BRAID GROUPS

ALLEN YUAN

1. Introduction

In the first lecture of our tutorial, the knot group of the trefoil was remarked to be the braid group B_3. There are, in general, many more connections between knot theory and braid groups. Furthermore, the study of these braid groups is also both important to mathematics and applicable to physics. This paper explores the topic of braid groups and braids. The braid groups and the pure braid groups will be introduced from both the algebraic and geometric perspectives. From these definitions, the connections that these braid groups have with knot theory are explored, culminating in a statement of Alexander's Theorem.

2. Motivations

The motivation of our study begins with the knot group of the trefoil which was computed to be a group with two generators, x and y, and the relation $xyx = yxy$. Another way this relation arises naturally is to consider the ways to "braid" three strings together, where three examples: x, y and x^{-1} are shown in Figure 1.

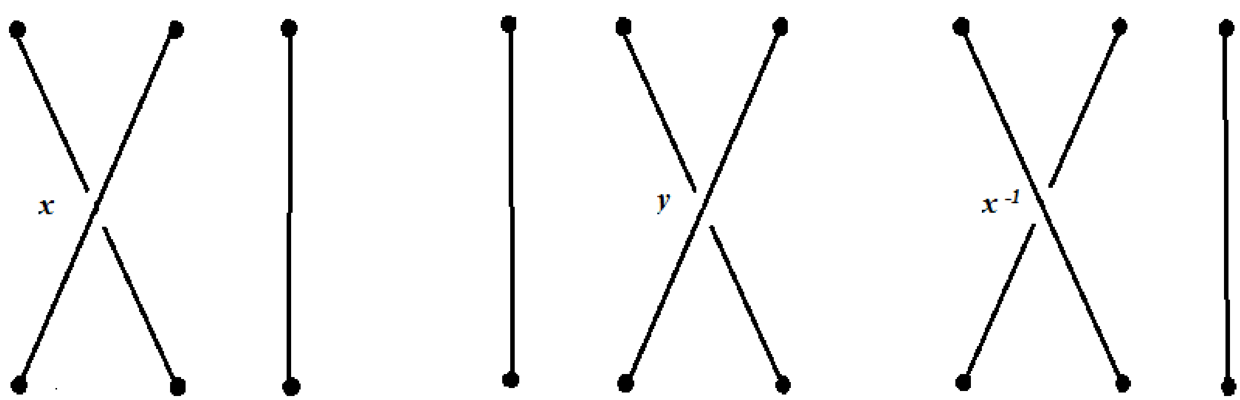

Figure 1. x, y, and x^{-1}

Indeed, note that if we can "compose" two braids by stacking them, we have that we can "move" or "deform" the configuration xyx to the configuration yxy smoothly by nudging the strings in Figure 2.

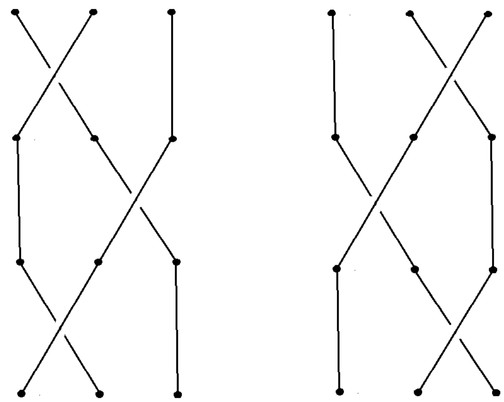

FIGURE 2. xyx and yxy

In a similar manner, we can see that the configuration obtained by composing x with x^{-1} can be deformed to three straight strings, what we might want to call the identity braid.

3. Geometric Braids

To make rigorous our intuition about these braid diagrams, we make the following definitions (due to [2]):

Definition 3.1. *A geometric braid on $n \geq 1$ strings is a set $b \subset \mathbb{R}^2 \times I$ formed by n disjoint topological intervals (called strings of b) such that the projection $\mathbb{R}^2 \times I \to I$ maps each string homeomorphically onto I and string k starts at $(k, 0, 0)$ and ends at $(k, 0, 1)$.*

We can therefore draw these geometric braids in 2-dimensions by using braid diagrams in the same way that knots are represented in the knot diagrams. An example is shown below in Figure 3. The vertices are labelled $(x, y, t) \in \mathbb{R}^2 \times I$ such that the x axis is horizontal, the y axis is perpendicular to the page, and the t axis goes vertically. In particular, note that at each of the finitely many crossings, exactly two strings meet transversely, and one goes above and the other goes below.

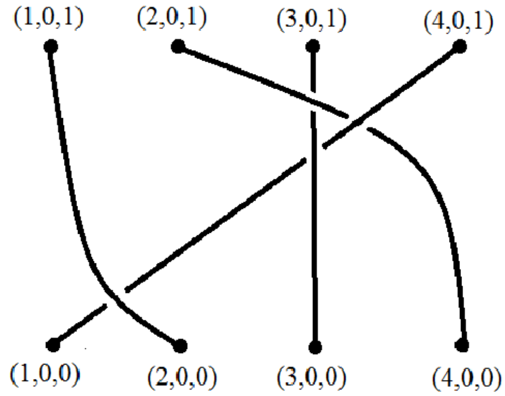

FIGURE 3. A braid on four strings

As with knots, we wish to define braids as equivalent if they can be continuously deformed to each other. In particular, we can define an equivalence relation on the geometric braids where two braids are equivalent if they are related by an isotopy where each intermediate object is a geometric braid. The equivalence classes of this relation will henceforth be referred to as braids. It is not difficult to see that each braid can be presented by a braid diagram and that each braid diagram presents a well-defined braid.

We would like to be able to determine whether two braids are isotopic just by looking at their braid diagrams. Therefore, we define two braid diagrams to be isotopic if there is an isotopy between them where each intermediate object is a braid diagram. However, this notion of equivalence does not suffice. Observe that, in particular, the two transformations pictured below cannot be obtained by isotopies of braid diagrams. They are known as the Reidemeister moves, Ω_2 and Ω_3, pictured in Figures 4 and 5, respectively.

It is known from [2] that:

Theorem 3.2. *Two braid diagrams present isotopic geometric braids if and only if the diagrams are related by a finite sequence of isotopies, Reidemeister moves and/or inverses of Reidemeister moves.*

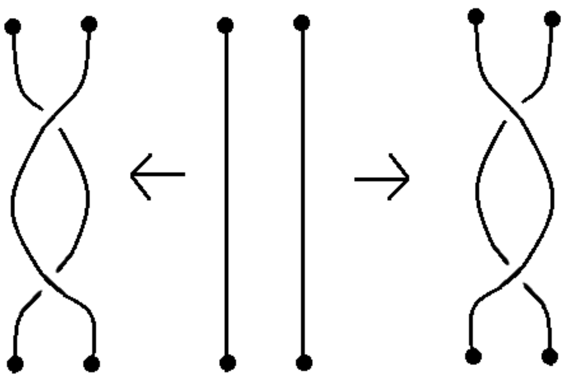

FIGURE 4. Ω_2

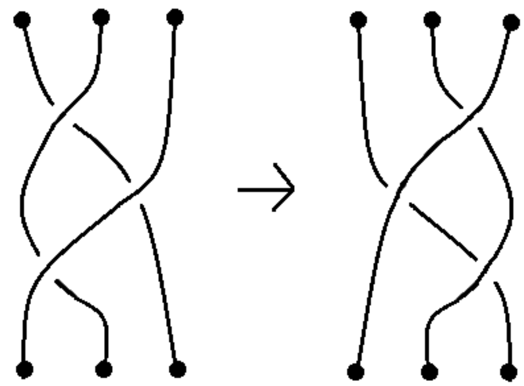

FIGURE 5. Ω_3

4. THE BRAID GROUPS

As alluded to in the motivation section, there is a notion of composition for braids. To compose a braid β_1 with a braid β_2, we can simply put β_1 on top of β_2 by matching up the bottoms of the strings in β_1 with the tops of the strings in β_2, and then shrinking each braid by a factor of 2 along the t axis (i.e., vertically). It can easily be seen that this is a well-defined composition law on the set of braids on n strings.

Now, define for each $1 \leq k \leq n-1$ the elementary braids σ_k^+ and σ_k^- to be the braids made by single crossings of the strands for k and $k+1$, as shown in Figure 6.

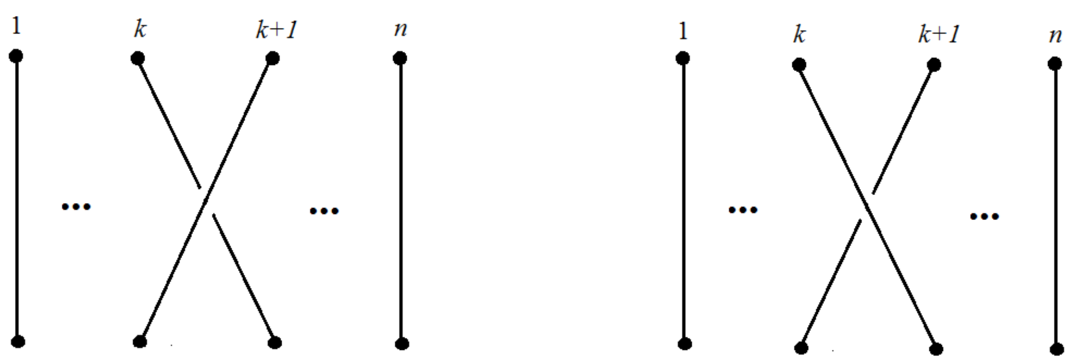

FIGURE 6. σ_k^+ and σ_k^-

Observe that $\sigma_k^+\sigma_k^- = \sigma_k^-\sigma_k^+ = 1$, where 1 denotes the braid with no crossings. Thus, the elementary braids σ_k^+ and σ_k^- for $k = 1, 2, \cdots, n-1$ generate a group. It is clear that every braid is in this group and the group consists only of braids. Therefore, the set of braids with the composition operation forms a group. This group is known as the braid group, B_n. Note that in this group, certainly $\sigma_i^+\sigma_j^+ = \sigma_j^+\sigma_i^+$ if i and j are not consecutive, since then the two crossings will use disjoint pairs of strings. Furthermore, because of Ω_3, we have that $\sigma_i^+\sigma_{i+1}^+\sigma_i^+ = \sigma_{i+1}^+\sigma_i^+\sigma_{i+1}^+$ (recall the motivation of Figure 2). Therefore, the following definition of the braid group via generators and relations can easily be deduced.

Definition 4.1. *The braid group B_n is generated by $n-1$ generators $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$ and has "braid" relations*

$$\sigma_i\sigma_j = \sigma_j\sigma_i$$

for $|i-j| \geq 2$, and

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

for $1 \leq i \leq n-1$.

For example, B_1 is trivial, $B_2 \cong \mathbb{Z}$, and we already know what B_3 is. The higher braid groups can be hard to describe fully. However, the braid relations appear naturally in the symmetric group; simply note that the adjacent transpositions $s_i = (i \;\; i+1)$ satisfy the braid relations, and thus, there is a map $\pi : B_n \to S_n$ taking $\sigma_i \mapsto s_i$. Geometrically, this map just takes a braid to the permutation mapping k to $\tau(k)$ where $\tau(k)$ is the integer corresponding to the end of the kth string. As a result of π being surjective and S_n being nonabelian for $n \geq 3$, we have:

Corollary 4.2. *B_n is nonabelian for $n \geq 3$.*

Furthermore, the *pure braid group* P_n is defined as the kernel of the map π. Geometrically, these pure braids can be seen as represented by braids where the ith string starts at $(i, 0, 0)$ and end at $(i, 0, 1)$.

We also have an obvious group homomorphism $\iota : B_n \to B_{n+1}$ where $\sigma_i \mapsto \sigma_i$ for $i = 1, 2, \cdots, n-1$. While this is an inclusion map, it is in fact unclear just from the algebra that this is injective. However, if we interpret the map ι geometrically as merely adding a non-intersecting straight string to a braid b on n strings, it is clear that if $\iota(b)$ and $\iota(b')$ are isotopic, then b and b' are isotopic just by restricting the isotopy.

5. Braids as Fundamental Groups

Recall our initial motivation of the braid group B_3 being the knot group of the trefoil. One might speculate that, in general, the braid groups occur naturally as fundamental groups. Indeed, they do and we shall present this interpretation with the notation of [1].

We may consider the *configuration space* of n points in the complex plane,

$$\mathcal{C}_{0,\hat{n}} = \{(z_1, \cdots, z_n) \in \mathbb{C} \times \cdots \times \mathbb{C} \mid z_i \neq z_j \text{ if } i \neq j\}.$$

The symmetric group acts on $\mathcal{C}_{0,\hat{n}}$ by permuting the coordinates, and so we may quotient out this action to get another space $\mathcal{C}_{0,n}$ and a map $\tau : \mathcal{C}_{0,\hat{n}} \to \mathcal{C}_{0,n}$. We then claim that the *pure braid groups* P_n and the *braid groups* B_n are the fundamental groups (where z is an arbitrary base point)

$$P_n = \pi_1(\mathcal{C}_{0,\hat{n}}, z), \qquad B_n = \pi_1(\mathcal{C}_{0,n}, \tau(z)).$$

To see why this is true intuitively, we first observe that the quotient map $\tau : \mathcal{C}_{0,\hat{n}} \to \mathcal{C}_{0,n}$ is a covering space map. Then, we think about a point in $\mathcal{C}_{0,n}$ as a set of n distinct points in $\mathbb{C}$; by lifting a loop to the covering space $\mathcal{C}_{0,\hat{n}}$, we get a unique path $I \to \mathcal{C}_{0,\hat{n}}$ which, through our identification of a point as n distinct points in $\mathbb{C}$ and the identification of $\mathbb{C}$ with $\mathbb{R}^2$, becomes n paths which never coincide, ending and starting at points in the same fiber of τ. This is precisely a braid. Observe that the notion of composition also matches with the first geometric definition.

6. Relationship to Knots

In the final section, we present results to which the reader may find proofs in [2].

One may see one obvious way to turn a braid into a knot. Indeed, if one identifies the points $(i, 0, 0)$ and $(i, 0, 1)$ for $i = 1, 2, \cdots, n$, any braid becomes a link. Furthermore, if we think of the original braid as being embedded in a solid cylinder (with the lines $t = 0$ and $t = 1$ being the diameters of the caps of the cylinder), then by extending this identification to the caps of the cylinder, this link is embedded inside of a solid torus. This process is known as closing a braid β, and the resulting link, denoted $\hat{\beta}$ is called the *closure* of the braid. Indeed, it is proven in [2] that if two braids are isotopic, then their closures are isotopic as links in the torus. Therefore, the isotopy class of the closure of a geometric braid depends only on its isotopy class as a braid. In fact, we have the correspondence:

Theorem 6.1. *For any $\beta, \beta' \in B_n$, their closures $\hat{\beta}, \hat{\beta}'$ are isotopic in the solid torus if and only if β and β' are conjugate in B_n.*

In general, a link L embedded in a solid torus $V = D \times S^1$ is called a *closed n-braid* if L meets every disk $D \times \{z\}$ for $z \in S^1$ at exactly n points, and each of these meetings is transverse. It is clear that every closed n-braid can be cut apart to make a braid. Note also that since any link in the solid torus can be given a canonical counter-clockwise orientation, the process of closing a braid creates an oriented link. The point of all the preceding discussion is so that we may state Alexander's Theorem.

Theorem 6.2. *(J. W. Alexander). Any oriented link in $\mathbb{R}^3$ is isotopic to the closure of a braid.*

This theorem highlights the importance of studying braids because it implies that by studying braids, one is, in effect, studying all links.

References

[1] J Birman and T. Brendle, Braids: A Survey, arXiv:math/0409205v2.

[2] C. Kassel and V. Turaev, Braid Groups *Graduate Texts in Mathematics*, (2008). Springer