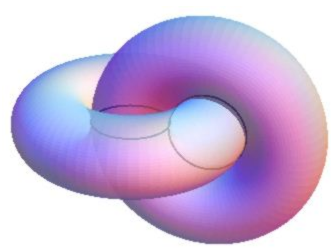

FIGURE 5. The complement of a solid torus in S^3 is homeomorphic to another solid torus.

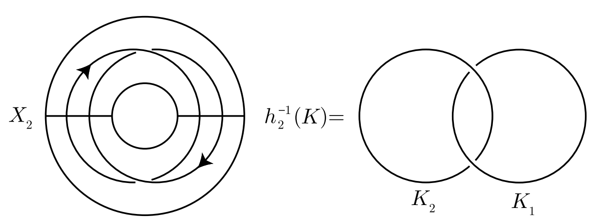

FIGURE 6. h^{-1} is a Hopf link in X_2.

Exercise 2.7. Let $K \cup L$ be the two-component link in Figure 7. What knot or link does K lift to in the two-sheeted cover X_2 of X_L?

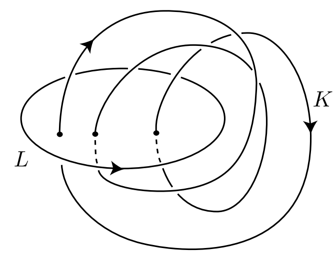

FIGURE 7

Exercise 2.8 (Optional)**.** Find a two-component link $K \cup L$ such that K lifts to a figure eight knot in the two-sheeted cover X_2 of X_L.