1. COVERING SPACES

In the previous lecture, we saw how one could obtain a presentation of the knot group from a knot projection. However, this is not the best way to study the knot group topologically, as the presentation obtained depends on the choice of knot projection. A more topological way of studying the knot group is provided by covering spaces.

Covering spaces enable one to study fundamental groups via their action on topological spaces, similarly to how group representations enable one to study abstract groups through their action on vector spaces. As we shall see, the fundamental group $\pi_1(X)$ of a space X can be thought of as a Galois (automorphism) group $\mathrm{Gal}(\tilde{X}/X)$, where $\tilde{X}$ is the universal covering space of X. This perspective gives rise to a fundamental analogy between topological and arithmetic fundamental/Galois groups in arithmetic topology.

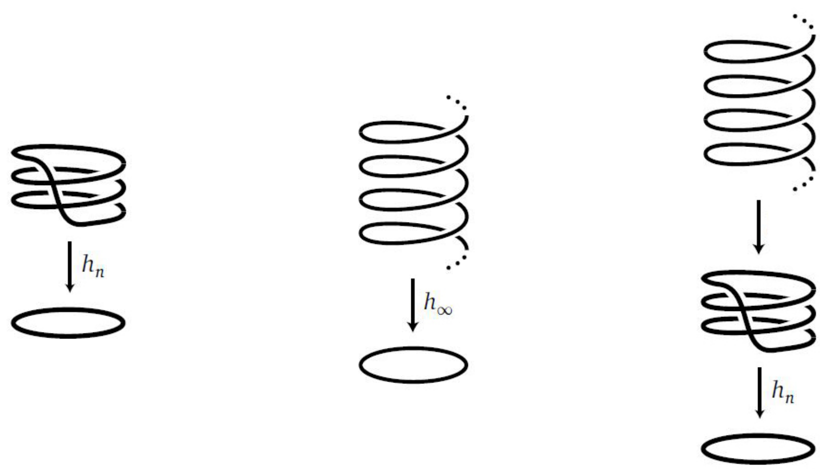
1.1. **Unramified coverings.** We recall the basic definitions and results regarding covering spaces. In what follows, we shall assume that the base space X is a connected topological manifold. (This will allow us to ignore technicalities about local path-connectedness and semi-locally simply-connectedness, which one can tell from the names will cause headaches.)

Definition 1.1. Let X be a space. A continuous map $h : Y \to X$ is called an *(unramified) covering* if for any $x \in X$, there is an open neighborhood U of x such that $h^{-1}(U)$ is a disjoint union of open sets in Y, each of which is mapped homeomorphically onto U by h. (Note that we do not require $h^{-1}(U_\alpha)$ to be non-empty, so h need not be surjective.) The set of automorphisms $Y \xrightarrow{\cong} Y$ over X forms a group, called the group of *covering transformations* of $h : Y \to X$, denoted by $\mathrm{Aut}(Y/X)$.

Example 1.2 (Coverings of S^1)**.**

- $h_n : S^1 \to S^1$, $h(z) = z^n$, where n is a positive integer and we view z as a complex number with $|z| = 1$. (Figure 1a shows $n = 3$.)
- $h_\infty : \mathbb{R}^1 \to S^1$, $h(t) = (\cos 2\pi t, \sin 2\pi t)$. (Figure 1b.)

In fact, the covering $h_\infty : \mathbb{R}^1 \to S^1$ actually covers the coverings $h_n : S^1 \to S^1$, as shown in Figure 1c. As we shall see later, $\mathbb{R}^1$ is the universal covering space of S^1, that is, it is a covering of any other (connected) covering space of S^1 (which turn out to be the finite coverings h_n).



(A) $h_n : S^1 \to S^1$ (B) $h_\infty : \mathbb{R}^1 \to S^1$ (C) h_n is a subcovering of h_∞

FIGURE 1. Coverings of S^1

Example 1.3. We consider a higher-dimensional example. Let Y be the closed orientable surface of genus 11, the "11-hole torus," as in Figure 2. This has 5-fold rotational symmetry generated by a rotation of angle $2\pi/5$, and hence an action of the cyclic group $\mathbb{Z}/5\mathbb{Z}$. The quotient space $X = Y/(\mathbb{Z}/5\mathbb{Z})$ is a surface of genus 3, obtained from one of the five subsurfaces by identifying two boundary circles C_i and C_{i+1}. Thus we have a covering space $M_{11} \to M_3$, where M_g denotes the closed orientable surface of genus g. This example clearly generalizes by replacing the 2 holes in each "arm" of M_{11} by m holes and the 5-fold symmetry by n-fold symmetry to give covering spaces $M_{mn+1} \to M_{m+1}$.

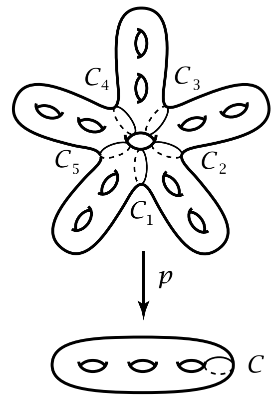


FIGURE 2. Covering of the 3-hole torus by the 11-hole torus.

In what follows, we shall restrict our attention to connected covering spaces, since a general covering space is just a disjoint union of connected ones.

Definition 1.4. A covering $h: Y \to X$ is called *Galois* or *normal* if for each $x \in X$ and each pair of lifts $\tilde{x}'$, $\tilde{x}'$ of x, there is a covering transformation taking $\tilde{x}$ to $\tilde{x}'$. For a Galois covering $h: Y \to X$, we call $\mathrm{Aut}(Y/X)$ the *Galois group* of Y over X and denote it by $\mathrm{Gal}(Y/X)$.

Intuitively, a Galois covering is one with maximal symmetry, in analogy with Galois extensions, which as splitting fields of polynomials, can be considered "maximally symmetric."

Recall that the main theorem of Galois theory gives a bijective correspondence between intermediate field extensions and subgroups of the Galois group. There is a similar version of the main theorem for coverings, which relates connected coverings of a given space X and subgroups of $\pi_1(X)$.

Theorem 1.5. *The induced map $h_*: \pi_1(Y, y) \to \pi_1(X, x)$ is injective, and there is a bijection*

$$\{\textit{connected coverings } h: Y \to X\}/\textit{isom.} \xrightarrow{\cong} \{\textit{subgroups of } \pi_1(X, x)\}/\textit{conj.}$$
$$(h: Y \to X) \mapsto h_*(\pi_1(Y, y)) \qquad\qquad (y \in h^{-1}(x))$$

with the property that $h: Y \to X$ is a Galois covering if and only if $h_*(\pi_1(Y, y))$ is a normal subgroup of $\pi_1(X, x)$. In this case $\mathrm{Gal}(Y/X) \cong \pi_1(X, x)/h_*(\pi_1(Y, y))$.

The covering $h: \tilde{X} \to X$ (up to isomorphism over X) which corresponds to the identity subgroup of $\pi_1(X, x)$ is called the *universal covering* of X; it is a covering space of any other covering space of X. Since the map $h_*: \pi_1(Y, y) \to \pi_1(X, x)$ is injective, $\pi_1(\tilde{X}) = 1$, i.e. the universal covering is simply-connected, and $\mathrm{Gal}(\tilde{X}/X) \cong \pi_1(X)$. The two most important types of covering spaces we shall consider are the universal covering space and the cyclic covering spaces.

Example 1.6. We return to Example 1.2, the coverings of S^1. We have seen that $\mathbb{R}^1 \to S^1$ is a covering. Since $\mathbb{R}^1$ is simply-connected, this tells us that this is in fact the universal covering of $\mathbb{R}^1 \to S^1$. Indeed, $\mathrm{Gal}(\mathbb{R}^1/S^1) \cong \mathbb{Z} \cong \pi_1(S^1)$, with a generator τ of $\mathrm{Gal}(\mathbb{R}^1/S^1)$ acting by a shift of the helix. Moreover, since the only quotient groups of $\mathbb{Z}$ are the finite cyclic groups $\mathbb{Z}/n\mathbb{Z}$ and we have found coverings $h_n: S^1 \to S^1$ with exactly these Galois groups, this tells us that these are the only coverings of S^1.

Theorem 1.5 tells us that instead of using the Wirtinger presentation, one can also study a knot group $G_K = \pi_1(X_K)$ by studying the covering spaces of the knot exterior X_K. While the universal covering space of a knot exterior depends on the particular knot, there is a uniform procedure for constructing the cyclic coverings of a knot exterior. The idea is to reverse the reasoning in Example 1.3: instead of taking one of several copies of a manifold with boundary and gluing the boundaries together, we want to slice the base space open and glue several copies of the resulting space together along the boundaries appropriately.

How can we slice the knot complement in a natural way? If K is the unknot, then there is an obvious way to slice the knot complement such that the knot figures prominently: we slice along the intersection of the knot complement with the disk bounded by the unknot. Is it possible to do this for a general knot K? The answer is in the affirmative: Seifert showed in 1934 that for every knot or link, there exists an compact oriented connected surface, called a *Seifert surface*, whose boundary is that knot or link.

Example 1.7.

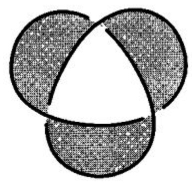


FIGURE 3. Möbius band with three half-twists, with boundary a trefoil. Note that the Möbius band is not considered to be a Seifert surface for the trefoil as it is not orientable.

Exercise 1.8. Figure 4 shows a Seifert surface for a knot, since this surface is compact, oriented and connected. What knot is this a Seifert surface of? What type of surface is this? (Hint: use Euler characteristic and the classification theorem for surfaces by genus, orientability and number of boundary components.)

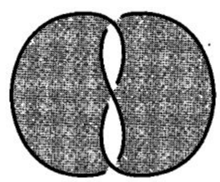


FIGURE 4

Example 1.9 (Infinite cyclic covering). Let $K \subseteq S^3$ be a knot. From a Wirtinger presentation for G_K, one sees that $x_1, \ldots, x_n$ are mutually conjugate. Hence the abelianization $G_K/[G_K, G_K]$ is an infinite cyclic group generated by the class of a meridian α of K. Let $\psi_\infty : G_K \to \mathbb{Z}$ be the surjective homomorphism sending α to 1, and let $h_\infty : X_\infty \to X_K$ be the covering corresponding to $\ker(\psi_\infty)$ in Theorem 1.5. The covering space X_∞ is independent of the choice of α and is called the *infinite cyclic covering* of X_K. It is constructed as follows. Let Σ_K be a Seifert surface of K. Let Y be the space obtained by cutting X_K along $X_K \cap \Sigma_K$, and let Σ^+ and Σ^- be the two surfaces homeomorphic to $X_K \cap \Sigma_K$ obtained from the cut, as in Figure 5.

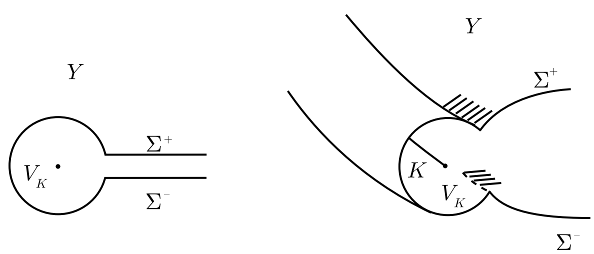


FIGURE 5. Copies Σ^+ and Σ^- of $X_K \cap \Sigma_K$ obtained by cutting along $X_K \cap \Sigma_K$.

Let Y_i ($i \in \mathbb{Z}$) be copies of Y. The space X_∞ is obtained from the disjoint union of all the Y_i's by identifying Σ_i^+ with Σ_{i+1}^- ($i \in \mathbb{Z}$), as in Figure 6, and a generator τ of $\mathrm{Gal}(X_\infty/X_K)$ is given by the shift sending Y_i to Y_{i+1} ($i \in \mathbb{Z}$).

Example 1.10 (Finite cyclic covering). For each $n \in \mathbb{N}$, let $\psi_n : G_K \to \mathbb{Z}/n\mathbb{Z}$ be the composite of ψ_∞ with the surjection $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, and let $h_n : X_n \to X_K$ be the covering corresponding to $\ker(\psi_n)$. Then $\mathrm{Gal}(X_n/X_K) \cong \mathbb{Z}/n\mathbb{Z}$. The covering spaces X_n are constructed similarly to X_∞, except that we now take n copies $Y_0, \ldots, Y_{n-1}$ of Y, and identify Σ_{n-1}^+ with Σ_0^- instead, as in Figure 7. A generator τ of $\mathrm{Gal}(X_n/X_K)$ corresponding to 1 mod $n\mathbb{Z}$ is given by the shift sending Y_i to Y_{i+1} ($i \in \mathbb{Z}/n\mathbb{Z}$).

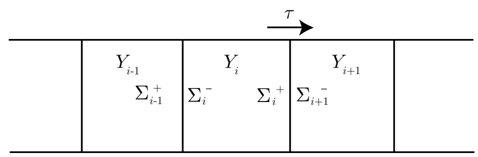


FIGURE 6. Space X_∞ obtained from the disjoint union of the Y_i's by identifying Σ_i^+ with Σ_{i+1}^- ($i \in \mathbb{Z}$)

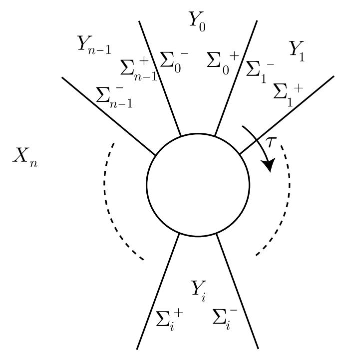


FIGURE 7. Space X_∞ obtained from the disjoint union of $Y_0, \ldots, Y_{n-1}$ by identifying Σ_i^+ with Σ_{i+1}^- ($i \in \mathbb{Z}/n\mathbb{Z}$)

1.2. **Ramified coverings.** Above, we considered (unramified) coverings of the knot exterior X_K. However, we may wish to consider a covering of the entire space S^3 extending the above coverings. This is accomplished by ramified or branched coverings.

Let M and N be n-manifolds ($n \geq 2$) and let $f : N \to M$ be a continuous map. Define $S_N := \{y \in N \mid f \text{ is not a homeomorphism in a neighborhood of } y\}$ and $S_M := f(S_N)$.

Definition 1.11. The map $f : N \to M$ is called a *covering ramified over* S_M if the following conditions are satisfied:

(1) $f|_{N \setminus S_N} : N \setminus S_N \to M \setminus S_M$ is an (unramified) covering, and

(2) for any $y \in S_N$, there exist neighborhoods V of y and U of $f(y)$, and homeomorphisms $\varphi : V \xrightarrow{\approx} D^2 \times D^{n-2}$ and $\psi : U \xrightarrow{\approx} D^2 \times D^{n-2}$, such that $(g_e \times \mathrm{id}_{D^{n-2}}) \circ \varphi = \psi \circ f$ for some positive integer $e = e(y)$, where $g_e(z) := z^e$ for $z \in D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$. (That is, f acts locally like the eth-power map.)

One can "complete" an unramified covering to obtain a ramified covering, as follows.

Example 1.12 (Fox completion). Consider the n-fold cyclic covering $h_n : X_n \to X_K$ from Example 1.10. The restriction $h_n|_{\partial X_n} : \partial X_n \to \partial X_K$ is an n-fold covering of tori and $n\alpha$ is a meridian of ∂X_n. Attach $V = D^2 \times S^1$ to X_n by gluing ∂V and ∂X_n in such a way that a meridian of ∂V coincides with $n\alpha$. Denote by M_n the closed 3-manifold obtained in this manner.

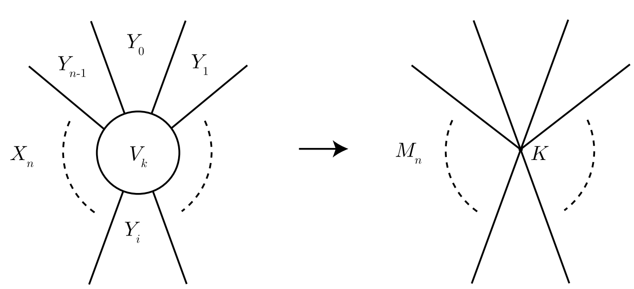


FIGURE 8. Fox completion M_n of X_n

Define $f_n : M_n \to S^3$ by $f_n|_{X_n} := h_n$ and $f_n|_V := f_n \times \mathrm{id}_{S^1}$. Then f_n is a covering ramified over K, which we call the *Fox completion* of $h_n : X_n \to X_K$.