1. The Alexander polynomial in terms of homology

Let K be a knot in S^3. We have noted that the first homology group $H_1(X_K)$ of the knot exterior X_K is not useful in distinguishing knots, since it is always isomorphic to $\mathbb{Z}$. However, the situation is different if we consider the infinite cyclic covering $h_\infty : X_\infty \to X_K$: since $\mathrm{Gal}(X_\infty/X_K) \cong \mathbb{Z}$, we see that h_∞ corresponds to the abelianization $\psi : G_K \to G_K^{\mathrm{ab}} \cong \mathbb{Z} \cong \mathrm{Gal}(X_\infty/X_K)$, so $\pi_1(X_\infty) \cong \ker\psi \cong [G_K, G_K]$ and $H_1(X_\infty) \cong [G_K, G_K]^{\mathrm{ab}}$ is in general not a constant group.

We can attempt to understand the group $H_1(X_\infty)$ by considering the long exact sequence in homology associated to the pair $(X_\infty, h_\infty^{-1}(x_0))$, where x_0 is a fixed basepoint in X_K:

$$\cdots \to H_1(h_\infty^{-1}(x_0)) \to H_1(X_\infty) \to H_1(X_\infty, h_\infty^{-1}(x_0)) \to H_0(h_\infty^{-1}(x_0)) \to H_0(X_\infty) \to H_0(X_\infty, h_\infty^{-1}(x_0)).$$

- $\underline{H_1(h_\infty^{-1}(x_0))}$: since $h_\infty^{-1}(x_0)$ consists of a discrete set of points, $H_1(h_\infty^{-1}(x_0)) = 0$.
- $\underline{H_0(h_\infty^{-1}(x_0))}$: $H_0(h_\infty^{-1}(x_0))$ is the free abelian group on the set of points $h_\infty^{-1}(x_0)$, which are indexed by elements of $G_K^{\mathrm{ab}} \cong \mathbb{Z}$, so $H_0(h_\infty^{-1}(x_0)) \cong \mathbb{Z}[G_K^{\mathrm{ab}}] \cong \mathbb{Z}[t, t^{-1}] =: \Lambda$. (Here t corresponds to the class of a meridian α that generates G_K^{ab}.)
- $\underline{H_0(X_\infty)}$: since X_∞ is connected, $H_0(X_\infty) \cong \mathbb{Z}$.
- $\underline{H_0(X_\infty, h_\infty^{-1}(x_0))}$: by the CW approximation theorem, we can approximate the pair $H_0(X_\infty, h_\infty^{-1}(x_0))$ by a CW pair (X, A) that is weakly homotopy equivalent and which will thus have the same homology groups. But $H_0(X_\infty, h_\infty^{-1}(x_0)) \oplus \mathbb{Z} \cong H_0(X, A) \oplus \mathbb{Z} \cong H_0(X/A) \cong Z$ since X/A is connected, so $H_0(X_\infty, h_\infty^{-1}(x_0)) = 0$.
- $\underline{H_1(X_\infty, h_\infty^{-1}(x_0))}$: here we use the fact that relative cycles are (equivalence classes of) chains in $C_n(X_\infty)$ whose boundary lies in $h_\infty^{-1}(x_0)$. For $g = [l] \in G_K$, let $\tilde{l}$ denote the lift of l with starting point $y_0 \in h_\infty^{-1}(x_0)$. Then $\tilde{l} \in C_1(X_\infty, h_\infty^{-1}(x_0))$ and we have a map
$$d : G_K \to H_1(X_\infty, h_\infty^{-1}(x_0)), \quad d(g) := [\tilde{l}].$$
For $g_1 = [l_1]$, $g_2 = [l_2] \in G_K$, let $\tilde{l_1}$, $\tilde{l}_2$ and $\widetilde{l_1 \cdot l_2}$ be the lifts of l_1, l_2 and $l_1 \cdot l_2$ respectively with starting point y_0, and let $\tilde{l}_2'$ be the lift of l_2 whose starting point is the ending point of l_1. Then $d(g_1 g_2) = [\widetilde{l_1 \cdot l_2}] = [\tilde{l}_1] + [\tilde{l}_2']$ and $d(g_1) + \psi(g_1)d(g_2) = [\tilde{l}_1] + \psi(g_1)[\tilde{l}_2] = [\tilde{l}_1] + [\tilde{l}_2']$. Hence $d(g_1 g_2) = d(g_1) + \psi(g_1)d(g_2)$ in $H_1(X_\infty, h_\infty^{-1}(x_0))$.

Moreover, each of the above are (left) $\mathbb{Z}[G_K^{\mathrm{ab}}]$-modules. This is an appropriate time to pause and introduce the following definition.

Definition 1.1. Let A_K be the quotient module of the (left) free $\mathbb{Z}[G_K^{\mathrm{ab}}]$-module $\bigoplus_{g \in G_K} \mathbb{Z}[G_K^{\mathrm{ab}}]dg$ on the symbols dg $(g \in G_K)$ by the (left) $\mathbb{Z}[G_K^{\mathrm{ab}}]$-submodule generated by elements of the form $d(g_1 g_2) - dg_1 - \psi(g_1)d(g_2)$ for g_1, $g_2 \in G_K$:

$$A_K := \left(\bigoplus_{g \in G_K} \mathbb{Z}[G_K^{\mathrm{ab}}]dg \right) / \langle d(g_1 g_2) - dg_1 - \psi(g_1)d(g_2)\,(g_1, g_2 \in G_K) \rangle_{\mathbb{Z}[G_K^{\mathrm{ab}}]}.$$

By definition, the map $d : G \to A_K$ defined by the correspondence $g \mapsto dg$ is a ψ-differential, namely for g_1, $g_2 \in G$, one has
$$d(g_1 g_2) = d(g_1) + \psi(g_1)d(g_2),$$
and A_K is universal for this property in the sense that for any (left) $\mathbb{Z}[G_K^{\mathrm{ab}}]$-module A and any ψ-differential $\partial : G \to A$, there exists a unique $\mathbb{Z}[G_K^{\mathrm{ab}}]$-homomorphism $\varphi : A_K \to A$ such that $\varphi \circ d = \partial$.

Fact 1.2. There is an exact sequence of (left) $\mathbb{Z}[G_K^{\mathrm{ab}}]$-modules

$$0 \to [G_K, G_K]^{\mathrm{ab}} \xrightarrow{\theta_1} A_K \xrightarrow{\theta_2} \mathbb{Z}[G_K^{\mathrm{ab}}] \xrightarrow{\epsilon_{\mathbb{Z}[G_K^{\mathrm{ab}}]}} \mathbb{Z} \to 0$$

called the *Crowell exact sequence*, where θ_1 is the homomorphism induced by $n \mapsto dn$ $(n \in [G_K, G_K])$, θ_2 is the homomorphism induced by $dg \mapsto \psi(g) - 1$ $(g \in G_K)$ and $\epsilon_{\mathbb{Z}[G_K^{\mathrm{ab}}]}(\sum a_g g) := \sum a_g$.

One can check that the isomorphisms above commute with the maps in the Crowell exact sequence and the homology exact sequence, and hence that the Crowell exact sequence is simply the homology exact sequence in another guise.

Definition 1.3. The $\mathbb{Z}[G_K^{\text{ab}}]$-module module A_K is called the *Alexander module* of the knot K.

Using the Fox free differential calculus, one can give an explicit resolution for the Alexander module A_K.

Definition 1.4. Let F be the free group on the generators $x_1, x_2, \ldots, x_m$. The *Fox free derivative* $\frac{\partial}{\partial_{x_i}} : \mathbb{Z}[F] \to \mathbb{Z}[F]$ is defined by the following axioms:

- $\dfrac{\partial}{\partial x_i}(u+v) = \dfrac{\partial}{\partial x_i}u + \dfrac{\partial}{\partial x_i}v$ for any $u, v \in \mathbb{Z}[F]$,
- $\dfrac{\partial}{\partial x_i}e = 0$,
- $\dfrac{\partial}{\partial x_i}x_j = \partial_{ij}$ where ∂_{ij} is the Kronecker delta,
- $\dfrac{\partial}{\partial x_i}(uv) = \dfrac{\partial}{\partial x_i}u + u\dfrac{\partial}{\partial x_i}v$ for any $u, v \in F$.

One can check that this system of axioms is consistent. As a consequence of the axioms, we also have the following formula for inverses:

$$\frac{\partial}{\partial x_i}u^{-1} = -u^{-1}\frac{\partial}{\partial x_i}u \quad \text{for any } u \in F.$$

Theorem 1.5. *Let $G_K = \langle x_1, \ldots, x_m \mid R_1, R_2, \ldots, R_{m-1} \rangle$ be a presentation of the knot group G_K (e.g. a Wirtinger presentation). Let F be the free group on $x_1, \ldots, x_m$, and let $\pi : F \to G_K$ be the natural homomorphism. (We shall also denote by π the induced map of group rings $\mathbb{Z}[F] \to \mathbb{Z}[G_K]$.) The Alexander module A_K has a free resolution over $\mathbb{Z}[G_K^{ab}]$:*

$$\mathbb{Z}[G_K^{ab}]^{m-1} \xrightarrow{Q} \mathbb{Z}[G_K^{ab}]^m \to A_K \to 0.$$

Here the $(m-1) \times m$ presentation matrix Q_K, called the Alexander matrix *of K, is given by*

$$Q_K = \left((\psi \circ \pi) \left(\frac{\partial R_i}{\partial x_j} \right) \right)_{ij} \in \mathbb{Z}[G_K^{ab}]^{(m-1)\times m} \cong \Lambda^{(m-1)\times m}.$$

Note that the Alexander matrix depends on a choice of presentation for G_K!

Example 1.6 (Alexander matrix for the trefoil)**.** Let us compute Alexander matrices for the trefoil using two different presentations of the knot group: $\langle a, b \mid aba - bab \rangle$ and $\langle x, y \mid x^3 - y^2 \rangle$.

- $\langle a, b \mid aba - bab \rangle$:
$$\frac{\partial}{\partial a} = 1 + ab - b, \quad \frac{\partial}{\partial b} = -1 - ba + a,$$
so the associated Alexander matrix is $[\psi(1+ab-b) \quad \psi(-1-ba-a)] = [1+t^2-t \quad -1-t^2+t]$. (Abelianizing sends both a and b to t.)
- $\langle x, y \mid x^3 - y^2 \rangle$:
$$\frac{\partial}{\partial x} = 1 + x + x^2, \quad \frac{\partial}{\partial y} = -1 - y,$$
so the associated Alexander matrix is $[\psi(1+x+x^2) \quad \psi(-1-y)] = [1+t^2+t^4 \quad -1-t^3]$. (Here one needs to be careful: abelianizing sends x to t^3 and y to t^2!)

For a commutative ring R and a finitely presented R-module M, let

$$R^s \xrightarrow{Q} R^r \to M \to 0$$

be a free resolution of M over R with presentation matrix Q, and define $E(M)$ to be the ideal of R generated by the maximal minors of Q.

Definition 1.7. The *Alexander ideal* is the ideal $E(A_K)$ of $\mathbb{Z}[G_K^{\text{ab}}] \cong \Lambda$ generated by the $(m-1)$-minors of Q_K.

Note that the Alexander ideal can be defined because G_K^{ab} is abelian by definition. It is a theorem of Crowell and Fox that the Alexander ideal is independent of a choice of a free resolution for A_K, unlike the Alexander matrix. Moreover, Alexander proved that the Alexander ideal is always a principal ideal (although $\mathbb{Z}[t, t^{-1}]$ is *not* a PID!). Thus a generator of the Alexander ideal is defined up to multiplication by a unit of Λ, namely $\pm t^n$ for some integer n.

Example 1.8 (Alexander ideal for the trefoil)**.** It is clear from the first part of Example 1.6 that the Alexander ideal of the trefoil is $\langle 1-t+t^2\rangle$. We can also see this from the second part of Example 1.6 by the calculations $1+t^2+t^4=(1-t+t^2)(1+t+t^2)$, $(1+t^3)=(1-t+t^2)(1+t)$.

Definition 1.9. The *Alexander polynomial* $\Delta_K(t)$ of a knot K is a generator of $E(A_K)$ (and hence is defined up to multiplication by $\pm t^n$ for some integer n).

Example 1.10. The Alexander polynomial of the trefoil is $\Delta_K(t) = 1-t+t^2$. Note that this matches our computation using skein relations up to a factor of t.

Exercise 1.11. Compute the Alexander polynomial of the figure eight knot, using both skein relations and the Fox free derivative.

Working through the definition of the Alexander ideal using a Wirtinger presentation for K, one can show that the effect of taking the mirror image of a knot K is to make the substitution $t \longleftrightarrow t^{-1}$. On the other hand, by the theorem of Crowell and Fox and the definition in terms of the first homology group of X_∞, the Alexander ideal does not depend on the chirality of K. Hence we conclude that the Alexander polynomial is symmetric, that is, up to multiplication by t^n, it has the form $a_r t^{-r}+\cdots+a_1t^{-1}+a_0+a_1t+\cdots+a_rt^r$.