THE JONES POLYNOMIAL AND KHOVANOV HOMOLOGY

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0. Introduction

A fundamental problem in knot theory is to determine whether two knots—or more generally, two links—are equivalent (up to ambient isotopy). A standard approach to this type of topological problem is to find suitable algebraic invariants of knots or links, e.g., the knot group and Alexander polynomial which we have seen in lecture. The Jones polynomial is another one of these invariants, a Laurent polynomial with integer coefficients like the Alexander polynomial, which can be assigned to any oriented link [5]. The first goal of the paper will be to define and understand the Jones polynomial; we do this in §2.

There is also a generalisation (a "categorification") of the Jones polynomial known as Khovanov homology. Here, instead of assigning a polynomial, we assign a complex of graded vector spaces to oriented links. It turns out that the homology of this complex is a link invariant, and that the graded Euler characteristic—an invariant of the homology, so also a link invariant—is in fact the Jones polynomial, up to a normalising factor [2]. The second goal of this paper will be to define and understand Khovanov homology and its relationship to the Jones polynomial; we do this in §3.

We perhaps should mention, finally, that it seems (cf. [7, 8]) that the Jones polynomial and Khovanov homology have great physical significance as well, which further motivates their study. Unfortunately though, this won't be discussed in this paper.

1. Basic notions

Before addressing the main topics of the paper in §§2–3, we include some notes here reviewing the basic definitions and facts from knot theory that will be used later on. Some of the notation and terminology is nonstandard, but seems to be more explicit than the standard abuse of notation and terminology; since the goal of this section of the paper is to clarify the foundational concepts of knot theory, being more explicit should hopefully be helpful and justified here. Proofs will not be included here, but can be found in, e.g., [3, 4, 1].

Convention 1.1. We fix throughout an (arbitrary) orientation on $\mathbf{R}^3$.

Definition 1.2. A *link* L (of $n \in \mathbf{N}$ *components*) is an embedding $L : \coprod_{i=1}^n S^1 \hookrightarrow \mathbf{R}^3$ of n disjoint circles into n disjoint closed curves $\gamma_i \subset \mathbf{R}^3$. A *knot* K is simply a link of 1 component, that is, an embedding $S^1 \hookrightarrow \mathbf{R}^3$. An *orientation* $\mathcal{O}$ on L is a choice of orientation on $\coprod_{i=1}^n S^1$, or equivalently a choice for each $1 \leq i \leq n$ of one of the two orientations on the curve γ_i. A pair $(L, \mathcal{O})$ of a link L and an orientation $\mathcal{O}$ on L is called an *oriented link*. We say a link L is *tame* if L has a tubular neighbourhood, i.e., there is an embedding $\widetilde{L} : \coprod_{i=1}^n S^1 \times D^2 \hookrightarrow \mathbf{R}^3$ of n disjoint solid tori into 3-space which extends L, in the obvious sense.

Convention 1.3. For the remainder of this paper, *all links will be assumed to be tame.*

Definition 1.4. Let L, L' be links. An *ambient isotopy* from L to L' consists of a family of homeomorphisms $\{h_t\}_{0 \leq t \leq 1}$ of $\mathbf{R}^3$ satisfying the following conditions:

(1) the map $h : \mathbf{R}^3 \times [0,1] \longrightarrow \mathbf{R}^3$ sending $(x,t) \longmapsto h_t(x)$ is continuous,
(2) $h_0 = \mathrm{id}_{\mathbf{R}^3}$, and
(3) $h_1 \circ L = L'$.

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Let $\mathcal{O}, \mathcal{O}'$ be orientations on L, L', respectively. An *ambient isotopy (of oriented links)* from $(L, \mathcal{O})$ to $(L', \mathcal{O}')$ consists of an ambient isotopy $\{h_t\}_{0 \le t \le 1}$ from L to L' such that the orientation $\mathcal{O}_{\mathrm{ind}}$ induced on L' by $\mathcal{O}$ and h_1 agrees with $\mathcal{O}'$.

The above clearly gives us an equivalence relation on links (resp. oriented links): L (resp. $(L, \mathcal{O})$) and L' (resp. $(L', \mathcal{O}')$) are *ambient isotopic*, or simply *equivalent*, if and only if there exists an ambient isotopy from L (resp. $(L, \mathcal{O})$) to L' (resp. $(L', \mathcal{O}')$). The equivalence class of L (resp. $(L, \mathcal{O})$) under ambient isotopy is called the *link type* (resp. *oriented link type*) of L (resp. $(L, \mathcal{O})$), and is denoted $[L]$ (resp. $[(L, \mathcal{O})]$).

Definition 1.5. Let $L : \coprod_{i=1}^{n} S^1 \hookrightarrow \mathbf{R}^3$ be a link. A *projection* $\pi : \mathrm{im}(L) \longrightarrow \mathbf{R}^2$ of L is a map induced by a projection of $\mathbf{R}^3$ onto a plane $\Pi \subset \mathbf{R}^3$. Let $\mathcal{X} := \{x \in \mathbf{R}^2 \mid |\pi^{-1}(x)| > 1\}$ be the set of *crossings* of π. We say π is a *regular projection* if

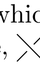
(1) $\mathcal{X}$ is finite, and
(2) for each $x \in \mathcal{X}$, $|\pi^{-1}(x)| = 2$, and the two curve segments in $\mathrm{im}(\pi)$ intersecting at x do so transversely.

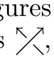
If $\mathrm{im}(L)$ is the disjoint union of the closed curves γ_i, then clearly $\mathrm{im}(\pi)$ will be the union of the closed (though perhaps self-intersecting) curves $\pi(\gamma_i)$. An *orientation* $\mathcal{O}$ on π is a choice of one of the two orientations, for each $1 \le i \le n$, on the curve $\pi(\gamma_i)$. A pair $(\pi, \mathcal{O})$ of a projection π of L and an orientation $\mathcal{O}$ on π is called an *oriented projection* of L; if π is regular, we say $(\pi, \mathcal{O})$ is a *regular oriented projection.* If $\mathcal{O}_L$ is an orientation on L, then clearly there is an induced orientation $\mathcal{O}_{\mathrm{ind}}$ on any projection π on L.

Remark 1.6. It is evident that even in regular projections of a link we lose some information about the link: the relative heights (with respect to the projection plane Π) of two points which project to a crossing. This leads us to the following definition.

Definition 1.7. Let L be a link. A *link diagram* of L is a pair $(\pi, \mathcal{A})$ of a regular projection π of L and a *crossing assignment* $\mathcal{A}$ on π, that is, for each crossing x of π, a choice of which of the two curve segments intersecting at x lies below the other.

Determining this data $\mathcal{A}$ from L is easy. Suppose π is induced by projection onto the plane $\Pi \subset \mathbf{R}^3$; since we have fixed an orientation on $\mathbf{R}^3$ there is a canonical choice of unit normal vector n to Π. Then if $p, q \in \mathrm{im}(L)$ are distinct points such that $\pi(p) = \pi(q)$, we must have $p - q = cn$ for some $c \in \mathbf{R} - \{0\}$. Of course then $c > 0$ implies p and its curve segment lie above q and its curve segment, and $c < 0$ implies the opposite.

We illustrate the assignment $\mathcal{A}$ in figures by breaking the curve which lies below the other at the crossing and leaving the other curve solid. E.g., if a diagram locally looks like, , then the crossing is the centre point and the segment passing between the top-left and bottom-right lies below the segment passing between the bottom-left and top-right.

An orientation $\mathcal{O}$ on a link diagram $(\pi, \mathcal{A})$ is simply an orientation $\mathcal{O}$ on π. We call the triple $(\pi, \mathcal{A}, \mathcal{O})$ an *oriented link diagram.* We illustrate orientation in figures using arrows to indicate direction in the obvious way. E.g., if the above local diagram is annotated as , then the top curve segment is oriented from the bottom-left to the top-right and the bottom curve segment is oriented from the bottom-right to the top-left.

Fact 1.8. *Every link (resp. oriented link) has a link diagram (resp. oriented link diagram).*

Remark 1.9. Continuing our thought from Remark 1.6: it is also evident, though, that for regular projections, this is the only information we lose. That is, if we have a link diagram, then we know the relative heights at each crossing. Since heights can clearly then be adjusted via ambient isotopy as long as the relative heights at crossings are maintained, it follows that a link diagram determines a link up to ambient isotopy. Of course this holds with orientations equipped as well. We restate this as the following fact.

Fact 1.10. *The link type* $[L]$ *(resp. oriented link type* $[(L, \mathcal{O})]$*) of a link (resp. oriented link)* L *(resp.* $(L, \mathcal{O})$*) is determined by any link diagram (resp. oriented link diagram) of* L *(resp.* $(L, \mathcal{O})$*).*

We have a notion of isotopy-equivalence for link diagrams completely analogous to the notion of ambient isotopy of links discussed earlier.

Definition 1.11. Let $(\pi, \mathcal{A}), (\pi', \mathcal{A}')$ be link diagrams. A *planar isotopy* from $(\pi, \mathcal{A})$ to $(\pi', \mathcal{A}')$ consists of a family of homeomorphisms $\{h_t\}_{0 \le t \le 1}$ of $\mathbf{R}^2$ satisfying the following conditions:

(1) the map $h : \mathbf{R}^2 \times [0, 1] \longrightarrow \mathbf{R}^2$ sending $(x, t) \longmapsto h_t(x)$ is continuous,
(2) $h_0 = \mathrm{id}_{\mathbf{R}^2}$, and

(3) $h_1 \circ \pi = \pi'$, such that the crossing assignment $\mathcal{A}_{\text{ind}}$ induced on π' by $\mathcal{A}$ and h_1 agrees with $\mathcal{A}'$.

Let $\mathcal{O}, \mathcal{O}'$ be orientations on π, π', respectively. A *planar isotopy (of oriented link diagrams)* from $(\pi, \mathcal{A}, \mathcal{O})$ to $(\pi', \mathcal{A}', , \mathcal{O}')$ consists of an ambient isotopy $\{h_t\}_{0 \le t \le 1}$ from $(\pi, \mathcal{A})$ to $(\pi', \mathcal{A}')$ such that the orientation $\mathcal{O}_{\text{ind}}$ induced on π' by $\mathcal{O}$ and h_1 agrees with $\mathcal{O}'$.

The above clearly gives us an equivalence relation on link diagrams (resp. oriented link diagrams): $(\pi, \mathcal{A})$ (resp. $(\pi, \mathcal{A}, \mathcal{O})$) and $(\pi', \mathcal{A}')$ (resp. $(\pi', \mathcal{A}', \mathcal{O}')$) are *planar isotopic* if and only if there exists a planar isotopy from $(\pi, \mathcal{A})$ (resp. $(\pi, \mathcal{A}, \mathcal{O})$) to $(\pi', \mathcal{A}')$ (resp. $(\pi', \mathcal{A}', \mathcal{O}')$). The equivalence class of $(\pi, \mathcal{A})$ (resp. $(\pi, \mathcal{A}, \mathcal{O})$) is denoted $[(\pi, \mathcal{A})]$ (resp. $[(\pi, \mathcal{A}, \mathcal{O})]$).

Notation 1.12. Denote by **Dgm** the set of equivalence classes of link diagrams modulo planar isotopy and by **OrDgm** the set of equivalence classes of oriented link diagrams modulo planar isotopy.

Now, clearly it is much easier to visualise and think about link diagrams than to do so about links themselves. It would be supremely convenient, given two link diagrams, to have a necessary and sufficient condition on the diagrams for their links to be equivalent. In fact, there indeed exists such a condition!

Definition 1.13. We define the *Reidemeister moves* $\Omega_{i,j}, 1 \le i \le 3, 1 \le j \le 2$ such that, for $D = [(\pi, \mathcal{A})] \in$ **Dgm**, $\Omega_{i,j}(D) = [(\pi'_{i,j}, \mathcal{A}'_{i,j})] \in$ **Dgm**, where $(\pi'_{i,j}, \mathcal{A}'_{i,j})$ is the link diagram which is identical to $(\pi, \mathcal{A})$ except for one local change, as depicted in the following figure.

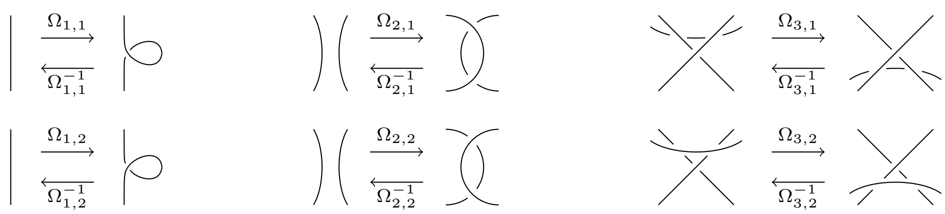


FIGURE 1. Reidemeister moves $\Omega_{i,j}$

For $D = [(\pi, \mathcal{A}, \mathcal{O})] \in$ **OrDgm**, we define the Reidemeister moves such that $\Omega_{i,j}(D) = [(\pi'_{i,j}, \mathcal{A}'_{i,j}, \mathcal{O}'_{i,j})] \in$ **OrDgm**, where $\pi'_{i,j}, \mathcal{A}'_{i,j}$ are as above and $\mathcal{O}'_{i,j}$ is the obvious orientation induced on $(\pi'_{i,j}, \mathcal{A}'_{i,j})$ by $\mathcal{O}$.

We use these Reidemeister moves to define an equivalence relation on **Dgm** and **OrDgm**.

Definition 1.14. Let $D, D' \in$ **Dgm** (resp. $D, D' \in$ **OrDgm**). We say D, D' are *equivalent* if and only if there is a finite sequence $(\Omega^{s_k}_{i_k, j_k})_{1 \le k \le N}$, $s_k \in \{\pm 1\}, 1 \le i_k \le 3, 1 \le j_k \le 2, N \in \mathbf{N}$, of Reidemeister moves and their inverses such that

$$(\Omega^{s_N}_{i_N, j_N} \circ \Omega^{s_{N-1}}_{i_{N-1}, j_{N-1}} \circ \cdots \circ \Omega^{s_1}_{i_1, j_1})(D) = D'.$$

It is clear that this is indeed an equivalence relation on **Dgm** (resp. **OrDgm**).

Fact 1.15. *Let L, L' (resp. $(L, \mathcal{O}), (L', \mathcal{O}')$) be links (resp. oriented links) with link diagrams $(\pi, \mathcal{A}), (\pi', \mathcal{A}')$ (resp. oriented link diagrams $(\pi, \mathcal{A}, \mathcal{O}_{\text{ind}}), (\pi', \mathcal{A}', \mathcal{O}'_{\text{ind}})$), respectively. Let $D := [(\pi, \mathcal{A})], D' := [(\pi', \mathcal{A}')]$ (resp. $D := [(\pi, \mathcal{A}, \mathcal{O}_{\text{ind}})], D' := [(\pi', \mathcal{A}', \mathcal{O}'_{\text{ind}})]$). Then L, L' are equivalent if and only if D, D' are equivalent.*

Due to this very useful characterisation of link equivalence, we will often be dealing with diagrams which are identical everywhere except in a small neighbourhood. Thus the following notational convention will be useful as well.

Notation 1.16. Suppose we have some set S and a (set) map F : **Dgm** $\longrightarrow S$. Assume there exists $G : S \longrightarrow S$ such that for $D = [(\pi, \mathcal{A})] \in$ **Dgm** we have a relation

$$(*) \qquad F(D) = G(F(D_1), F(D_2), \ldots, F(D_r)),$$

where, for $1 \le i \le r$, $D_i = [(\pi'_i, \mathcal{A}'_i)]$ such that $(\pi'_i, \mathcal{A}'_i)$ is identical to $(\pi, \mathcal{A})$ except for some local change Ψ_i (in the same sense of the Reidemeister moves $\Omega_{i,j}$). Then we will denote this relation for general D by replacing D and $D_i, 1 \le i \le r$ in $(*)$ with an illustration of the local part of the diagram which varies among (the equivalence class representatives of) D, D_i. See condition (2) of Definition 2.1 for the first example of this.

Notation 1.17. We will denote by $\bigcirc$ the equivalence class of the standard diagram of the trivial *unknot* in **Dgm**.

Definition 1.18. Let $D = [(\pi, \mathcal{A})], D' = [(\pi', \mathcal{A}')] \in \mathbf{Dgm}$. Define the *disjoint union* of D, D', denoted $D \amalg D'$, by the equivalence class of the link diagram given by the disjoint union (in the obvious sense) of $(\pi, \mathcal{A})$ and $(\pi', \mathcal{A}')$ in $\mathbf{R}^2$.

We now have all the definitions, notations and facts stated to move on and discuss the Jones polynomial.

2. The Jones polynomial

We will characterise the Jones polynomial via an auxiliary polynomial known as the Kauffman bracket.

Definition 2.1. The *Kauffman bracket* is the unique assignment $\langle - \rangle : \mathbf{Dgm} \longrightarrow \mathbf{Z}[q, q^{-1}]$ of Laurent polynomial (in q) to each link diagram $D \in \mathbf{Dgm}$ which satisfies the following properties:

(1) $\langle \bigcirc \rangle = 1$,
(2) $\langle \times \rangle = \langle \asymp \rangle - q \, \langle \,) \, (\, \rangle$, and
(3) $\langle \bigcirc \amalg D \rangle = (q + q^{-1}) \, \langle D \rangle$ for all $D \in \mathbf{Dgm}$.

If $\widetilde{D} = [(\pi, \mathcal{A}, \mathcal{O})] \in \mathbf{OrDgm}$ then we write $\langle \widetilde{D} \rangle$ for $\langle D \rangle$ with $D = [(\pi, \mathcal{A})] \in \mathbf{Dgm}$ the unoriented link diagram equivalence class underlying $\widetilde{D}$.

Definition 2.2. Let $D = [(\pi, \mathcal{A})] \in \mathbf{Dgm}$. Let $\mathcal{X}$ be the set of crossings of π. For $x \in \mathcal{X}$ call the equivalence class of the link diagram locally changed at this crossing from $\times$ to $\asymp$ (resp. $)\,($) the *0-smoothing* (resp. *1-smoothing*) of D at x. A *smoothing* of D is the equivalence class of the link diagram locally changed at each crossing from $\times$ to either $\asymp$ or $)\,($, i.e., a choice of 0-smoothing or 1-smoothing of each $x \in \mathcal{X}$. Of course it follows that there are $2^{|\mathcal{X}|}$ smoothings of D, in bijection with $\{0,1\}^{\mathcal{X}}$. And clearly a smoothing will be equal to the equivalence class of a disjoint union $\coprod_{i=1}^{u} \bigcirc$ of some $u \in \mathbf{N}$ unknots.

Notation 2.3. In the situation of the above definition. For $\alpha \in \{0,1\}^{\mathcal{X}}$, define $|\alpha| := \sum_{x \in \mathcal{X}} \alpha(x)$ and $u_\alpha \in \mathbf{N}$ such that the smoothing of D corresponding to α is the equivalence class of $\coprod_{i=1}^{u_\alpha} \bigcirc$.

Proposition 2.4. *Let $D = [(\pi, \mathcal{A})] \in \mathbf{Dgm}$. Let $\mathcal{X}$ be the set of crossings of π. Then*

$$\langle D \rangle = \sum_{\alpha \in \{0,1\}^{\mathcal{X}}} (-1)^{|\alpha|} q^{|\alpha|} (q + q^{-1})^{u_\alpha}.$$

Proof. We see this by induction on $n := |\mathcal{X}|$. If $n = 0$, that is, $X = \emptyset$, then we must have $D = [\coprod_{i=1}^{u} \bigcirc]$ for some $u \in \mathbf{N}$. By (1) and (3) of Definition 2.1 it is clear then that $\langle D \rangle = (q + q^{-1})^u$. Since $\{0,1\}^{\mathcal{X}} = \{\alpha\}$ in this case, with $|\alpha| = 0$ (the empty sum, by convention) and $u_\alpha = u$ by definition, the claim clearly holds for $n = 0$. Now assume $n > 0$ and choose an arbitrary $x \in \mathcal{X}$. Let D_0 be the 0-smoothing and D_1 the 1-smoothing of D at x. Then D_0, D_1 both have sets of crossings $\mathcal{X}' := \mathcal{X} - \{x\}$ with order $n - 1$. Then by induction and (2) of Definition 2.1 we have

$$\langle D \rangle = \langle D_0 \rangle - q \, \langle D_1 \rangle = \sum_{\beta \in \{0,1\}^{\mathcal{X}'}} (-1)^{|\beta|} q^{|\beta|} (q + q^{-1})^{u_\beta} + \sum_{\gamma \in \{0,1\}^{\mathcal{X}'}} (-1)^{|\gamma|+1} q^{|\gamma|+1} (q + q^{-1})^{u_\gamma}.$$

Now clearly to each $\beta \in \{0,1\}^{\mathcal{X}'}$ we can associate an $\alpha \in \{0,1\}^{\mathcal{X}}$ with $\alpha(y) = \beta(y)$ if $y \neq x$ and $\alpha(x) = 0$; then $|\beta| = |\alpha|$ and since D_0 results from a 0-smoothing at x of course $u_\beta = u_\alpha$. We can similarly associate to each $\gamma \in \{0,1\}^{\mathcal{X}'}$ an $\alpha \in \{0,1\}^{\mathcal{X}}$ with $\alpha(y) = \beta(y)$ if $y \neq x$ and $\alpha(x) = 1$; then $|\alpha| = |\gamma| + 1$ and $u_\gamma = u_\alpha$. It follows that

$$\langle D \rangle = \sum_{\substack{\{\alpha \in \{0,1\}^{\mathcal{X}} \mid \\ \alpha(x)=0\}}} (-1)^{|\alpha|} q^{\alpha} (q + q^{-1})^{u_\alpha} + \sum_{\substack{\{\alpha \in \{0,1\}^{\mathcal{X}} \mid \\ \alpha(x)=1\}}} (-1)^{|\alpha|} q^{|\alpha|} (q + q^{-1})^{u_\alpha} = \sum_{\alpha \in \{0,1\}^{\mathcal{X}}} (-1)^{|\alpha|} q^{|\alpha|} (q + q^{-1})^{u_\alpha}.$$

Thus the claim holds for n, and by induction we are done. $\square$

Remark 2.5. The above proposition makes it evident that the Kauffman bracket is indeed well-defined and unique.

Definition 2.6. Let $D = [(\pi, \mathcal{A}, \mathcal{O})] \in \mathbf{OrDgm}$. Let x be a crossing of π. If the orientation $\mathcal{O}$ is such that $(\pi, \mathcal{A}, \mathcal{O})$ locally looks like at x, then we call x a *positive crossing*, and if locally it looks like , then we call x a *negative crossing*.

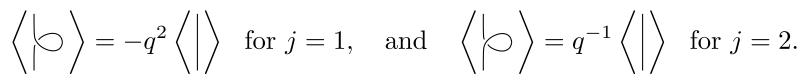
Definition 2.7. Let $D = [(\pi, \mathcal{A}, \mathcal{O})] \in \mathbf{OrDgm}$ and $\mathcal{X}$ the set of crossings of π. Let $n := |\mathcal{X}|$, n_+ the number of positive crossings in $\mathcal{X}$, and n_- the number of negative crossings in $\mathcal{X}$. Define $J(D) \in \mathbf{Z}[q, q^{-1}]$ by

$$J(D) := (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle \,.$$

Proposition 2.8. *If $D, D' \in$ **OrDgm** are equivalent then $J(D) = J(D')$.*

Proof. By definition of equivalence in **OrDgm** and symmetry it suffices to show this when $D' = \Omega_{i,j}(D)$ for some $1 \le i \le 3, 1 \le j \le 2$. Denote by n'_+ (resp. n'_-) the number of positive (resp. negative) crossings of D'.

First, $i = 1$. Applying (2) of Definition 2.1 twice gives us



We observe that regardless of the orientation on D, D' we have $n'_- = n_- + 1, n'_+ = n_+$ for $j = 1$ and $n'_+ = n_+ + 1, n'_- = n_-$ for $j = 2$. Thus we have

$$J(D') = (-1)^{n'_-} q^{n'_+ - 2n'_-} \langle D' \rangle = (-1)^{n_- + 1} q^{n_+ - 2n_- - 2}(-q^2 \langle D \rangle) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle = J(D)$$

for $j = 1$, and

$$J(D') = (-1)^{n'_-} q^{n'_+ - 2n'_-} \langle D' \rangle = (-1)^{n_-} q^{n_+ + 1 - 2n_-}(q^{-1} \langle D \rangle) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle = J(D)$$

for $j = 2$. So $J(D)$ is invariant under $\Omega^s_{1,j}, s \in \{\pm 1\}, 1 \le j \le 2$.

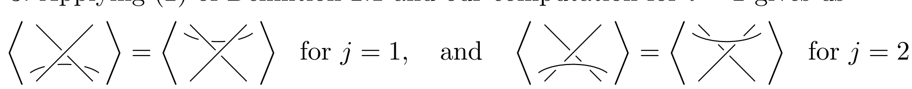
Now, $i = 2$. Applying (2) of Definition 2.1 and our computation for $i = 1$ gives us

We observe that regardless of the orientation on D, D' we have $n'_- = n_- + 1, n'_+ = n_+ + 1$ for $j = 1$ and $j = 2$. Thus we have

$$J(D') = (-1)^{n'_-} q^{n'_+ - 2n'_-} \langle D' \rangle = (-1)^{n_- + 1} q^{n_+ + 1 - 2n_- - 2}(-q \langle D \rangle) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle = J(D)$$

for $j = 1$ and $j = 2$. So $J(D)$ is invariant under $\Omega^s_{2,j}, s \in \{\pm 1\}, 1 \le j \le 2$.

Finally, $i = 3$. Applying (2) of Definition 2.1 and our computation for $i = 2$ gives us



We observe that regardless of the orientation on D, D' we have $n'_- = n_-, n'_+ = n_+$ for $j = 1$ and $j = 2$. Thus we have

$$J(D') = (-1)^{n'_-} q^{n'_+ - 2n'_-} \langle D' \rangle = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle = J(D)$$

for $j = 1$ and $j = 2$. So $J(D)$ is invariant under $\Omega^s_{3,j}, s \in \{\pm 1\}, 1 \le j \le 2$. $\square$

This invariance coupled Fact 1.15 allows us to make the following definition.

Definition 2.9. For any oriented link $(L, \mathcal{O})$ we have $J(D) = J(D')$ for all oriented link diagram classes of $(L, \mathcal{O})$. We call this common Laurent polynomial the *Jones polynomial* of $(L, \mathcal{O})$, denoted $J(L, \mathcal{O})$, or just $J(L)$ if an orientation is implicit. Moreover if $(L, \mathcal{O})$ and $(L', \mathcal{O}')$ are equivalent we have $J(L, \mathcal{O}) = J(L', \mathcal{O}')$. In this sense, the Jones polynomial is an invariant of oriented links.

3. Khovanov homology

Now, there is a general idea of *categorification*: replace set-theoretic notions—e.g., elements, functions, equations of functions—with category-theoretic notions—e.g., objects, functors, natural transformations of functors—and hope to get new theory or a deeper understanding of things. Khovanov did this for the Jones polynomial, finding a more "categorical" link invariant which generalises the Jones polynomial [6]. Here we will give the construction of this invariant, called Khovanov homology. However we will not give a proof of invariance; this can be found in the original paper or in the exposition [2].

We first introduce some definitions and terminology regarding graded vector spaces, which are the objects we will use to categorify polynomials. Fix throughout a field k, which all vector spaces will be over.

Definition 3.1. A *graded vector space* V is a vector space of the form $V = \bigoplus_{i\in\mathbf{Z}} V_i$, where each V_i is a vector space called the i-th *homogenous component* of V. An element $v_i \in V_i$ is said to have *degree* i for $i \in \mathbf{Z}$. The *graded dimension* of V is the formal power series $\dim_q(V) := \sum_{i\in\mathbf{Z}} q^i \dim(V_i)$, which is a Laurent polynomial in $\mathbf{Z}[q, q^{-1}]$ if $\dim(V_i) > 0$ for only finitely many $i \in \mathbf{Z}$.

Definition 3.2. Let $V = \bigoplus_i V_i$ and $W = \bigoplus_i W_i$ be graded vector spaces. The *(graded) tensor product* $V \otimes W$ of V and W is the graded vector space $\bigoplus_i (V\otimes W)_i$, where we define

$$(V \otimes W)_i := \bigoplus_{\{\alpha,\beta\in\mathbf{Z}|\alpha+\beta=i\}} V_\alpha \otimes W_\beta \quad \text{for } i \in \mathbf{Z}.$$

It is clear that $\dim_q(V \otimes W) = \dim_q(V) \cdot \dim_q(W)$.

Definition 3.3. Let $V = \bigoplus_i V_i$ and $W = \bigoplus_i W_i$ be graded vector spaces. A *graded map of degree* $d \in \mathbf{Z}$ is a linear map $\varphi : V \longrightarrow W$ which satisfies

$$\varphi(V_i) \subset W_{i+d} \quad \text{for } i \in \mathbf{Z}.$$

Definition 3.4. A *complex* of vector spaces $V = (V^\bullet, d^\bullet)$ is a sequence of vector spaces V^i and linear maps $d^i : V^i \longrightarrow V^{i+1}$, $i \in \mathbf{Z}$, such that $d^{i+1} \circ d^i = 0$ for $i \in \mathbf{Z}$. If V^i is graded and d^i a graded map of degree 0 for each $i \in \mathbf{Z}$ we say V is a *complex of graded vector spaces*. Since $\mathrm{im}(d^i) \subset \ker(d^{i+1})$, it makes sense to define the *homology spaces* $\mathrm{H}^i(V) := \ker(d^{i+1})/\,\mathrm{im}(d^i)$ for $i \in \mathbf{Z}$, which are furthermore graded if V is graded.

Definition 3.5. Let $V = (V^\bullet, d^\bullet)$ be a complex of graded vector spaces. The *graded Euler characteristic* of V is defined to be the formal power series

$$\chi_q(V) := \sum_{i\in\mathbf{Z}} (-1)^i \dim_q(\mathrm{H}^i(V)).$$

Assuming V^i is finite dimensional for $i \in \mathbf{Z}$, it is an easy consequence of the rank-nullity theorem that we also have

$$\chi_q(V) := \sum_{i\in\mathbf{Z}} (-1)^i \dim_q(V^i).$$

Definition 3.6. For $d \in \mathbf{Z}$ we define the *degree shift operation* $\cdot\{r\}$ on graded vector spaces $V = \bigoplus_i V_i$ by $V\{r\}_i := V_{i-r}$ for $i \in \mathbf{Z}$, so that $\dim_q(V\{r\}) = q^r \cdot \dim_q(V)$. This also clearly induces a degree shift operation on complexes of graded vector spaces $V = (V^\bullet, d^\bullet)$, which we also denote by $\cdot\{r\}$. For $s \in \mathbf{Z}$ we define the *height shift operation* $\cdot[s]$ on complexes $V = (V^\bullet, d^\bullet)$ by $V[s]^i := V^{i-s}$ and $d[s]^i := d^{i-s}$ for $i \in \mathbf{Z}$.

Construction 3.7. We now give a construction of the *Khovanov bracket* associated to a link diagram. Let V be the graded vector space with basis $\{v_{\pm 1}\}$, where v_σ has degree σ for $\sigma \in \{\pm 1\}$. Then $\dim_q(V) = q + q^{-1}$.

Let $D = [(\pi, \mathcal{A})] \in \mathbf{Dgm}$ and $\mathcal{X}$ be the set of crossings of π. Recall Notation 2.3. For $\alpha \in \{0,1\}^{\mathcal{X}}$ define $V_\alpha(D) := V^{\otimes u_\alpha}\{|\alpha|\}$, the u_α-th graded tensor power of V shifted by $|\alpha|$. To make the bijection between the factors of the disjoint union $\coprod_{i=1}^{u_\alpha} \bigcirc$ and the factors of the tensor power $V^{\otimes u_\alpha}$ concrete, fix a labelling $\bigcirc_1^\alpha, \bigcirc_2^\alpha, \ldots, \bigcirc_{u_\alpha}^\alpha$ of the factors of $\coprod_{i=1}^{u_\alpha} \bigcirc$. Now, define

$$[\![D]\!]^i := \bigoplus_{\{\alpha\in\{0,1\}^{\mathcal{X}}||\alpha|=i\}} V_\alpha(D) \quad \text{for } i \in \mathbf{Z},$$

where we take the empty direct sum to be the zero vector space. Next let $x \in \mathcal{X}$ and $\xi \in \{0,1\}^{\mathcal{X}-\{x\}}$. Let $\alpha, \beta \in \{0,1\}^{\mathcal{X}}$ be defined by $\alpha(y) := \xi(y), \beta(y) := \xi(y)$ for $y \in \mathcal{X}'$ and $\alpha(x) = 0, \beta(x) = 1$. We define a map $d_\xi : V_\alpha(D) \longrightarrow V_\beta(D)$. Let $A := \coprod_{i=1}^{u_\alpha} \bigcirc_i^\alpha$ and $B := \coprod_{i=1}^{u_\beta} \bigcirc_i^\beta$. One easily sees that we have only the following two possibilities for the relationship between A and B.

(1) $(u_\beta = u_\alpha + 1)$ We locally have $\bigcirc_b^\beta \amalg \bigcirc_c^\beta, 1 \le b < c \le u_\beta$ in B in place of $\bigcirc_a^\alpha, 1 \le a \le u_\alpha$ in A, and otherwise A and B are identical, so we have a bijective correspondence $\varphi_\xi : \{1, 2, \ldots, u_\alpha\} - \{a\} \longrightarrow \{1, 2, \ldots, u_\beta\} - \{b, c\}$. Define $\mu : V \longrightarrow V \otimes V$ by sending

$$v_+ \longmapsto v_+ \otimes v_- + v_- \otimes v_+ \quad \text{and} \quad v_- \longmapsto v_- \otimes v_-$$

and extending linearly. Then μ induces a map $d'_\xi : V^{\otimes u_\alpha} \longrightarrow V^{\otimes u_\beta}$ which acts as the identity between the i-th factor in $V^{\otimes u_\alpha}$ and the $\varphi_\xi(i)$-th factor in $V^{\otimes u_\beta}$ for $1 \le i \le u_\alpha, i \ne a$, and which acts as μ between the a-th factor in $V^{\otimes u_\alpha}$ and the b-th and c-th factors of $V^{\otimes u_\beta}$. This finally induces a map

$d_\xi : V^{\otimes u_\alpha}\{|\alpha|\} \longrightarrow V^{\otimes u_\beta}\{|\beta|\}$. Since clearly d'_ξ is a graded map of degree -1 and $|\beta| = |\alpha| + 1$, d_ξ is a graded map of degree 0.

(2) ($u_\beta = u_\alpha - 1$) We have $\bigcirc^\beta_c, 1 \leq c \leq u_\beta$ in B in place of $\bigcirc^\alpha_a \amalg \bigcirc^\alpha_b, 1 \leq a < b \leq u_\alpha$ in A, and otherwise A and B are identical, so we have a bijective correspondence $\varphi_\xi : \{1, 2, \ldots, u_\alpha\} - \{a, b\} \longrightarrow \{1, 2, \ldots, u_\beta\} - \{c\}$. Define $\nu : V \otimes V \longrightarrow V$ by sending

$$v_+ \otimes v_- \longmapsto v_-, \quad v_+ \otimes v_+ \longmapsto v_+, \quad v_- \otimes v_+ \longmapsto v_-, \quad \text{and} \quad v_- \otimes v_- \longmapsto 0$$

and extending linearly. Then ν induces a map $d'_\xi : V^{\otimes u_\alpha} \longrightarrow V^{\otimes u_\beta}$ which acts as the identity between the i-th factor in $V^{\otimes u_\alpha}$ and the $\varphi_\xi(i)$-th factor in $V^{\otimes u_\beta}$ for $1 \leq i \leq u_\alpha, i \notin \{a, b\}$, and which acts as ν between the a-th and b-th factors in $V^{\otimes u_\alpha}$ and the c-th factor of $V^{\otimes u_\beta}$. This finally induces a map $d_\xi : V^{\otimes u_\alpha}\{|\alpha|\} \longrightarrow V^{\otimes u_\beta}\{|\beta|\}$. Since clearly d'_ξ is a graded map of degree -1 and $|\beta| = |\alpha| + 1$, d_ξ is a graded map of degree 0.

Now, fix an ordering $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$ of the crossings. Let $1 \leq j \leq n$ such that $x = x_j$ and $p_\xi := \sum_{i=1}^{j-1} \xi(x_j)$.

For $0 \leq i < n$ we now define $d^i : [\![D]\!]^i \longrightarrow [\![D]\!]^{i+1}$ as the graded map of degree 0 which acts on $v \in V_\alpha(D)$ for $\alpha \in \{0,1\}^{\mathcal{X}}, |\alpha| = i$ by

$$v \longmapsto \sum_{\{x \in \mathcal{X} | \alpha(x) = 0\}} (-1)^{p_{\xi_{\alpha,x}}} d_{\xi_{\alpha,x}}(v),$$

where $\xi_{\alpha,x} \in \{0,1\}^{\mathcal{X}-\{x\}\}}$, $x \in \mathcal{X}, \alpha(x) = 0$, is defined for $y \in \mathcal{X} - \{x\}$ by $\xi_{\alpha,x}(y) = \alpha(y)$. And for $i < 0$ and $i \geq n$, since either $[\![D]\!]^i$ or $[\![D]\!]^{i+1}$ is the zero vector space, we set d^i to be the forced map. We claim $d^{i+1} \circ d^i = 0$ for $i \in \mathbf{Z}$, so $[\![D]\!] = ([\![D]\!]^\bullet, d^\bullet)$ is a complex of graded vector spaces. This is obvious for $i < 0$ and $i \geq n - 1$ so assume $0 \leq i \leq n - 2$. Let $v \in V_\alpha(D)$ with $\alpha \in \{0,1\}^{\mathcal{X}}, |\alpha| = i$. By definition we will have

$$(d^{i+1} \circ d^i)(v) = \sum_{\{(x,x') \in \mathcal{X}^2 | \alpha(x) = \alpha(x') = 0\}} (-1)^{p_{\xi_{\alpha,x}} + p_{\xi_{\alpha_x,x'}}} (d_{\xi_{\alpha_x,x'}} \circ d_{\xi_{\alpha,x}})(v),$$

where $\alpha_x \in \{0,1\}^{\mathcal{X}-\{x'\}}$, $(x, x') \in \mathcal{X}^2, \alpha(x) = \alpha(x') = 0$, is defined for $y \in \mathcal{X} - \{x, x'\}$ by $\alpha_x(y) = \alpha(y)$ and $\alpha_x(x) = 1$. Since the maps d_ξ defined above only depend on the link diagram locally, it is clear that

$$d_{\xi_{\alpha_x,x'}} \circ d_{\xi_{\alpha,x}} = d_{\xi_{\alpha_{x'},x}} \circ d_{\xi_{\alpha,x'}}$$

for $(x, x') \in \mathcal{X}^2, \alpha(x) = \alpha(x') = 0$. On the other hand, if $x = x_s, x' = x_t$ with $1 \leq s < t \leq n$, then

$$p_{\xi_{\alpha,x}} + p_{\xi_{\alpha_x,x'}} = \sum_{j=1}^{s-1} \xi_{\alpha,x}(x_j) + \sum_{j=1}^{t-1} \xi_{\alpha_x,x'}(x_j) = 2\sum_{j=1}^{s-1} \alpha(x_j) + 1 + \sum_{j=s+1}^{t-1} \alpha(x_j),$$

while

$$p_{\xi_{\alpha,x'}} + p_{\xi_{\alpha_{x'},x}} = \sum_{j=1}^{t-1} \xi_{\alpha,x'}(x_j) + \sum_{j=1}^{s-1} \xi_{\alpha_{x'},x}(x_j) = 2\sum_{j=1}^{s-1} \alpha(x_j) + \sum_{j=s+1}^{t-1} \alpha(x_j).$$

Thus $(-1)^{p_{\xi_{\alpha,x}} + p_{\xi_{\alpha_x,x'}}} = -(-1)^{p_{\xi_{\alpha,x'}} + p_{\xi_{\alpha_{x'},x}}}$; by symmetry the same result holds if $t < s$. From these two observations it follows that

$$(-1)^{p_{\xi_{\alpha,x}} + p_{\xi_{\alpha_x,x'}}} (d_{\xi_{\alpha_x,x'}} \circ d_{\xi_{\alpha,x}})(v) = -(-1)^{p_{\xi_{\alpha,x'}} + p_{\xi_{\alpha_{x'},x}}} (d_{\xi_{\alpha_{x'},x}} \circ d_{\xi_{\alpha,x'}})(v),$$

which finally implies $(d^{i+1} \circ d^i)(v) = 0$, and hence $d^{i+1} \circ d^i = 0$.

Notation 3.8. As with the Kauffman bracket, if $\widetilde{D} = [(\pi, \mathcal{A}, \mathcal{O})] \in \mathbf{OrDgm}$, we denote by $[\![\widetilde{D}]\!]$ the complex $[\![D]\!]$, where $D = [(\pi, \mathcal{A})] \in \mathbf{Dgm}$.

Definition 3.9. Let $D = [(\pi, \mathcal{A}, \mathcal{O})] \in \mathbf{OrDgm}$ and n, n_+, n_- as in Definition 2.7. We define the *Khovanov complex* of D, denoted $\mathcal{V}(D)$, to be the complex $[\![D]\!][-n_-]\{n_+ - 2n_-\}$ of graded vector spaces.

Proposition 3.10. *Let D, n, n_+, n_- as in the above definition. Then $\chi_q(\mathcal{V}(D)) = J(D)$.*

Proof. This is immediate from the construction of $\mathcal{V}(D)$, Proposition 2.4, and the second characterisation of χ_q given in Definition 3.5. □

Thus we can define $\chi_q(L, \mathcal{O})$ to be the link invariant $\chi_q(\mathcal{V}(D))$ for any oriented link diagram D of $(L, \mathcal{O})$. In fact, as advertised above, we get an even stronger invariant out of this construction.

Theorem 3.11. *Suppose $D, D' \in$ **OrDgm** are equivalent. Then we have an isomorphism $\mathrm{H}^i(\mathcal{V}(D)) \simeq \mathrm{H}^i(\mathcal{V}(D'))$ for $i \in \mathbf{Z}$. It follows that for $(L, \mathcal{O}) \in$ **OrDgm** we can define $\mathrm{H}^i(L, \mathcal{O})$ for $i \in \mathbf{Z}$ to be $\mathrm{H}^i(\mathcal{V}(D))$ for any oriented link diagram D of $(L, \mathcal{O})$, and these homology spaces are link invariants.*

Proof. Also as advertised, the proof is not included here. See [6]. $\square$

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