FINE DELIGNE–LUSZTIG VARIETIES AND ARITHMETIC FUNDAMENTAL LEMMAS

XUHUA HE, CHAO LI, AND YIHANG ZHU

ABSTRACT. We prove a character formula for some closed fine Deligne–Lusztig varieties. We apply it to compute fixed points for fine Deligne–Lusztig varieties arising from the basic loci of Shimura varieties of Coxeter type. As an application, we prove an arithmetic intersection formula for certain diagonal cycles on unitary and GSpin Rapoport–Zink spaces arising from the arithmetic Gan–Gross–Prasad conjectures. In particular, we prove the arithmetic fundamental lemma in the minuscule case, without assumptions on the residual characteristic.

CONTENTS

1. INTRODUCTION

1.1. **The AFL conjecture.** The *arithmetic Gan–Gross–Prasad conjectures* (AGGP) generalize the celebrated Gross–Zagier formula to higher dimensional Shimura varieties of orthogonal or unitary type ([GGP12, §27], [Zha12, §3.2], [RSZ17a]). The *arithmetic fundamental lemma conjecture* (AFL) arises from Zhang's relative trace formula approach towards the AGGP conjecture for the group $\mathrm{U}(1, n-2) \times \mathrm{U}(1, n-1), n \geq 2$. It relates a derivative of orbital integrals on symmetric spaces to an arithmetic intersection number of cycles on unitary Rapoport–Zink spaces,

$$\omega(\gamma) \cdot \partial \operatorname{Orb}(\gamma, \mathbf{1}_{S_n(\mathcal{O}_F)}) = -\operatorname{Int}(g) \cdot \log q. \tag{1.1.1}$$

For the precise definitions of the quantities appearing in the identity, see [RSZ17b, §1]. The left-hand side of (1.1.1) is known as the *analytic side* and the right-hand side is known as the *arithmetic-geometric* side.

Let us briefly recall the definition of the arithmetic-geometric side. Let p be an odd prime. Let F be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$ and a uniformizer π. Let E be an unramified quadratic extension of F. Let $\breve{E}$ be the completion of the maximal unramified extension of E. Let $k = \overline{\mathbb{F}}_q$. For any integer $n \geq 1$, the *unitary Rapoport–Zink space* $\mathcal{N}_n$ is the formal scheme over $S = \operatorname{Spf} \mathcal{O}_{\breve{E}}$ parameterizing deformations up to quasi-isogeny of height 0 of unitary $\mathcal{O}_F$-modules of signature $(1, n-1)$. Fix an integer $n \geq 2$. There is a natural closed immersion $\delta : \mathcal{N}_{n-1} \to \mathcal{N}_n$. Denote by $\Delta \subset \mathcal{N}_{n-1} \times_S \mathcal{N}_n$ the image of $(\mathrm{id}, \delta) : \mathcal{N}_{n-1} \to \mathcal{N}_{n-1} \times_S \mathcal{N}_n$.

Date: August 17, 2019.
2010 *Mathematics Subject Classification.* 11G18, 14G17; secondary 20G40.

Let C_{n-1} be a non-split Hermitian space of dimension $n-1$, for the quadratic extension E/F. Here non-split means that the discriminant has odd valuation. Define a non-split Hermitian space of dimension n by $C_n := C_{n-1} \oplus Eu$, where the direct sum is orthogonal and u has norm 1. The unitary group $J(F) := \mathrm{U}(C_n)(F)$ acts on $\mathcal{N}_n$ in a natural way. Let $g \in J(F)$. The arithmetic-geometric side of the AFL conjecture (1.1.1) concerns the arithmetic intersection number of the diagonal cycle Δ and its translate by $\mathrm{id} \times g$, defined as (see [Zha12, §2.2])

$$\mathrm{Int}(g) := \chi(\mathcal{N}_{n-1} \times_S \mathcal{N}_n, \mathcal{O}_\Delta \otimes^{\mathbb{L}} \mathcal{O}_{(\mathrm{id} \times g)\Delta}).$$

When Δ and $(\mathrm{id} \times g)\Delta$ intersect properly, namely when the formal scheme

$$\Delta \cap (\mathrm{id} \times g)\Delta \cong \delta(\mathcal{N}_{n-1}) \cap \mathcal{N}_n^g \tag{1.1.2}$$

is an Artinian scheme (where $\mathcal{N}_n^g$ denotes the fixed points of g), the arithmetic intersection number $\mathrm{Int}(g)$ is simply the $\mathcal{O}_{\breve{E}}$-length of the Artinian scheme (1.1.2) (see [RTZ13, Proposition 4.2 (iii)]).

Recall that $g \in J(F)$ is called *regular semi-simple* if

$$L(g) := \mathcal{O}_E \cdot u + \mathcal{O}_E \cdot gu + \cdots + \mathcal{O}_E \cdot g^{n-1}u$$

is a full-rank $\mathcal{O}_E$-lattice in C_n. In this case, the *invariant* of g is the unique sequence of integers

$$\mathrm{inv}(g) := (r_1 \geqslant r_2 \geqslant \ldots \geqslant r_n)$$

characterized by the condition that there exists a basis $\{e_i\}$ of the lattice $L(g)$ such that $\{\pi^{-r_i}e_i\}$ is a basis of the dual lattice $L(g)^\vee$. It turns out that the "bigger" $\mathrm{inv}(g)$ is, the more difficult it is to compute the intersection. With this in mind, recall that a regular semi-simple element g is called *minuscule* if $r_1 = 1$ and $r_n \geqslant 0$.

1.2. **The AFL in the minuscule case.**

In the minuscule case, the analytic side is relatively straightforward to evaluate. One of our main results is an explicit formula for the arithmetic-geometric side $\mathrm{Int}(g)$ when g is minuscule, which allows us to establish new cases of the AFL conjecture.

Theorem 1.2.1 (Corollary 5.1.4)**.** *The arithmetic fundamental lemma holds when g is minuscule.*

Remark 1.2.2. When $F = \mathbb{Q}_p$ and $p > \frac{n+1}{2}$, this theorem was first proved by Rapoport–Terstiege–Zhang [RTZ13] (see also a simplified proof in [LZ17]). The same methods together with [Cho18] should prove the theorem for any p-adic field F with the size of its residue field $q > \frac{n+1}{2}$. However, potential global applications to the AGGP conjectures require the truth of AFL at *all* unramified places, thus it is desirable to remove the assumption that $q > \frac{n+1}{2}$. Our proof is different from [RTZ13] and treats all local fields F (with odd residue characteristic, in order to define the Rapoport–Zink spaces) uniformly.

Remark 1.2.3. After this work is done, Zhang [Zha19] has recently announced a proof of the arithmetic fundamental lemma when $F = \mathbb{Q}_p$ and $p > n$ (without assuming that g is minuscule).

To state the explicit formula for $\mathrm{Int}(g)$, assume g is minuscule and $\mathcal{N}_n^g \neq \emptyset$. Then it can be shown that g stabilizes both $L(g)^\vee$ and $L(g)$, and acts as an unitary operator on $\mathbb{V} := L(g)^\vee/L(g)$, which has a natural structure of a Hermitian space over $\mathbb{F}_{q^2}$. Let $\bar{g} \in \mathrm{U}(\mathbb{V})(\mathbb{F}_q)$ be the induced element.

For any monic polynomial $Q \in \mathbb{F}_{q^2}[\lambda]$ with $Q(0) \neq 0$, we define its *reciprocal polynomial* Q^* by replacing each root $x \in k^\times$ of Q with x^{-q} (with multiplicities). We say Q is *self-reciprocal* if $Q = Q^*$.

Let $f \in \mathbb{F}_{q^2}[\lambda]$ be the characteristic polynomial of $\bar{g}$. Then f is self-reciprocal. For any monic irreducible $Q \in \mathbb{F}_{q^2}[\lambda]$, we denote the multiplicity of Q in f by m_Q.

Theorem 1.2.4 (Theorem 5.1.2)**.** *Assume g is minuscule and* $\mathrm{Int}(g) \neq 0$. *Then there is a unique monic irreducible self-reciprocal $Q_0 \in \mathbb{F}_{q^2}[\lambda]$ such that m_{Q_0} is odd. We have*

$$\mathrm{Int}(g) = \frac{m_{Q_0}+1}{2} \cdot \deg Q_0 \cdot \prod_{\{Q,Q^*\}} (1+m_Q).$$

Here the product is over pairs $\{Q, Q^*\}$ of monic irreducible non-self-reciprocal polynomials in $\mathbb{F}_{q^2}[\lambda]$ with non-zero constant terms.

Theorem 1.2.1 then follows immediately from Theorem 1.2.4 and the explicit formula for the analytic side given in [RTZ13, Proposition 8.2].

Remark 1.2.5. Theorem 1.2.4 is also used to prove the minuscule case of Liu's arithmetic fundamental lemma for Fourier-Jacobi cycles, see [Liu18, Appendix E].

Remark 1.2.6. In Theorem 5.2.4 we also establish an analogous arithmetic intersection formula for GSpin Rapoport–Zink spaces arising from the AGGP conjectures for orthogonal groups. This provides a new proof of the main result of [LZ18], and also removes the assumption that $p \geqslant \frac{n+1}{2}$ in *loc. cit.*

1.3. **Computing the arithmetic intersection.** The starting point of the proof of Theorem 1.2.4 is the observation made in [LZ17, Proposition 4.1.2] that, in the minuscule case, the formal scheme (1.1.2) can be identified with the fixed point scheme $\mathcal{V}^{\bar{g}}$ of an explicitly given smooth projective variety $\mathcal{V}$ over k, under a finite-order automorphism $\bar{g}$. It also turns out that $\mathcal{V}^{\bar{g}}$ is an Artinian scheme. Hence $\mathrm{Int}(g)$ is given by the k-length of $\mathcal{V}^{\bar{g}}$.

In order to compute the k-length of $\mathcal{V}^{\bar{g}}$, there are two apparent approaches. One approach, taken in [LZ17], is to explicitly study all the local equations. The other approach, which we take in the current paper, is to compute it using the Lefschetz trace formula. Thus we obtain

$$\mathrm{Int}(g) = \mathrm{tr}\left(\bar{g}, \mathbf{H}^*(\mathcal{V})\right), \tag{1.3.1}$$

where $\mathbf{H}^*(\mathcal{V})$ denotes the étale $\mathbb{Q}_\ell$-cohomology of $\mathcal{V}$, for a fixed prime $\ell \neq p$.

To compute the right hand side of (1.3.1), we utilize the fact that the variety $\mathcal{V}$ is the closure of a generalized Deligne–Lusztig variety in a partial flag variety of the unitary group $\mathbb{G} = \mathrm{U}(\mathbb{V})$ over $\mathbb{F}_q$. To be precise, let $G := \mathbb{G}_k$, and let σ be the Frobenius automorphism of k over $\mathbb{F}_q$. Then $\mathcal{V}$ is the closure inside G/P of the generalized Deligne–Lusztig variety

$$X_P(w) := \{hP \in G/P : h^{-1}\sigma(h) \in PwP\},$$

for a certain standard parabolic subgroup $P \subset G$ and a certain $w \in W_P \backslash W / W_P$. Here W denotes the Weyl group of G and W_P denotes the parabolic subgroup of W corresponding to P. The automorphism $\bar{g}$ of $\mathcal{V}$ is given by the natural action of the group element $\bar{g} \in \mathbb{G}(\mathbb{F}_q)$.

Vollaard [Vol10, Theorem 2.15] constructed a nice stratification

$$\mathcal{V} = \bigsqcup_i X_i \tag{1.3.2}$$

of $\mathcal{V}$ into finitely many locally closed strata X_i, where each X_i is the image in G/P of a generalized Deligne–Lusztig variety in G/P_i for a different parabolic subgroup $P_i \subset G$. This stratification is remarkable because it is different from the naïve decomposition

$$\mathcal{V} = \overline{X_P(w)} = \bigsqcup_{w' \in W_P \backslash W / W_P, w' \leq w} X_P(w').$$

In fact, the stratification (1.3.2) is a special example of *stratification into fine Deligne–Lusztig varieties*, which will be discussed in the next subsection §1.4. Now each X_i turns out to be a fine Deligne–Lusztig variety in G/P, and can be related via parabolic induction to a classical Deligne–Lusztig variety in the full flag variety of a Levi subgroup of G. In this way, the computation of the right hand side of (1.3.1) reduces to computing the characters on the cohomology with compact support $\mathbf{H}_c^*(X_i)$ for each X_i, and eventually reduces to the classical Deligne–Lusztig character formula in [DL76].

We thus place the problem of computing the right hand side of (1.3.1) into the more general framework of developing a character formula for fine Deligne–Lusztig varieties and their closures.

1.4. **A character formula for fine Deligne–Lusztig varieties.** Let $\mathbb{F}_q$ be a finite field. Let $k = \overline{\mathbb{F}}_q$, and let σ be the Frobenius automorphism of k over $\mathbb{F}_q$. Let $\mathbb{G}$ be a connected reductive group over $\mathbb{F}_q$. Let $G = \mathbb{G}_k$, and let W be the Weyl group of G. Let J be a subset of the simple reflections in W. Let W_J be the subgroup of W generated by J, and let P_J be the corresponding standard parabolic subgroup of G. Let ${}^JW$ be the set of minimal length coset representatives of $W_J\backslash W$. For $w \in {}^JW$, we have the associated *fine Deligne–Lusztig variety*

$$X_{J,w} = \{gP_J \in G/P_J; g^{-1}\sigma(g) \in P_J \cdot_\sigma BwB\},$$

where $\cdot_\sigma$ is the σ-conjugation action. When $J = \emptyset$, $X_{\emptyset,w}$ recovers the classical Deligne–Lusztig variety X_w inside the full flag variety of G, associated to w.

In Definition 2.4.1, we will introduce the notion of a *σ-unbranched datum* $(J, \mathscr{L})$, where J is a set of simple reflections in W, and $\mathscr{L}$ is a sub-diagram of the Dynkin diagram of G satisfying certain axioms with respect to J. Associated to such $(J, \mathscr{L})$, we will construct canonically a finite sequence of elements $w_i \in {}^JW$, such that we have the following simple closure relation (see Corollary 2.4.6)

$$\overline{X_{J,w_1}} = \bigsqcup_i X_{J,w_i}. \tag{1.4.1}$$

The above stratification subsumes (1.3.2) as a special case. Moreover, for each i we will construct a rational parabolic subgroup $\mathbb{P}_i \subset \mathbb{G}$, and a projection to a reductive group $\mathbb{P}_i \to \mathbb{G}_i$ over $\mathbb{F}_q$, such that w_i can be naturally viewed as an element of the Weyl group W_i of $G_i := \mathbb{G}_{i,k}$. We show that each fine Deligne–Lusztig variety X_{J,w_i} is related via parabolic induction to the classical Deligne–Lusztig variety $X_{w_i}^{\mathbb{G}_i}$ in the full flag variety of G_i associated to w_i (see Proposition 2.5.1):

$$X_{J,w_i} \cong \mathbb{G}(\mathbb{F}_q) \times^{\mathbb{P}_i(\mathbb{F}_q)} X_{w_i}^{\mathbb{G}_i}.$$

For each i, we fix a σ-stable maximal torus $T_i \subset G_i$ of type w_i. Now we are ready to state our main character formula.

Theorem 1.4.1 (Theorem 2.8.1)**.** *Assume $(J, \mathscr{L})$ is a σ-unbranched datum. Let $w_i, \mathbb{P}_i, \mathbb{G}_i, T_i$ be as above. Let $g \in \mathbb{G}(\mathbb{F}_q)$ be a regular element. Then*

$$\operatorname{tr}(g, \mathbf{H}^*(\overline{X_{J,w_1}})) = \sum_i \operatorname{tr}(g, \mathbf{H}_c^*(X_{J,w_i})) = \sum_i \sum_{\gamma\in\Gamma_i} \#\mathcal{M}_i^{g,\gamma} \cdot \frac{|\mathbb{G}_{i,\gamma}(\mathbb{F}_q)|}{|\mathbb{G}_{i,\gamma}^0(\mathbb{F}_q)|} \cdot \left|T_i \cap ({}^{\mathbb{G}_i(\mathbb{F}_q)}\gamma_i)\right|. \tag{1.4.2}$$

Here we have

- *Γ_i is a complete set of representatives of elements in $T_i(\mathbb{F}_q)$ modulo $\mathbb{G}_i(\mathbb{F}_q)$-conjugacy.*
- *$\mathcal{M}_i^g := \{r \in \mathbb{G}(\mathbb{F}_q)/\mathbb{P}_i(\mathbb{F}_q); r^{-1}gr \in \mathbb{P}_i(\mathbb{F}_q)\}$, and $\mathcal{M}_i^{g,\gamma} \subset \mathcal{M}_i^g$ consists of those $r \in \mathcal{M}_i^g$ such that the semi-simple part of the projection of $r^{-1}gr$ to $\mathbb{G}_i$ is $\mathbb{G}_i(\mathbb{F}_q)$-conjugate to γ.*
- *${}^{\mathbb{G}_i(\mathbb{F}_q)}\gamma_i$ is the $\mathbb{G}_i(\mathbb{F}_q)$-conjugacy class of γ_i.*

1.5. **Four families of fine Deligne–Lusztig varieties.** In §4, we apply Theorem 1.4.1 to fine Deligne–Lusztig varieties that arise from the basic loci of Shimura varieties of Coxeter type [GH15]. There are four infinite families of such fine Deligne–Lusztig varieties, where the $\mathbb{F}_q$-groups $\mathbb{G}$ are respectively the even non-split special orthogonal group, the odd special orthogonal group, the symplectic group, and the odd unitary group.

In all these cases, we obtain an explicit formula for $\mathrm{tr}(g, \mathbf{H}^*(\overline{X_{J,w_1}}))$, for $g \in \mathbb{G}(\mathbb{F}_q)$ whose image under the standard representation is regular. Our formula is in terms of the characteristic polynomial of g, subsuming the formula in Theorem 1.2.4 as a special case. See Theorems 4.3.3, 4.4.3, 4.5.4, 4.6.3. The odd unitary cases and the even non-split special orthogonal cases are relevant to the AGGP conjectures for unitary and orthogonal groups respectively, and our formulas lead to the arithmetic intersection formulas in Theorem 1.2.4 and Remark 1.2.6.

1.6. **Further remarks on Theorem 1.2.4.** Arguably the most difficult part of Theorem 1.2.4 is to compute the intersection multiplicity at each point of intersection in (1.1.2). The computation in [RTZ13] uses Zink's theory of windows and displays to compute the local equations of (1.1.2). It requires explicitly writing down the window of the universal deformation of p-divisible groups. The assumption $p > \frac{n+1}{2}$ made in *loc. cit.* ensures that the ideal of local equations is admissible (see the last paragraph of [RTZ13, p. 1661]), which is crucial in order to construct the frames for the relevant windows needed in Zink's theory.

As mentioned above, the starting point of the simplified proof in [LZ17] is that the intersection (1.1.2) can be identified with $\mathcal{V}^{\bar{g}}$, and thus a deformation-theoretic problem of p-divisible groups is transformed to a purely algebro-geometric problem over k. When $p > \frac{n+1}{2}$, the computation of $\mathcal{V}^{\bar{g}}$ is further reduced in [LZ17] to a more elementary fixed point problem of a linear transformation on a projective space. However, when $p \leqslant \frac{n+1}{2}$ the multiplicities remain mysterious.

Our proof of Theorem 1.2.4 shares the same starting point as [LZ17]. The new observation is the inductive structure of fine Deligne–Lusztig varieties, which allows us to exploit the full power of the classical character formula of Deligne–Lusztig. Our approach circumvents the need to analyze the local structure of (1.1.2), and gives the desired formula without the extra assumption on p.

Finally, we remark that in the computation in [RTZ13] or [LZ17], the number $\frac{m_{Q_0}+1}{2}$ in Theorem 1.2.4 appears as the common intersection multiplicity at each point of intersection. In our current computation, we obtain a different geometric interpretation of this number, as the *number of the strata* X_i whose $\mathbf{H}_c^*$ contribute non-trivially to the trace (1.3.1). (In the proofs of Theorem 4.3.3 and Theorem 4.6.3, this number appears as $|\mathscr{I}|$.) As a simple illustration of this phenomenon, consider the automorphism $f(x) = x + 1$ of order p on $\mathbb{P}^1$ over k. The only fixed point is ∞, which has multiplicity 2. On the other hand, we have an f-stable stratification $\mathbb{P}^1 = \mathbb{A}^1 \sqcup \{\infty\}$, which gives $\mathrm{tr}(f, \mathbf{H}^*(\mathbb{P}^1)) = \mathrm{tr}(f, \mathbf{H}_c^*(\mathbb{A}^1)) + \mathrm{tr}(f, \mathbf{H}_c^*(\infty))$. Note that $\mathrm{tr}(f, \mathbf{H}_c^*(\mathbb{A}^1)) = \mathrm{tr}(f, \mathbf{H}_c^*(\infty)) = 1$. Thus the multiplicity 2 also appears as the number of contributing strata.

1.7. **Organization of the paper.** In §2, we introduce the notion of a σ-unbranched datum, and study the closure relation and inductive structure for the fine Deligne–Lusztig varieties associated to a σ-unbranched datum, culminating in the proof of the general character formula Theorem 1.4.1 (Theorem 2.8.1). In §3, we recall the four infinite families of fine Deligne–Lusztig varieties arising from basic loci of Coxeter type in Shimura varieties. In each case we identify the unique σ-unbranched datum. In §4, we apply the general character formula to each of the four families in §3, obtaining explicit character formulas in terms of characteristic polynomials (Theorems 4.3.3, 4.4.3, 4.5.4, 4.6.3). In §5, we apply the results in §4 to obtain the arithmetic intersection formulas in Theorem 1.2.4 and Remark 1.2.6 (Theorem 5.1.2 and Theorem 5.2.4).

1.8. **Notations and conventions.** Let k be an algebraically closed field. For a smooth scheme X over k, we denote by $\mathbf{H}^*(X)$ and $\mathbf{H}^*_c(X)$ the étale $\mathbb{Q}_\ell$-cohomology and the étale $\mathbb{Q}_\ell$-cohomology with compact support respectively, for a fixed prime ℓ which is invertible in k.

For any linear algebraic group G over k, we identify G with its k-points. If a subfield k_0 of k and a k_0-form $\mathbb{G}$ of G are given in the context, we often abuse notation to write $G(k_0)$ for $\mathbb{G}(k_0)$.

By convention, a *quadratic space* means a finite-dimensional vector space over a field equipped with a non-degenerate quadratic form. Since we will never consider characteristic 2 fields, we shall specify the quadratic form by specifying its associated bi-linear pairing. Thus the quadratic form is recovered from the bilinear pairing $[\cdot,\cdot]$ as $x \mapsto [x,x]/2$. Similarly, *Hermitian forms* and *symplectic forms* are all understood to be non-degenerate.

For any field F, we denote by $F[\lambda]^{\mathrm{monic}}$ the set of monic polynomials in the polynomial ring $F[\lambda]$.

1.9. **Acknowledgments.** X. H. was partially supported by the NSF grant DMS-1801352. C. L. was partially supported by an AMS travel grant for ICM 2018 and the NSF grant DMS-1802269. Y. Z. was partially supported by the NSF grant DMS-1802292. We would like to thank the Hausdorff Center for Mathematics for the hospitality, during the Conference on the Occasion of Michael Rapoport's 70th Birthday. We would also like to thank the referees for careful reading and useful comments.

2. Fine Deligne–Lusztig varieties

2.1. **Basic setting and notations.** Fix an odd prime p, and let q be a power of p. Let $k = \overline{\mathbb{F}}_q$ and σ be the Frobenius automorphism of k over $\mathbb{F}_q$.

Let $\mathbb{G}$ be a connected reductive group over $\mathbb{F}_q$, and let $G = \mathbb{G}_k$. We fix a σ-stable Borel subgroup B of G, with a Levi decomposition $B = TU$ which is also σ-stable. Let W be the *canonical Weyl group* of G equipped with the canonical action of the Frobenius σ, as in [DL76, §1.1]. Then using the pair (T, B) we identify W with $N_G(T)/T$, and the identification is σ-equivariant.

Let $\mathbb{S}$ be the set of simple reflections in W. For any $J \subset \mathbb{S}$, let $P_J \supset B$ be the standard parabolic subgroup of G associated to J, and let L_J be the standard Levi subgroup of P_J. Denote by W_J the subgroup of W generated by J (called a parabolic subgroup of W). Thus W_J is the Weyl group of L_J.

For $w \in W$, we denote by $\mathrm{supp}(w)$ the support of w, i.e., the set of simple reflections that occur in some (or equivalently, any) reduced expression of w. We define

$$\mathrm{supp}_\sigma(w) := \bigcup_{i\in\mathbb{Z}} \sigma^i(\mathrm{supp}(w)).$$

We recall the notion of Coxeter elements following [Spr74, 7.3]. For each σ-orbit in $\mathbb{S}$, we pick a simple reflection. Let c be the product of these simple reflections in any given order. We call such c a *σ-twisted Coxeter element* of W. More generally, for a σ-stable subset $\Sigma \subset \mathbb{S}$, we may consider σ-twisted Coxeter elements of the parabolic subgroup W_Σ. If c is such an element, then $\mathrm{supp}_\sigma(c) = \Sigma$, and $\mathrm{supp}(c)$ is a complete set of representatives of the σ-orbits in Σ.

2.2. **Classical Deligne–Lusztig varieties.** For $w \in W$, the (classical) Deligne–Lusztig variety X_w in the full flag variety G/B is defined by

$$X_w = \{gB \in G/B; g^{-1}\sigma(g) \in BwB\}.$$

These Deligne–Lusztig varieties give a partition of the full flag variety

$$G/B = \bigsqcup_{w\in W} X_w.$$

The closure relation is given by the Bruhat order $\leq$ of the Weyl group, i.e. for any $w \in W$,

$$\overline{X_w} = \bigsqcup_{w' \leqslant w} X_{w'}.$$

2.3. **Fine Deligne–Lusztig varieties.** Let $J \subset \mathbb{S}$. Let G/P_J be the partial flag variety of type J. In 1977, Lusztig introduced a partition of G/P_J into fine Deligne–Lusztig varieties.

We follow the approach in [He09, §3]. Let ${}^J W$ be the set of minimal length coset representatives of $W_J \backslash W$. For any $w \in {}^J W$, we set

$$X_{J,w} = \{gP_J \in G/P_J; g^{-1}\sigma(g) \in P_J \cdot_\sigma BwB\},$$

where $\cdot_\sigma$ is the σ-conjugation action, i.e., $x \cdot_\sigma y := xy\sigma(x)^{-1}$. When $J = \emptyset$, we have $X_{\emptyset,w} = X_w$.

Then we have a partition

$$G/P_J = \bigsqcup_{w \in {}^J W} X_{J,w}$$

into locally closed sub-varieties.

The partial order $\leqslant_{J,\sigma}$ on ${}^J W$ is introduced in [He07a, Proposition 3.8] (see also [He07b, 4.7]). For $w, w' \in {}^J W$, we write

$$w \leqslant_{J,\sigma} w'$$

if $uw\sigma(u)^{-1} \leqslant w'$ for some $u \in W_J$. By [He07a, Proposition 3.13] and [He07b, Corollary 4.6], $\leqslant_{J,\sigma}$ is a partial order on ${}^J W$. Now we have

Theorem 2.3.1. [He09, Theorem 3.1] *For $w \in {}^J W$,*

$$\overline{X_{J,w}} = \bigsqcup_{w' \in {}^J W; w' \leqslant_{J,\sigma} w} X_{J,w'}. \qquad \square$$

2.4. **The σ-unbranched datum.** We would like to single out certain cases where the right hand side of Theorem 2.3.1 has a relatively simple description.

Definition 2.4.1. We say that a subset $J \subset \mathbb{S}$ is *σ-unbranched* if the following conditions hold.

(1) The set $\mathbb{S} - J$ is contained in one σ-orbit in $\mathbb{S}$.

(2) There exists a sub-diagram $\mathscr{L}$ of the Dynkin diagram of $(\mathbb{G}, W, \mathbb{S})$ satisfying the following conditions.
- The diagram $\mathscr{L}$ is connected and without branching;
- The nodes of $\mathscr{L}$ form a complete set of representatives of the σ-orbits in $\mathbb{S}$.
- One (and hence exactly one) end-node of $\mathscr{L}$ is in $\mathbb{S} - J$.

We call a pair $(J, \mathscr{L})$ as above a *σ-unbranched datum* for $\mathbb{G}$. When we would like to emphasize the group $\mathbb{G}$, we write $(\mathbb{G}, J, \mathscr{L})$.

2.4.2. From now on we assume the existence of a σ-unbranched subset $J \subset \mathbb{S}$, and fix a σ-unbranched datum $(J, \mathscr{L})$ once and for all. Let a be the number of nodes in $\mathscr{L}$. By assumption $\mathscr{L}$ is connected and without branching, with exactly one end-node in $\mathbb{S} - J$. Hence we may canonically list the consecutive nodes in $\mathscr{L}$ as

$$\mathfrak{r}_1, \mathfrak{r}_2, \cdots, \mathfrak{r}_a \in \mathbb{S}, \tag{2.4.1}$$

with $\mathfrak{r}_a \in \mathbb{S} - J$. Write $i_{\max} = a + 1$.

For each $1 \leq i \leq i_{\max}$, define

$$w_i := \mathfrak{r}_a \mathfrak{r}_{a-1} \cdots \mathfrak{r}_i.$$

Here by convention $w_{i_{\max}} := 1$. We also define

$$\Sigma_i^\flat := \operatorname{supp}_\sigma w_i = \bigcup_{j=i}^{a} \text{the } \sigma\text{-orbit of } \mathfrak{r}_j,$$

$$\Sigma_i := \begin{cases} \text{the } \sigma\text{-orbit of } \mathfrak{r}_{i-1}, & \text{if } 2 \le i \le i_{\max}, \\ \emptyset, & \text{if } i = 1, \end{cases}$$

$$\Sigma_i^\sharp := \mathbb{S} - (\Sigma_i^\flat \cup \Sigma_i).$$

Lemma 2.4.3. *For all $3 \le i \le a$ and $m \in \mathbb{Z}$, the sets $\{\sigma^m(\mathfrak{r}_{i-2}), \sigma^m(\mathfrak{r}_{i-3}), \cdots, \sigma^m(\mathfrak{r}_1)\}$ and $\{\mathfrak{r}_a, \mathfrak{r}_{a-1}, \cdots, \mathfrak{r}_i\}$ are disconnected from each other.*

Proof. Suppose not. Then there exist j, l, with $i \le j \le a$ and $1 \le l \le i-2$, such that $\mathfrak{r}_j$ is connected with $\sigma^m(\mathfrak{r}_l)$. Choose $n \in \mathbb{N}$ such that $\sigma^{nm}(\mathfrak{r}_j) = \mathfrak{r}_j$. Then in the list

$$\mathfrak{r}_j, \sigma^m\mathfrak{r}_l, \sigma^m\mathfrak{r}_{l+1}, \cdots, \sigma^m\mathfrak{r}_j, \sigma^{2m}\mathfrak{r}_l, \cdots, \sigma^{2m}\mathfrak{r}_j, \cdots, \sigma^{nm}\mathfrak{r}_l, \cdots, \sigma^{nm}\mathfrak{r}_j,$$

each member is connected with its predecessor, and the last member is equal to the first member. Since the Dynkin diagram does not contain loops, there must exist two adjacent members in the above list that are equal. Thus there exist integers α, β, with $l \le \beta < j$, such that $\sigma^{\alpha m}\mathfrak{r}_\beta = \sigma^{\alpha m}\mathfrak{r}_{\beta+1}$ or $\sigma^{\alpha m}\mathfrak{r}_j = \sigma^{(\alpha+1)m}\mathfrak{r}_l$. The first case is impossible because $\mathfrak{r}_\beta \neq \mathfrak{r}_{\beta+1}$. In the second case, we have $\mathfrak{r}_j = \sigma^m\mathfrak{r}_l$, which contradicts with the axiom that $\mathfrak{r}_j$ and $\mathfrak{r}_l$ represent different σ-orbits in $\mathbb{S}$. □

Lemma 2.4.4. *For each $1 \le i \le i_{\max}$, we have*

$$\mathbb{S} = \Sigma_i^\flat \sqcup \Sigma_i \sqcup \Sigma_i^\sharp.$$

The sets $\Sigma_i^\flat, \Sigma_i, \Sigma_i^\sharp$ are all σ-stable. Moreover $\Sigma_i^\flat$ is disconnected from $\Sigma_i^\sharp$.

Proof. The first assertion holds because $\mathfrak{r}_1, \cdots, \mathfrak{r}_a$ lie in distinct σ-orbits in $\mathbb{S}$. The second assertion follows easily from the definition. The third assertion follows from Lemma 2.4.3. □

Note that each w_i is σ-twisted Coxeter in $W_{\Sigma_i^\flat}$, and $W_{\Sigma_1^\flat} = W_{\mathbb{S}} = W$. We further have the following result.

Lemma 2.4.5. *For each $1 \le i \le i_{\max}$, we have $w_i \in {}^J W$. Moreover*

$$\{w \in {}^J W; w \le_{J,\sigma} w_1\} = \{w_1, w_2, \cdots, w_{i_{\max}}\}.$$

Proof. Since $\mathscr{L}$ is connected and since $\mathfrak{r}_a \in \mathbb{S} - J$, we have $w_i \in {}^J W$. By definition, $w_i \leqslant w_1$ for any i.

On the other hand, let $w \in {}^J W$ with $w \leqslant_{J,\sigma} w_1$. Then by [He07a, Proposition 3.8], there exists $u \in W_J$ with $\ell(w\sigma(u)^{-1}) = \ell(w) - \ell(u)$ and $uw\sigma(u)^{-1} \leqslant w_1$. Then we have $w\sigma(u)^{-1} \in {}^J W$ and $w\sigma(u)^{-1} = w_i$ for some $1 \leqslant i \leqslant i_{\max}$. Then $uw_i \leqslant w_1$. Since $u \in W_J$ and $w_i \in {}^J W$, we have $\ell(uw_i) = \ell(u) + \ell(w_i)$. Note that $\mathfrak{r}_{i-1}w_i \not\leqslant w_1$, so we have $u \leqslant \mathfrak{r}_{i-2}\mathfrak{r}_{i-3}\cdots\mathfrak{r}_1$. By Lemma 2.4.3, the sets $\{\sigma(\mathfrak{r}_{i-2}), \sigma(\mathfrak{r}_{i-3}), \ldots, \sigma(\mathfrak{r}_1)\}$ and $\{\mathfrak{r}_a, \mathfrak{r}_{a-1}, \ldots, \mathfrak{r}_i\}$ are disconnected from each other. Hence $w = w_i\sigma(u) = \sigma(u)w_i$. Since $w \in {}^J W$, we have $\sigma(u) = 1$ and hence $w = w_i$. □

By the above lemma, the fine Deligne–Lusztig variety X_{J,w_i} is defined for each $1 \le i \le i_{\max}$.

Corollary 2.4.6. *We have*

$$\overline{X_{J,w_1}} = \bigsqcup_{1 \le i \le i_{\max}} X_{J,w_i}.$$

Proof. This follows from Theorem 2.3.1 and Lemma 2.4.5. □

Given $g \in G^{\mathrm{reg}} \cap G(\mathbb{F}_q)$, our goal in this section is to compute

$$\mathrm{tr}(g, J, \mathscr{L}) := \mathrm{tr}(g, \mathbf{H}^*(\overline{X_{J,w_1}})).$$

Corollary 2.4.7. *For $g \in G^{\mathrm{reg}} \cap G(\mathbb{F}_q)$, we have*

$$\mathrm{tr}(g, J, \mathscr{L}) = \sum_{i=1}^{i_{\max}} \mathrm{tr}(g, \mathbf{H}_c^*(X_{J,w_i})).$$

Proof. This follows from Corollary 2.4.6. □

2.5. **Parabolic induction.** We keep the setting of §2.4. Fix $1 \leq i \leq i_{\max}$. Denote

$$P_i := P_{\Sigma_i^\flat \sqcup \Sigma_i^\sharp}, \quad L_i := L_{\Sigma_i^\flat \sqcup \Sigma_i^\sharp}, \quad G_i^{\mathrm{ad}} := (L_{\Sigma_i^\flat})^{\mathrm{ad}}, \quad H_i^{\mathrm{ad}} := (L_{\Sigma_i^\sharp})^{\mathrm{ad}}.$$

Since $\Sigma_i^\flat$ is disconnected from $\Sigma_i^\sharp$ (see Lemma 2.4.4), we have a canonical isomorphism

$$L_i^{\mathrm{ad}} \cong G_i^{\mathrm{ad}} \times H_i^{\mathrm{ad}}.$$

Let $L_i \to L_i^\natural$ be the central isogeny with the smallest kernel such that $L_i^\natural$ is the direct product of the inverse images in $L_i^\natural$ of G_i^{ad} and H_i^{ad}. We denote by G_i (resp. H_i) the inverse image of G_i^{ad} (resp. H_i^{ad}) in $L_i^\natural$. Then G_i^{ad} (resp. H_i^{ad}) is indeed the adjoint group of G_i (resp. H_i), so the notation is compatible.

Thus we have $L_i^\natural = G_i \times H_i$. Moreover, since $\Sigma_i^\flat, \Sigma_i^\sharp$ are σ-stable, the groups $P_i, L_i, L_i^\natural, G_i, H_i$, as well as the central isogeny $L_i \to L_i^\natural$ and the decomposition $L_i^\natural = G_i \times H_i$, are all defined over $\mathbb{F}_q$. When we would like to emphasize the reductive groups over $\mathbb{F}_q$ underlying P_i, L_i, etc., we shall write $\mathbb{P}_i, \mathbb{L}_i$, etc. We let π_i denote the projection $P_i \to L_i \to L_i^\natural \to G_i$, and let π_i' denote the projection $P_i \to L_i \to L_i^\natural \to H_i$.

Let $W_i := W_{\Sigma_i^\flat}$. Then W_i is identified with the Weyl group of G_i, inside which w_i is a σ-twisted Coxeter element. Let $X_{w_i}^{\mathbb{G}_i}$ be the classical Deligne–Lusztig variety associated to the element $w_i \in W_i$ in the full flag variety of G_i. Then we have a natural action of $G_i(\mathbb{F}_q)$ on $X_{w_i}^{\mathbb{G}_i}$. Define the action of the group $P_i(\mathbb{F}_q)$ on $G(\mathbb{F}_q) \times X_{w_i}^{\mathbb{G}_i}$ by

$$p \cdot (g, x) = (gp^{-1}, \pi_i(g) \cdot x).$$

Let $G(\mathbb{F}_q) \times^{P_i(\mathbb{F}_q)} X_{w_i}^{\mathbb{G}_i}$ be the quotient space. As a k-variety this is just a finite disjoint union of isomorphic copies of $X_{w_i}^{\mathbb{G}_i}$.

Proposition 2.5.1. *For each $1 \leq i \leq i_{\max}$, we have a $G(\mathbb{F}_q)$-equivariant isomorphism*

$$G(\mathbb{F}_q) \times^{P_i(\mathbb{F}_q)} X_{w_i}^{\mathbb{G}_i} \xrightarrow{\sim} X_{J,w_i}, \qquad (g, g'(G_i \cap B)) \longmapsto gg'P_J.$$

Proof. We fix $1 \leq i \leq i_{\max}$. We claim that $\Sigma_i^\sharp$ is the maximal subset of J that is stable under $\mathrm{Ad}(w_i) \circ \sigma$. In fact, by definition $\Sigma_i^\sharp$ is a σ-stable subset of J (see Lemma 2.4.4). Since $\Sigma_i^\sharp$ is disconnected from $\Sigma_i^\flat$ by Lemma 2.4.4, $\Sigma_i^\sharp$ is also stable under $\mathrm{Ad}(w_i)$. Now let $K \subset J$ be a $\mathrm{Ad}(w_i) \circ \sigma$-stable subset. If $i = 1$, then $\Sigma_i = \emptyset$ by definition. If $2 \leq i \leq i_{\max}$, then $\mathrm{Ad}(w_i)\mathfrak{r}_{i-1} \notin \mathbb{S}$, and for any $\mathfrak{r}$ in the σ-orbit of $\mathfrak{r}_{i-1}$, either $\mathrm{Ad}(w_i)\mathfrak{r} = \mathfrak{r}$ or $\mathrm{Ad}(w_i)\mathfrak{r} \notin \mathbb{S}$. Hence $\Sigma_i \cap K = \emptyset$ in all cases. Similarly, for any integer j with $i \leq j \leq a$, the following holds. On one hand either $\mathrm{Ad}(w_i)\mathfrak{r}_j = \mathfrak{r}_{j-1}$ or $\mathrm{Ad}(w_j) \notin \mathbb{S}$, and on the other hand, for any $\mathfrak{r} \neq \mathfrak{r}_j$ that is in the σ-orbit of $\mathfrak{r}_j$, either $\mathrm{Ad}(w_i)\mathfrak{r} = \mathfrak{r}$ or $\mathrm{Ad}(w_i)\mathfrak{r} \notin \mathbb{S}$. (In fact we always have $\mathrm{Ad}(w_i)\mathfrak{r}_i \notin \mathbb{S}$). Using this and by induction on j, we see that K does not contain any element in the σ-orbit of $\mathfrak{r}_j$, for all $j \geq i$. Therefore $K \cap \Sigma_i^\flat = \emptyset$. We already saw $K \cap \Sigma_i = \emptyset$, so $K \subset \Sigma_i^\sharp$. This proves our claim that $\Sigma_i^\sharp$ is the maximal subset of J that is stable under $\mathrm{Ad}(w_i) \circ \sigma$.

By the above claim and by [Lus07, 4.2(d)] (see also [He09, §3]), the projection map $G/P_{\Sigma_i^\sharp} \to G/P_J$ induces an isomorphism

$$X_{\Sigma_i^\sharp, w_i} \xrightarrow{\sim} X_{J,w_i}.$$

Note that $P_{\Sigma_i^\sharp} \cdot_\sigma Bw_iB \subset P_i$. Thus $gP_{\Sigma_i^\sharp} \in X_{\Sigma_i^\sharp, w_i}$ implies that $g^{-1}\sigma(g) \in P_i$. By Lang's theorem, $g^{-1}\sigma(g) \in P_i$ is equivalent to $g \in G(\mathbb{F}_q)P_i$. The projection map $G/P_{\Sigma_i^\sharp} \to G/P_i$ induces an isomorphism

$$X_{\Sigma_i^\sharp, w_i} \xrightarrow{\sim} G(\mathbb{F}_q) \times^{P_i(\mathbb{F}_q)} X',$$

where X' is the sub-variety of $P_i/P_{\Sigma_i^\sharp}$ given by

$$X' = \{pP_{\Sigma_i^\sharp} \in P_i/P_{\Sigma_i^\sharp};\ p^{-1}\sigma(p) \in P_{\Sigma_i^\sharp} \cdot_\sigma Bw_iB\}.$$

Recall that π_i denotes the projection $P_i \to L_i \to L_i^\natural \to G_i$. Note that

$$P_i/P_{\Sigma_i^\sharp} \cong L_i/(L_i \cap P_{\Sigma_i^\sharp}) \cong L_i^\natural/(\pi_i(B) \times H_i) \cong G_i/\pi_i(B),$$

where $G_i/\pi_i(B)$ is the full flag variety of G_i. Under this isomorphism, the sub-variety X' of $P_i/P_{\Sigma_i^\sharp}$ is identified to $X_{w_i}^{\mathbb{G}_i}$. The proposition is proved. □

Corollary 2.5.2. *For each $1 \le i \le i_{\max}$, we have an isomorphism of virtual $G(\mathbb{F}_q)$-representations*

$$\mathbf{H}_c^*(X_{J,w_i}) \cong \mathrm{Ind}_{P_i(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \mathbf{H}_c^*(X_{w_i}^{\mathbb{G}_i}),$$

where $P_i(\mathbb{F}_q)$ acts on $X_{w_i}^{\mathbb{G}_i}$ via the projection $\pi_i : P_i(\mathbb{F}_q) \to G_i(\mathbb{F}_q)$.

Proof. This follows immediately from Proposition 2.5.1. □

2.6. **Review of regular elements.** We recall the definition of regular elements and some standard facts. Let G be a reductive group over k.

Definition 2.6.1. An element $g \in G$ is called *regular*, if the centralizer G_g of g in G has dimension equal to the rank of G. The set of regular elements is denoted by G^{reg}.

If G is semi-simple, the above definition is the same as [Ste65]. In general, one easily checks that $g \in G$ is regular in the above sense if and only if the image of g in G^{ad} is regular. Thus we can easily transport the results from [Ste65], which only discusses semi-simple groups, to reductive groups.

Theorem 2.6.2. *An element $g \in G$ is regular if and only if there are only finitely many Borel subgroups of G that contain g.*

Proof. This follows from [Ste65, Theorem 1.1] applied to G^{ad}. □

Proposition 2.6.3. *Assume G' is a reductive group over k that contains G as a closed subgroup. Then ${G'}^{\mathrm{reg}} \cap G \subset G^{\mathrm{reg}}$.*

Proof. Fix a Borel subgroup $B' \subset G'$ that contains B. By Theorem 2.6.2, it suffices to show that the natural map between flag varieties $G/B \to G'/B'$ is finite-to-one (at the level of k-points). For this, it suffices to show that B is of finite index in $B' \cap G$. Note that the identity component $(B' \cap G)^0$ of $B' \cap G$ is a connected solvable closed subgroup of G which contains B. Hence $(B' \cap G)^0 = B$. But we know that $(B' \cap G)^0$ has finite index in $B' \cap G$ because the latter is a linear algebraic group over k. □

Proposition 2.6.4. *Let $P = P_J$ be a standard parabolic subgroup of G, with standard Levi subgroup $L = L_J$. The projection $P \to L$ maps $P \cap G^{\mathrm{reg}}$ into L^{reg}.*

Proof. The projection $P \to L$ induces a bijection from the set of Borel subgroups of G contained in P to the set of Borel subgroups of L. Thus the proposition follows from Theorem 2.6.2. □

The following proposition is well known and elementary to verify.

Proposition 2.6.5. *Let V be a finite dimensional k-vector space. An element $g \in \mathrm{GL}(V)$ is regular if and only if each eigenspace of g is one dimensional.* □

2.7. **The character formula on a classical Deligne–Lusztig variety.** Let $g \in G(\mathbb{F}_q)$ and let $g = su$ be the Jordan decomposition of g. Assume g is regular in G. Let $w \in W$. Let (T_w, B_w) be the pair associated to w as in [DL76, Lemma 1.13]. Namely, T_w is a σ-stable maximal torus of G, and B_w is a Borel subgroup of G containing T_w such that B_w and $\sigma(B_w)$ have relative position w. The pair (T_w, B_w) is well defined up to $G(\mathbb{F}_q)$-conjugation, but we fix a representative. We denote by ${}^{G}s$ the conjugacy class in $G(k)$ of s, and denote by ${}^{G(\mathbb{F}_q)}s$ the conjugacy class in $G(\mathbb{F}_q)$ of s.

Proposition 2.7.1. *In the above setting, we have*

$$\operatorname{tr}(g, \mathbf{H}_c^*(X_w)) = \frac{|G_s(\mathbb{F}_q)|}{|G_s^0(\mathbb{F}_q)|} \cdot \left|T_w \cap {}^{G(\mathbb{F}_q)}s\right|. \tag{2.7.1}$$

Proof. By [DL76, Theorem 4.2], we have

$$\operatorname{tr}(g, \mathbf{H}_c^*(X_w)) = \frac{1}{|G_s^0(\mathbb{F}_q)|} \sum_{g' \in G(\mathbb{F}_q); g' T_w (g')^{-1} \subset G_s^0} Q_{g' T_w (g')^{-1}, G_s^0}(u),$$

where $Q_{g' T_w (g')^{-1}, G_s^0}$ is the Green function. Since g is regular in G, we know that u is regular in G_s^0. Hence by [DL76, Theorem 9.16], we have $Q_{g' T_w (g')^{-1}, G_s^0}(u) = 1$ for every g' that appears in the above summation. Therefore we have

$$\operatorname{tr}(g, \mathbf{H}_c^*(X_w)) = \frac{1}{|G_s^0(\mathbb{F}_q)|} \#\{g' \in G(\mathbb{F}_q); g' T_w (g')^{-1} \subset G_s^0\}.$$

Now for $g' \in G(\mathbb{F}_q)$, the condition $g' T_w (g')^{-1} \subset G_s^0$ is equivalent to the condition $s \in g' T_w (g')^{-1}$, which is equivalent to the condition $(g')^{-1} s g' \in T_w \cap {}^{G(\mathbb{F}_q)}s$. Therefore we have

$$\#\{g' \in G(\mathbb{F}_q); g' T_w (g')^{-1} \subset G_s^0\} = |G_s(\mathbb{F}_q)| \cdot \left|T_w \cap {}^{G(\mathbb{F}_q)}s\right|$$

by the orbit-stabilizer relation. The proposition follows. □

Definition 2.7.2. For each $\gamma \in T_w(\mathbb{F}_q)$, define

$$\mathcal{T}(w, \gamma) := \frac{|G_\gamma(\mathbb{F}_q)|}{|G_\gamma^0(\mathbb{F}_q)|} \cdot \left|T_w \cap {}^{G(\mathbb{F}_q)}\gamma\right|.$$

Since T_w is well defined up to $G(\mathbb{F}_q)$-conjugation, the above definition indeed only depends on w and γ.

Corollary 2.7.3. *Let $g \in G(\mathbb{F}_q) \cap G^{\mathrm{reg}}$ and $w \in W$. Let $g = su$ be the Jordan decomposition. We have*

$$\operatorname{tr}(g, \mathbf{H}_c^*(X_w)) = \begin{cases} 0, & \text{if } T_w \cap {}^{G(\mathbb{F}_q)}s = \emptyset, \\ \mathcal{T}(w, \gamma), & \text{if } T_w \cap {}^{G(\mathbb{F}_q)}s \neq \emptyset. \end{cases}$$

In the second case, γ is any element of $T_w \cap {}^{G(\mathbb{F}_q)}s$.

Proof. This follows from Proposition 2.7.1, by noting that the right hand side of (2.7.1) only depends on the $G(\mathbb{F}_q)$-conjugacy class of s. □

2.7.4. Let $w \in W$ and $\gamma \in T_w(\mathbb{F}_q)$. We will give a more explicit formula for $\mathcal{T}(w, \gamma)$, under the assumption that G_γ is connected. For example, if G^{der} is simply connected, then our assumption is always satisfied, by a result of Steinberg [Ste68, Corollary 8.5] (cf. [Kot82, p. 788] or [Car93, Theorem 3.5.6]).

Assume G_γ is connected. We canonically identify W with $N_G(T_w)/T_w$ via the pair (T_w, B_w) fixed before. Then the Weyl group of G_γ is a canonical subgroup $W(\gamma)$ of W, generated by the reflections associated to roots α in $\Phi(T_w, G)$ such that $\alpha(\gamma) = 1$ (see [Car93, Theorem 3.5.4]). Denote by F_w the automorphism $\mathrm{Ad}(w) \circ \sigma$ of W. Then $W(\gamma)$ is stable under F_w, as γ is an $\mathbb{F}_q$-point of T_w.

Lemma 2.7.5. *In the setting of §2.7.4, we have*

$$\mathcal{T}(w,\gamma) = \#\{{}^x\gamma; x \in W, {}^x\gamma \in G(\mathbb{F}_q)\} = \#(W/W(\gamma))^{F_w}.$$

Proof. Since G_γ is connected, it follows from the Lang–Steinberg theorem that $H^1(\mathbb{F}_q, G_\gamma) = 0$, and so ${}^{G(\mathbb{F}_q)}\gamma = {}^G\gamma \cap G(\mathbb{F}_q)$. Therefore

$$\mathcal{T}(w,\gamma) = \left|T_w(\mathbb{F}_q) \cap {}^G\gamma\right|.$$

Now assume $h \in G$ satisfies $h\gamma h^{-1} \in T_w$. Then $h^{-1}T_w h \subset G_\gamma$. Sine $h^{-1}T_w h$ and T_w are two maximal tori of G_γ, there exists $c \in G_\gamma$ such that $h^{-1}T_w h = cT_w c^{-1}$. Then we have

$$h\gamma h^{-1} = (hc)\gamma(hc)^{-1}, \quad hc \in N_G(T_w).$$

The above analysis shows that,

$$\left|T_w(\mathbb{F}_q) \cap {}^G\gamma\right| = \#\{{}^x\gamma; x \in W, {}^x\gamma \in G(\mathbb{F}_q)\}.$$

This proves the first equality in the lemma. To prove the second equality, note that

$$\#\{{}^x\gamma; x \in W, {}^x\gamma \in G(\mathbb{F}_q)\} = \#(W/W_\gamma)^{F_w},$$

where W_γ is the stabilizer of γ in W. Since G_γ is connected, we have $W_\gamma = W(\gamma)$, see [Car93, Theorem 3.5.3]. □

2.8. **Combining the results.** Keep the setting of §2.4. For each $1 \le i \le i_{\max}$, fix a σ-stable maximal torus T_i in G_i of type w_i. Fix $\Gamma_i \subset T_i(\mathbb{F}_q)$ to be a complete set of representatives of elements in $T_i(\mathbb{F}_q)$ modulo $G_i(\mathbb{F}_q)$-conjugacy. Fix $g \in G(\mathbb{F}_q)$. For each $1 \le i \le i_{\max}$ and each $\gamma \in \Gamma_i$, define

$$\widetilde{\mathcal{M}}_i^g := \left\{r \in G(\mathbb{F}_q); r^{-1}gr \in P_i(\mathbb{F}_q)\right\}, \qquad \widetilde{\mathcal{M}}_i^{g,\gamma} := \{r \in \widetilde{\mathcal{M}}_i^g; (\pi_i(r^{-1}gr))_s \in {}^{G_i(\mathbb{F}_q)}\gamma\}.$$

Here $(\pi_i(r^{-1}gr))_s$ denotes the semi-simple part of $\pi_i(r^{-1}gr) \in G_i(\mathbb{F}_q)$ in the Jordan decomposition. Note that $\widetilde{\mathcal{M}}_i^g$ and $\widetilde{\mathcal{M}}_i^{g,\gamma}$, if non-empty, are stable under right multiplication by $P_i(\mathbb{F}_q)$. We denote

$$\mathcal{M}_i^g := \widetilde{\mathcal{M}}_i^g/P_i(\mathbb{F}_q), \qquad \mathcal{M}_i^{g,\gamma} := \widetilde{\mathcal{M}}_i^{g,\gamma}/P_i(\mathbb{F}_q).$$

For $\gamma \in \Gamma_i \subset T_i(\mathbb{F}_q)$, we also define $\mathcal{T}(w_i,\gamma)$ as in Definition 2.7.2, with respect to $\mathbb{G}_i$ and $w_i \in W_i$.

Theorem 2.8.1. *Fix $g \in G(\mathbb{F}_q) \cap G^{\mathrm{reg}}$. Then*

$$\mathrm{tr}(g, J, \mathscr{L}) = \sum_{i=1}^{i_{\max}} \sum_{\gamma \in \Gamma_i} \#\mathcal{M}_i^{g,\gamma} \cdot \mathcal{T}(w_i,\gamma).$$

Proof. By Corollary 2.4.7 and Corollary 2.5.2, we have

$$\mathrm{tr}(g, J, \mathscr{L}) = \sum_{i=1}^{i_{\max}} |P_i(\mathbb{F}_q)|^{-1} \sum_{r \in \widetilde{\mathcal{M}}_i^g} \mathrm{tr}(\pi_i(r^{-1}gr), \mathbf{H}_c^*(X_{w_i}^{\mathbb{G}_i})). \tag{2.8.1}$$

Fix $1 \le i \le i_{\max}$. For any $r \in \widetilde{\mathcal{M}}_i^g$, it follows from Proposition 2.6.4 that the image of $r^{-1}gr$ under $P_i \to L_i$ is regular in L_i. It easily follows that $\pi_i(r^{-1}gr)$ is regular in G_i. We may hence apply Corollary 2.7.3 to get

$$\sum_{r \in \widetilde{\mathcal{M}}_i^g} \mathrm{tr}(\pi_i(r^{-1}gr), \mathbf{H}_c^*(X_{w_i}^{\mathbb{G}_i})) = \sum_{\gamma \in \Gamma_i} \sum_{r \in \widetilde{\mathcal{M}}_i^{g,\gamma}} \mathcal{T}(w_i,\gamma) = \sum_{\gamma \in \Gamma_i} \#\widetilde{\mathcal{M}}_i^{g,\gamma} \cdot \mathcal{T}(w_i,\gamma). \tag{2.8.2}$$

Combining (2.8.1) and (2.8.2), we obtain

$$\mathrm{tr}(g, J, \mathscr{L}) = \sum_{i=1}^{i_{\max}} \sum_{\gamma \in \Gamma_i} |P_i(\mathbb{F}_q)|^{-1} \#\widetilde{\mathcal{M}}_i^{g,\gamma} \cdot \mathcal{T}(w_i,\gamma) = \sum_{i=1}^{i_{\max}} \sum_{\gamma \in \Gamma_i} \#\mathcal{M}_i^{g,\gamma} \cdot \mathcal{T}(w_i,\gamma).$$ □

3. Basic loci of Shimura varieties of Coxeter type

The notion of basic loci of Coxeter type in Shimura varieties is introduced in [GH15]. The basic loci in these cases can be decomposed into a finite union of Ekedahl–Oort strata indexed by the set $\mathrm{EO}^K_{\sigma,\mathrm{cox}}$ defined in [GH15, §5.1], and each Ekedahl–Oort stratum is a union of classical Deligne–Lusztig varieties of Coxeter type. We have the following classification theorem.

Table 1. σ-unbranched data

Enhanced Tits datum	σ-unbranched datum $(\mathbb{G}, J, \mathscr{L} = (\mathfrak{r}_1, \mathfrak{r}_2, \cdots, \mathfrak{r}_a))$	
$(A_n, \omega_1^\vee, \mathbb{S})$	(trivial group, $\emptyset, \emptyset$)	
$(B_n, \omega_1^\vee, \mathbb{S})$	$({}^2D_n, \mathbb{S} - \{s_{n-1}\}, (s_1, \cdots, s_{n-1}))$	
$(B_n, \omega_1^\vee, \tilde{\mathbb{S}} - \{n\})$	$(B_{n-1}, \mathbb{S} - \{s_{n-1}\}, (s_1, \cdots, s_{n-1}))$	
$(B\text{-}C_n, \omega_1^\vee, \mathbb{S})$	$({}^2D_n, \mathbb{S} - \{s_{n-1}\}, (s_1, \cdots, s_{n-1}))$	
$(B\text{-}C_n, \omega_1^\vee, \tilde{\mathbb{S}} - \{n\})$	$(B_{n-1}, \mathbb{S} - \{s_{n-1}\}, (s_1, \cdots, s_{n-1}))$	
$(C\text{-}B_n, \omega_1^\vee, \mathbb{S})$	$(B_n, \mathbb{S} - \{s_n\}, (s_1, \cdots, s_n))$	
$(C\text{-}BC_n, \omega_1^\vee, \mathbb{S})$	$(B_n, \mathbb{S} - \{s_n\}, (s_1, \cdots, s_n))$	
$(C\text{-}BC_n, \omega_1^\vee, \tilde{\mathbb{S}} - \{n\})$	$(C_n, \mathbb{S} - \{s_n\}, (s_1, \cdots, s_n))$	
$(D_n, \omega_1^\vee, \mathbb{S})$	$({}^2D_{n-1}, \mathbb{S} - \{s_{n-2}\}, (s_1, \cdots, s_{n-2}))$	
$({}^2A'_n, \omega_1^\vee, \mathbb{S})$	$({}^2A_{2m}, \mathbb{S} - \{s_m\}, (s_1, \cdots, s_m)),\ m := \lfloor \frac{n-1}{2} \rfloor$	
$({}^2B_n, \omega_1^\vee, \tilde{\mathbb{S}} - \{n\})$	$(B_n, \mathbb{S} - \{s_n\}, (s_1, \cdots, s_n))$	
$({}^2B\text{-}C_n, \omega_1^\vee, \tilde{\mathbb{S}} - \{n\})$	$(C_n, \mathbb{S} - \{s_n\}, (s_1, \cdots, s_n))$	
$({}^2D_n, \omega_1^\vee, \mathbb{S})$	$({}^2D_n, \mathbb{S} - \{s_{n-1}\}, (s_1, \cdots, s_{n-1}))$	
$(A_3, \omega_2^\vee, \mathbb{S})$	$({}^2(A_1 \times A_1), \{s_1\}, (s_2))$	*
$({}^2A'_3, \omega_2^\vee, \mathbb{S})$	$({}^2A_3, \{s_2, s_3\}, (s_2, s_1))$	*
$(C_2, \omega_2^\vee, \mathbb{S})$	$({}^2(A_1 \times A_1), \{s_1\}, (s_2))$	*
$(C_2, \omega_2^\vee, \tilde{\mathbb{S}} - \{1\})$	$(A_1, \emptyset, (s_1))$	
$({}^2C_2, \omega_2^\vee, \tilde{\mathbb{S}} - \{1\})$	$(B_2, \{s_1\}, (s_1, s_2))$	
$({}^2C\text{-}B_2, \omega_1^\vee, \tilde{\mathbb{S}} - \{1\})$	$(B_2, \{s_2\}, (s_2, s_1))$	*

Theorem 3.0.2. [GH15, Theorem A] *The irreducible enhanced Tits data of Coxeter type for σ-stable maximal K are classified in the first column of Table 1.*

We list in the second column of Table 1 the associated σ-unbranched data. In each case, let w be the maximal element in $\mathrm{EO}^K_{\sigma,\mathrm{cox}}$ computed in [GH15, §6]. Then the reductive group $\mathbb{G}$ over $\mathbb{F}_q$ is the reductive quotient of the parahoric subgroup associated to $\mathrm{supp}_\sigma(w)$, and we have $J = K \cap \mathrm{supp}_\sigma(w)$. In each case it turns out that J is σ-unbranched, and that there is a unique σ-unbranched datum of the form $(J, \mathscr{L})$. In table Table 1 we record the type of $\mathbb{G}$, the set J, and the nodes $(\mathfrak{r}_1, \cdots, \mathfrak{r}_a)$ of the unique $\mathscr{L}$ in the order as in (2.4.1). We let $s_i \in \mathbb{S}$ denote the i-th node, according to Bourbaki's numbering [Bou68]. In all except the four cases marked with $*$, we have $\mathfrak{r}_i = s_i$ for all $1 \le i \le a$.

Consequently, the associated fine Deligne–Lusztig varieties come in four infinite families:

(1) $\mathbb{G}$ is the non-split even special orthogonal group SO_{2n}, $J = \mathbb{S} - \{s_{n-1}\}$, $\mathscr{L} = (s_1, \cdots, s_{n-1})$.

(2) $\mathbb{G}$ is the odd special orthogonal group SO_{2n+1}, $J = \mathbb{S} - \{s_n\}$, $\mathscr{L} = (s_1, \cdots, s_n)$.

(3) $\mathbb{G}$ is the symplectic group Sp_{2n}, $J = \mathbb{S} - \{s_n\}$, $\mathscr{L} = (s_1, \cdots, s_n)$.

(4) $\mathbb{G}$ is the odd unitary group U_{2n+1}, $J = \mathbb{S} - \{s_n\}$, $\mathscr{L} = (s_1, \cdots, s_n)$.

4. Explicit character formulas

In this section, we use Theorem 2.8.1 to compute $\mathrm{tr}(g, J, \mathscr{L})$ for the four infinite families specified at the end of §3. We shall only consider $g \in G(\mathbb{F}_q)$ whose image in GL_N under the standard representation is regular. This is a stronger hypothesis than requiring g to be regular in G, except for the unitary case. However, for the known arithmetic applications this is enough (see §5). We first need some preparations in §4.1 and §4.2.

4.1. **Reciprocal of polynomials.** We shall work with the base field $\mathbb{F}_q$, but we shall consider polynomials $f(\lambda)$ in $\mathbb{F}_q[\lambda]$ or $\mathbb{F}_{q^2}[\lambda]$. These will appear as characteristic polynomials of elements in orthogonal or symplectic groups over $\mathbb{F}_q$, or unitary groups of $\mathbb{F}_{q^2}/\mathbb{F}_q$-Hermitian spaces. Recall that σ is the Frobenius automorphism of $k = \overline{\mathbb{F}}_q$ over $\mathbb{F}_q$. For $x \in k$, we write x^σ for the image of x under σ, i.e., $x^\sigma := x^q$.

Definition 4.1.1. For a polynomial $f \in \mathbb{F}_{q^2}[\lambda]$ with $f(0) \neq 0$, we define its *reciprocal polynomial* as

$$f^*(\lambda) := (f(0)^\sigma)^{-1} \cdot \lambda^{\deg f} \cdot f(1/\lambda)^\sigma \in \mathbb{F}_{q^2}[\lambda].$$

We call $f \in \mathbb{F}_{q^2}[\lambda]$ *self-reciprocal*, if $f(0) \neq 0$ and $f = f^*$. (In particular, self-reciprocal polynomials are monic.) These definitions restrict to polynomials in $\mathbb{F}_q[\lambda]$.

Remark 4.1.2. If $f(\lambda) \in \mathbb{F}_{q^2}[\lambda]$ is monic and has factorization $f(\lambda) = \prod_j (\lambda - \lambda_j)$ with each $\lambda_j \in k^\times$, we have $f^*(\lambda) = \prod_j (\lambda - (\lambda_j^\sigma)^{-1})$. If in addition $f(\lambda) \in \mathbb{F}_q[\lambda]$, then we also have $f^*(\lambda) = \prod_j (\lambda - \lambda_j^{-1})$.

Definition 4.1.3. We denote by $\mathsf{Irr}^\times$ the set of monic irreducible polynomials in $\mathbb{F}_q[\lambda]$ with non-zero constant terms. We let $\mathsf{SR} \subset \mathsf{Irr}^\times$ be the subset of self-reciprocal irreducible polynomials, and let $\mathsf{NSR} := (\mathsf{Irr}^\times - \mathsf{SR})/*$ be the set of unordered pairs $\{Q, Q^*\}$ of non-self-reciprocal monic irreducible polynomials with non-zero constant terms. Similarly, we denote by $\mathsf{Irr}_2^\times$ the set of monic irreducible polynomials in $\mathbb{F}_{q^2}[\lambda]$ with non-zero constant terms. We let $\mathsf{SR}_2 \subset \mathsf{Irr}_2^\times$ be the subset of self-reciprocal irreducible polynomials, and let $\mathsf{NSR}_2 := (\mathsf{Irr}_2^\times - \mathsf{SR}_2)/*$.

Lemma 4.1.4. *If $f \in \mathbb{F}_q[\lambda]$ is self-reciprocal, then its irreducible factorization is of the form*

$$f = \prod_{Q \in \mathsf{SR}} Q^{m_Q(f)} \prod_{\{Q,Q^*\} \in \mathsf{NSR}} (QQ^*)^{m_{\{Q,Q^*\}}(f)}, \tag{4.1.1}$$

for unique non-negative integers $m_Q(f), m_{\{Q,Q^*\}}(f)$. Similarly, if $f \in \mathbb{F}_{q^2}[\lambda]$ is self-reciprocal, then we have

$$f = \prod_{Q \in \mathsf{SR}_2} Q^{m_Q(f)} \prod_{\{Q,Q^*\} \in \mathsf{NSR}_2} (QQ^*)^{m_{\{Q,Q^*\}}(f)}, \tag{4.1.2}$$

for unique non-negative integers $m_Q(f), m_{\{Q,Q^*\}}(f)$.

Proof. This easily follows from unique factorization in $\mathbb{F}_q[\lambda]$ and $\mathbb{F}_{q^2}[\lambda]$. □

Definition 4.1.5. Let $f \in \mathbb{F}_q[\lambda]$ be self-reciprocal. Define $m_Q(f), m_{\{Q,Q^*\}}(f)$ as in (4.1.1). Define

$$\mathscr{M}(f) := \prod_{\{Q,Q^*\} \in \mathsf{NSR}} (1 + m_{\{Q,Q^*\}}(f)).$$

Similarly, let $f \in \mathbb{F}_{q^2}[\lambda]$ be self-reciprocal. Define $m_Q(f), m_{\{Q,Q^*\}}(f)$ as in (4.1.2). Define

$$\mathscr{M}_2(f) := \prod_{\{Q,Q^*\} \in \mathsf{NSR}_2} (1 + m_{\{Q,Q^*\}}(f)).$$

Lemma 4.1.6. *Let $f \in \mathbb{F}_q[\lambda]$ be self-reciprocal. Assume there is a unique element $Q_0 \in \mathsf{SR}$ such that $m_{Q_0}(f)$ is odd. Let m be an odd integer such that $1 \le m \le m_{Q_0}(f)$. Then*

$$\#\left\{U \in \mathbb{F}_q[\lambda]^{\mathrm{monic}}; UU^* = f/Q_0^m\right\} = \mathscr{M}(f).$$

Similarly, let $f \in \mathbb{F}_{q^2}[\lambda]$ be self-reciprocal. Assume there is a unique element $Q_0 \in \mathsf{SR}_2$ such that $m_{Q_0}(f)$ is odd. Let m be an odd integer such that $1 \le m \le m_{Q_0}(f)$. Then

$$\#\left\{U \in \mathbb{F}_{q^2}[\lambda]^{\mathrm{monic}}; UU^* = f/Q_0^m\right\} = \mathscr{M}_2(f).$$

Proof. We only prove the statement about $\mathscr{M}(f)$, the other statement being similar. Write $h := f/Q_0^m$. For any $Q \in \mathsf{SR}$, $m_Q(h)$ is even. For any $\{Q, Q^*\} \in \mathsf{NSR}$, $m_{\{Q,Q^*\}}(h) = m_{\{Q,Q^*\}}(f)$. Now any $U \in \mathbb{F}_q[\lambda]^{\mathrm{monic}}$ with $UU^* = h$ is given by

$$U = \prod_{Q \in \mathsf{SR}} Q^{\frac{m_Q(h)}{2}} \prod_{\{Q,Q^*\} \in \mathsf{NSR}} U_{\{Q,Q^*\}},$$

where each $U_{\{Q,Q^*\}} = Q^i (Q^*)^j$, for any of the $1 + m_{\{Q,Q^*\}}(h)$ possible choices of pairs of non-negative integers (i, j) satisfying $i + j = m_{\{Q,Q^*\}}(h)$. □

Definition 4.1.7. Let $f \in \mathsf{SR}$ of even degree d. By an *admissible enumeration* of the roots of f, we mean an enumeration of the d distinct roots of f in $k^\times$ of the form $\lambda_1, \cdots, \lambda_{\frac{d}{2}}, \lambda_1^{-1}, \cdots, \lambda_{\frac{d}{2}}^{-1}$ such that

$$\lambda_1^\sigma = \lambda_2, \lambda_2^\sigma = \lambda_3, \cdots, \lambda_{\frac{d}{2}-1}^\sigma = \lambda_{\frac{d}{2}}, \lambda_{\frac{d}{2}}^\sigma = \lambda_1^{-1}.$$

Lemma 4.1.8. *Let $f \in \mathsf{SR}$ of degree d. Then either d is even or $f(\lambda) = \lambda \pm 1$. When d is even, there are precisely d distinct admissible enumerations of the roots of f, all obtained from a given one by powers of a cyclic permutation of order d.*

Proof. The map $x \mapsto x^{-1}$ induces an involution on the set of all d distinct roots of f. If d is odd, this involution has a fixed point, which means 1 or -1 is a root of f. Hence $f = \lambda \pm 1$.

We assume d is even. We first prove the existence of one admissible enumeration. The d distinct roots of f are of the form $\lambda_1, \cdots, \lambda_{d/2}, \lambda_1^{-1}, \cdots, \lambda_{d/2}^{-1}$. Since they form precisely one σ-orbit, we may reorder the λ_i's or switch the roles of λ_i and λ_i^{-1}, to arrange that $\lambda_2 = \lambda_1^\sigma, \cdots, \lambda_{d/2} = \lambda_{d/2-1}^\sigma$. We claim that we must then have $\lambda_{d/2}^\sigma = \lambda_1^{-1}$. In fact, since the d distinct roots form precisely one σ-orbit, we have $\lambda_{d/2}^\sigma = \lambda_j^{-1}$ for a unique $1 \le j \le d/2$. If $j \ge 2$, then

$$\lambda_{\frac{d}{2}}, \lambda_j^{-1}, \lambda_{j+1}^{-1}, \cdots, \lambda_{\frac{d}{2}}^{-1}, \lambda_j, \lambda_{j+1}, \cdots, \lambda_{\frac{d}{2}-1}$$

already form one σ-orbit, which does not contain λ_1, a contradiction. Thus we have shown the existence of an admissible enumeration. The rest of the lemma is clear. □

Definition 4.1.9. Let $d \ge 2$ be an even integer. Given a tuple $\Lambda = (\lambda_1, \cdots, \lambda_{\frac{d}{2}}) \in (k^\times)^{\oplus \frac{d}{2}}$, we define

$$\Lambda^{-1} := (\lambda_1^{-1}, \cdots, \lambda_{\frac{d}{2}}^{-1}), \quad \bar{\Lambda} := (\lambda_1, \cdots, \lambda_{\frac{d}{2}-1}, \lambda_{\frac{d}{2}}^{-1}), \quad \Lambda[1] := (\lambda_{\frac{d}{2}}, \lambda_1, \cdots, \lambda_{\frac{d}{2}-1}).$$

Let Λ be as above and let $f \in \mathsf{SR}$ have degree d. We say that Λ is *admissible with respect to* f, if (Λ, Λ^{-1}) is an admissible enumeration of the roots of f in the sense of Definition 4.1.7.

Definition 4.1.10. Let $f \in \mathsf{SR}_2$ of odd degree d. By an *admissible enumeration* of the roots of f, we mean an enumeration $\lambda_1, \cdots, \lambda_d$ of the d distinct roots of f such that

$$\lambda_1^{\sigma^2} = \lambda_2, \cdots, \lambda_{d-1}^{\sigma^2} = \lambda_d, \lambda_d^{\sigma^2} = \lambda_1.$$

Lemma 4.1.11. *Let $f \in \mathsf{SR}_2$ of odd degree d.*

(1) There are precisely d distinct admissible enumerations of the roots, all obtained from a given one by powers of a cyclic permutation of order d.

(2) Assume $d \geq 3$. Let $\lambda_1, \cdots, \lambda_d$ be an admissible enumeration of the roots of f. For any integer j we define λ_j to be $\lambda_{j'}$, for $1 \leq j' \leq d$ such that $j \equiv j' \mod d$. Then for all $j \in \mathbb{Z}$ we have

$$(\lambda_j^{-1})^\sigma = \lambda_{j+\frac{d+1}{2}}. \tag{4.1.3}$$

Proof. Part (1) follows immediately from the fact that the d distinct roots form precisely one σ^2-orbit. We prove part (2). Since for all j we have $\lambda_j = \sigma^{2(j-1)}(\lambda_1)$, it suffices to prove (4.1.3) for $j = 1$. Since the set of the roots is closed under the map $x \mapsto (x^{-1})^\sigma$, we have $(\lambda_1^{-1})^\sigma = \lambda_l$ for some $1 \leq l \leq d$. We get

$$\lambda_1^{\sigma^2} = (((\lambda_1^{-1})^\sigma)^{-1})^\sigma = (\lambda_l^{-1})^\sigma = \sigma^{2(l-1)}[(\lambda_1^{-1})^\sigma] = \sigma^{2(l-1)}(\lambda_l) = \lambda_{l+(l-1)}.$$

On the other hand $\lambda_1^{\sigma^2} = \lambda_2$, so $2l - 1 \equiv 2 \mod d$. Since $1 \leq l \leq d$ and $d \geq 3$ is odd, the only solution of this congruence is $l = (d + 3)/2$, as desired. □

4.2. **Eigenvalues ± 1.** Fix a non-degenerate quadratic space $(V, [\cdot, \cdot])$ over k. We would like to control the multiplicities of the eigenvalues ± 1, for elements $g \in \mathrm{O}(V) \cap \mathrm{GL}(V)^{\mathrm{reg}}$. For $g \in \mathrm{GL}(V)$ and $\lambda \in k$, we write $V(g, \lambda)$ for the generalized eigenspace of g belonging to λ, i.e., $V(g, \lambda) = \ker(g - \lambda)^{\dim V}$.

Proposition 4.2.1. *Let $g \in \mathrm{O}(V) \cap \mathrm{GL}(V)^{\mathrm{reg}}$. Let $j = 1$ or -1. Then $\dim V(g, j)$ is either zero or odd.*

Proof. Firstly, it is easy to see that $V(g, j)$ is orthogonal to $V(g, \lambda)$ for any $\lambda \in k - \{j\}$. In particular, the quadratic form restricted to $V(g, j)$ is non-degenerate, and we obtain a quadratic space $(V(g, j), [\cdot, \cdot])$. By Proposition 2.6.5, $g|_{V(g,j)}$ is in $\mathrm{GL}(V(g, j))^{\mathrm{reg}}$. Thus we may and shall assume that $V = V(g, j)$.

Assume that $\dim V = \dim V(g, j) = 2n$, with $n \geq 1$, and we are to deduce a contradiction. Under this assumption we have $g \in \mathrm{SO}(V)$ (since $\det g = j^{2n} = 1$). In particular g lies in a Borel subgroup of $\mathrm{SO}(V)$, and so g stabilizes a maximal totally isotropic subspace $M \subset V$. Let N be a maximal totally isotropic subspace of V such that $V = M \oplus N$. Since $g \in \mathrm{GL}(V)^{\mathrm{reg}}$, the Jordan canonical form of $g|_M \in \mathrm{GL}(M)$ must be one Jordan block of eigenvalue j (see Proposition 2.6.5). We thus find a k-basis $e_1, \cdots, e_n$ of M, such that $(g - j)$ sends each e_α to $e_{\alpha-1}$ (with $e_0 := 0$). Let $f_1, \cdots, f_n$ be the basis of N satisfying $[e_\alpha, f_\beta] = \delta_{\alpha,\beta}$. Using $g \in \mathrm{SO}(V)$ it is easy to see that

$$gf_n = jf_n + \sum_{\alpha=1}^{n} \eta_\alpha e_\alpha$$

for some $\eta_\alpha \in k$. Then we have

$$0 = [f_n, f_n] = [gf_n, gf_n] = 2j\eta_n.$$

Hence $\eta_n = 0$. It follows that $(g - j)$ maps the k-span of $e_1, \cdots, e_n, f_n$ into the k-span of $e_1, \cdots, e_{n-1}$. Hence the nullity of $(g - j)$ is at least 2, a contradiction (see Proposition 2.6.5). □

4.3. **The non-split even special orthogonal group.** In this subsection we consider case (1) in §3.

We fix a non-degenerate non-split $2n$-dimensional quadratic space $(\mathbb{V}, [\cdot, \cdot])$ over $\mathbb{F}_q$, with $n \geq 1$ (the case $n = 0$ being trivial). Let $\mathbb{G} = \mathrm{SO}(\mathbb{V}, [\cdot, \cdot])$. Let $V := \mathbb{V} \otimes_{\mathbb{F}_q} k$. By the classification of quadratic forms over $\mathbb{F}_q$ (see see [Kit93, §1.3], also cf. [DM91, §15.3]) there exists a k-basis $e_1, \cdots, e_n, f_1, \cdots, f_n$ of V, satisfying

$$[e_\alpha, e_\beta] = [f_\alpha, f_\beta] = 0, \quad [e_\alpha, f_\beta] = \delta_{\alpha,\beta}, \quad \forall\ 1 \leq \alpha, \beta \leq n;$$

$$e_\alpha^\sigma = e_\alpha, \quad f_\alpha^\sigma = f_\alpha, \quad \forall\ 1 \leq \alpha \leq n - 1;$$

$$e_n^\sigma = f_n, \quad f_n^\sigma = e_n.$$

For each $1 \leq i \leq n$, we define

$$V_i := \mathrm{span}_k(e_i, e_{i+1}, \cdots, e_n, f_i, f_{i+1}, \cdots, f_n) \subset V, \qquad W_i := \mathrm{span}_k(e_1, \cdots, e_i) \subset V.$$

For each $1 \leq i \leq n-1$, we have $W_i = W_i^\sigma$, and we write $\mathbb{W}_i$ for the $\mathbb{F}_q$-form of W_i. For $1 \leq i \leq n$, we have $V_i = V_i^\sigma$, and we write $\mathbb{V}_i$ for the $\mathbb{F}_q$-form of V_i.

Let $G = \mathbb{G}_k$. Let $B \subset G$ be the common stabilizer of either of the following two flags in V:

$$W_1 \subset W_2 \subset \cdots \subset W_{n-1} \subset W_n,$$

$$W_1 \subset W_2 \subset \cdots \subset W_{n-1} \subset W_n^\sigma.$$

Then B is a σ-stable Borel subgroup of G. Let T be the intersection of G with the diagonal torus in $\mathrm{GL}(V)$ under the basis $e_1, \cdots, e_n, f_1, \cdots, f_n$. Then T is the maximal torus of G contained in B.

We number the simple roots of (G, B, T) according to Bourbaki [Bou68]. We consider the σ-unbranched datum $(J = \mathbb{S} - \{s_{n-1}\}, \mathscr{L} = (s_1, \cdots, s_{n-1}))$. Following the notation of §2.4 and §2.5, we have $i_{\max} = n$, and for $1 \leq i \leq n$ we have

$$\mathbb{P}_i = \mathrm{Stab}_{\mathbb{G}}(\mathbb{W}_{i-1}), \qquad \mathbb{L}_i = \mathbb{L}_i^\natural = \mathrm{GL}(\mathbb{W}_{i-1}) \times \mathrm{SO}(\mathbb{V}_i)$$

$$\mathbb{G}_i = \mathrm{SO}(\mathbb{V}_i) = \mathrm{SO}_{2(n+1-i)} \text{ (non-split)}, \qquad \mathbb{H}_i = \mathrm{GL}(\mathbb{W}_{i-1}) = \mathrm{GL}_{i-1}.$$

Here by convention $\mathbb{W}_0 = 0$ and $\mathrm{GL}_0 = \{1\}$. As in §2.5, we have natural projections $\pi_i : \mathbb{P}_i \to \mathbb{G}_i$ and $\pi_i' : \mathbb{P}_i \to \mathbb{H}_i$.

For any $h \in G_i(k)$, we denote by $f_h \in k[\lambda]$ the characteristic polynomial of h acting on V_i, which has degree $2(n+1-i)$. Thus if $h \in G_i(\mathbb{F}_q)$, then f_h is self-reciprocal in $\mathbb{F}_q[\lambda]$. Similarly, for any $h \in H_i(k)$, we denote by $f_h(\lambda) \in k[\lambda]$ the characteristic polynomial of h acting on W_i, which has degree $i-1$.

Theorem 4.3.1. *We fix $1 \leq i \leq n$. Write n' for $n+1-i$. Thus $G_i = \mathrm{SO}_{2n'}$, with $n' \geq 1$. We have the following statements about $T_i(\mathbb{F}_q)$.*

(1) If $\gamma \in T_i(\mathbb{F}_q)$, then $f_\gamma = Q^m$ for some $Q \in \mathsf{SR}$, and some positive integer m. Moreover, either $Q(\lambda) = \lambda \pm 1$, or m is odd.

(2) Let $Q \in \mathsf{SR}$. Assume m is an odd integer such that $m \deg Q = 2n'$. (In particular $Q(\lambda) \neq \lambda \pm 1$). Then there exists $\gamma \in T_i(\mathbb{F}_q)$ with $f_\gamma = Q^m$.

(3) Let Q and m be as in part (2). Let $\gamma \in G_i(k)$ be a semi-simple element such that $f_\gamma = Q^m$. Then γ is $G_i(k)$-conjugate to an element of $T_i(\mathbb{F}_q)$.

(4) For any $\gamma \in T_i(\mathbb{F}_q)$, the centralizer $G_{i,\gamma}$ is connected.

(5) Let $\gamma \in T_i(\mathbb{F}_q)$. Write $f_\gamma = Q^m$ as in part (1). Assume $Q(\lambda) \neq \lambda \pm 1$. Then $\mathcal{T}(w_i, \gamma) = (\deg Q)/2$. Here $\mathcal{T}(w_i, \gamma)$ is defined in Definition 2.7.2.

Proof. On T_i we have coordinates

$$(k^\times)^{\oplus n'} \xrightarrow{\sim} T_i, \qquad (\lambda_1, \cdots, \lambda_{n'}) \longmapsto \gamma(\lambda_1, \cdots, \lambda_{n'}),$$

such that the eigenvalues (with multiplicities) of $\gamma(\lambda_1, \cdots, \lambda_{n'})$ acting on $V_i \cong k^{2n'}$ are

$$\lambda_1, \cdots, \lambda_{n'}, \lambda_1^{-1}, \cdots, \lambda_{n'}^{-1},$$

and such that

$$\gamma(\lambda_1, \cdots, \lambda_{n'})^\sigma = \gamma((\lambda_{n'}^{-1})^\sigma, \lambda_1^\sigma, \lambda_2^\sigma, \cdots, \lambda_{n'-1}^\sigma). \tag{4.3.1}$$

(1) Let $(\lambda_1, \cdots, \lambda_{n'})$ be the coordinates of γ. Since $\gamma^\sigma = \gamma$, it follows from (4.3.1) that we have the following equality between two $2n'$-tuples in $k^\times$:

$$(\lambda_1, \lambda_1^\sigma, \cdots, \lambda_1^{\sigma^{2n'-1}}) = (\lambda_1, \cdots, \lambda_{n'}, \lambda_1^{-1}, \cdots, \lambda_{n'}^{-1}). \tag{4.3.2}$$

We remark that (4.3.2) is valid even for $i = i_{\max} = n$. In fact, in that case $T_i = G_i$ is the kernel of the norm map $\mathrm{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathbb{G}_m \to \mathbb{G}_m$, and (4.3.2) reads $\lambda_1^\sigma = \lambda_1^{-1}$.

Therefore all eigenvalues of γ are in one σ-orbit. It follows that f_γ has a unique monic irreducible factor Q. Since f_γ is self-reciprocal, so is Q.

Now assume m is even. Then $d := \deg Q$ divides n'. Since (4.3.2) holds and since there are precisely d distinct eigenvalues of γ, we know that $\lambda_1^{\sigma^d} = \lambda_1$. Since d divides n', it follows that $\lambda_1^{\sigma^{n'}} = \lambda_1$. By (4.3.2) $\lambda_1^{\sigma^{n'}} = \lambda_1^{-1}$. Hence $\lambda_1 = \lambda_1^{-1}$, and so $\lambda_1 = \pm 1$. It follows that $Q(\lambda) = \lambda \pm 1$.

(2) Let $d = \deg Q$. Then d is even since dm is even. We fix a tuple $\Lambda \in (k^\times)^{\oplus \frac{d}{2}}$ admissible with respect to Q, see Definition 4.1.9. Then $\gamma := \gamma(\Lambda, \Lambda^{-1}, \cdots, \Lambda, \Lambda^{-1}, \Lambda)$ (with m total appearances of Λ and Λ^{-1}) is an element of $T_i(\mathbb{F}_q)$ satisfying $f_\gamma = Q^m$.

(3) Let $d = \deg Q$. We know d is even. We assume without loss of generality that $\gamma \in T_i(k)$. Since $f_\gamma = Q^m$, the n' coordinates of γ must contain elements $\lambda_1, \cdots, \lambda_{\frac{d}{2}}$ such that all roots of Q are given by $\lambda_1, \cdots, \lambda_{\frac{d}{2}}, \lambda_1^{-1}, \cdots, \lambda_{\frac{d}{2}}^{-1}$. We temporarily assume $m > 1$. By Lemma 4.1.8, there exists an admissible tuple Λ with respect to $Q(\lambda)$, obtained by permuting $\lambda_1, \cdots, \lambda_{d/2}$ and replacing some of them with their inverses. Up to the action of W_i, we may arbitrarily permute the coordinates of γ, and we may replace an arbitrary even number of coordinates of γ by their inverses. As $m > 1$, we may therefore conjugate γ by W_i to arrange that either

$$\gamma = \gamma(\Lambda, \Lambda^{-1}, \cdots, \Lambda, \Lambda^{-1}, \Lambda) \quad \text{(with } m \text{ total appearances of } \Lambda \text{ and } \Lambda^{-1})$$

or

$$\gamma = \gamma(\Lambda, \Lambda^{-1}, \cdots, \Lambda, \Lambda^{-1}, \bar{\Lambda}) \quad \text{(with } m-1 \text{ total appearances of } \Lambda \text{ and } \Lambda^{-1}).$$

In the first case we already have $\gamma \in T_i(\mathbb{F}_q)$. Assume we are in the second case. Since m is odd, we may simultaneously replace each of the first $m-1$ appearances of Λ or Λ^{-1} by its bar, i.e., γ is W_i-conjugate to

$$\gamma(\bar{\Lambda}, \overline{\Lambda^{-1}}, \cdots, \bar{\Lambda}, \overline{\Lambda^{-1}}, \bar{\Lambda}) = \gamma(\bar{\Lambda}, \bar{\Lambda}^{-1}, \cdots, \bar{\Lambda}, \bar{\Lambda}^{-1}, \bar{\Lambda}).$$

But the above element is W_i-conjugate to

$$\gamma(\bar{\Lambda}[1], \bar{\Lambda}^{-1}[1], \cdots, \bar{\Lambda}[1], \bar{\Lambda}^{-1}[1], \bar{\Lambda}[1]) = \gamma(\Omega, \Omega^{-1}, \cdots, \Omega, \Omega^{-1}, \Omega),$$

where $\Omega := \bar{\Lambda}[1]$. Note that Ω is admissible with respect to Q, and using this fact it is easy to check that the above element is in $T_i(\mathbb{F}_q)$.

Now we treat the case $m = 1$. In this case γ is W_i-conjugate to either $\gamma(\Lambda)$ or $\gamma(\bar{\Lambda})$, for a tuple Λ admissible with respect to Q. The element $\gamma(\Lambda)$ is already in $T_i(\mathbb{F}_q)$. The element $\gamma(\bar{\Lambda})$ is W_i-conjugate to $\gamma(\bar{\Lambda}[1])$, which is in $T_i(\mathbb{F}_q)$ since $\bar{\Lambda}[1]$ is admissible with respect to Q.

(4) We claim that any element $x \in W_i$ fixing γ is a certain product of reflections associated to roots that send γ to 1. Once the claim is proved, it will follows that $G_{i,\gamma}$ is connected, see [Car93, Theorem 3.5.3]. We now prove the claim.

Fix a $\mathbb{Z}$-basis $\epsilon_1, \cdots, \epsilon_{n'}$ of $X^*(T_i)$, such that the roots are $\pm\epsilon_\alpha \pm \epsilon_\beta$, $\alpha \neq \beta$. Then W_i can be identified with $(\{\pm 1\}^{\times n'})' \rtimes S_n$ acting on the set $\{\pm\epsilon_1, \cdots, \pm\epsilon_{n'}\}$. Here $(\{\pm 1\}^{\times n'})'$ denotes the subgroup of $\{\pm 1\}^{\times n'}$ consisting of elements with an even number of -1's. For any $x \in W_i$, define $A(x) := \{\alpha; 1 \leq \alpha \leq n', x(\epsilon_\alpha) \notin \{\pm\epsilon_\alpha\}\}$. Assume x fixes γ, and assume $A(x) \neq \emptyset$. Take $\alpha \in A(x)$. Then $x(\epsilon_\alpha) = \pm\epsilon_\beta$ for some $\beta \neq \alpha$. If $x(\epsilon_\alpha) = \epsilon_\beta$, then we left multiply x by the reflection $\epsilon_\alpha \mapsto \epsilon_\beta, \epsilon_\beta \mapsto \epsilon_\alpha$. If $x(\epsilon_\alpha) = -\epsilon_\beta$, then we left multiply x by the reflection $\epsilon_\alpha \mapsto -\epsilon_\beta, \epsilon_\beta \mapsto -\epsilon_\alpha$. In either case, we have left multiplied x by a reflection associated to a root (i.e. $\epsilon_\alpha - \epsilon_\beta$ in the first case and $\epsilon_\alpha + \epsilon_\beta$ in the second case) which sends γ to 1, and the product is an element $y \in W_i$ which also fixes γ and which satisfies $\#A(y) < \#A(x)$. In this way, we reduce to the case where $A(x) = \emptyset$. Now assume $A(x) = \emptyset$, and let $B(x) = \{\alpha; 1 \leq \alpha \leq n', x(\epsilon_\alpha) \neq \epsilon_\alpha\}$. Then $x \in (\{\pm 1\}^{\times n'})' \subset W_i$, with -1's appearing at the places indexed by $B(x)$. In particular, $\#B(x)$ is even. Since x fixes γ, we know $\epsilon_\alpha(\gamma) = \pm 1$ for each $\alpha \in B(x)$. By part (1) we know that ± 1 cannot simultaneously be eigenvalues of γ, so these $\epsilon_\alpha(\gamma)$ must all be 1 or all be -1. Write $B(x) = \{\alpha_1, \cdots, \alpha_l, \beta_1, \cdots, \beta_l\}$ arbitrarily. Then for each $1 \leq j \leq l$, the root $\epsilon_{\alpha_j} + \epsilon_{\beta_j}$ sends γ to 1. We easily see that x is the product of the reflections associated to the roots $\epsilon_{\alpha_j} + \epsilon_{\beta_j}$, for $1 \leq j \leq l$. The claim is proved.

(5) Let $d = \deg Q$. By part (1) we know that m is odd and d is even. Write $\gamma = \gamma(\lambda_1, \cdots, \lambda_{n'})$. Since (4.3.2) holds, we know that $\lambda_1, \cdots, \lambda_d$ are the d distinct roots of $Q(\lambda)$. Since $m(d/2) = n'$ and m is odd, we have $\lambda_{n'} = \lambda_{d/2}$, and in particular $\lambda_{d/2}^\sigma = \lambda_1^{-1}$. It then follows from (4.3.2) that $\Lambda := (\lambda_1, \cdots, \lambda_{d/2})$ is an admissible tuple with respect to Q, and that $\gamma = \gamma(\Lambda, \Lambda^{-1}, \cdots, \Lambda)$.

By part (4) and Lemma 2.7.5, we have

$$\mathcal{T}(w_i, \gamma) = \#\{\gamma' \in T_i(\mathbb{F}_q); \gamma' = {}^x\gamma \text{ for some } x \in W_i\}.$$

By the above argument, any such γ' must be given by $\gamma' = \gamma(\Lambda', (\Lambda')^{-1}, \cdots, \Lambda')$, for a tuple Λ' admissible with respect to Q. By Lemma 4.1.8 there are precisely d distinct admissible tuples Λ'. On the other hand it is clear that precisely $d/2$ such Λ' are such that $\gamma(\Lambda', (\Lambda')^{-1}, \cdots, \Lambda')$ equals ${}^x\gamma$ for some $x \in W_i$. It follows that $\mathcal{T}(w_i, \gamma) = d/2$ as desired. □

Lemma 4.3.2. *Let $g \in G(\mathbb{F}_q) \cap \mathrm{GL}(V)^{\mathrm{reg}}$. For each $1 \leq i \leq n$, let $\mathcal{M}_i^g$ be as in §2.8. We have a bijection*

$$\mathcal{M}_i^g \longrightarrow \left\{U \in \mathbb{F}_q[\lambda]^{\mathrm{monic}}; \deg U = i-1, UU^* \text{ divides } f_g \text{ in } \mathbb{F}_q[\lambda]\right\}, \quad rP_i(\mathbb{F}_q) \longmapsto f_{\pi_i'(r^{-1}gr)}.$$

Proof. Let $(\mathcal{M}_i^g)'$ be the set of g-stable $(i-1)$-dimensional totally isotropic $\mathbb{F}_q$-subspaces of $\mathbb{V}$. We know that all $(i-1)$-dimensional totally isotropic $\mathbb{F}_q$-subspaces of $\mathbb{V}$ are in the same $G(\mathbb{F}_q)$-orbit, because $i - 1 < n$. [1] Thus we have a bijection

$$\mathcal{M}_i^g \xrightarrow{\sim} (\mathcal{M}_i^g)', \quad rP_i(\mathbb{F}_q) \longmapsto rW_i.$$

Now given $W \in (\mathcal{M}_i^g)'$ corresponding to $rP_i(\mathbb{F}_q) \in \mathcal{M}_i^g$, the characteristic polynomial $f_{g|_W}$ of $g|_W$ is equal to $f_{\pi_i'(r^{-1}gr)}$. Hence it suffices to show that the map

$$(\mathcal{M}_i^g)' \longrightarrow \left\{U \in \mathbb{F}_q[\lambda]^{\mathrm{monic}}; \deg U = i-1, UU^* \text{ divides } f_g \text{ in } \mathbb{F}_q[\lambda]\right\}, \quad W \longmapsto f_{g|_W} \tag{4.3.3}$$

(which is obviously well-defined) is a bijection.

Given any element $U(\lambda)$ of the right hand side of (4.3.3), we obtain the $\mathbb{F}_q$-subspace $\ker U(g) \subset \mathbb{V}$, which is g-stable. Let $S := f_g/(UU^*) \in \mathbb{F}_q[\lambda]$. We now claim that $\ker U(g)$ has dimension $i-1$ and is totally isotropic. To check this it suffices to replace $\ker U(g)$ by its base change to k. Since $g \in \mathrm{GL}(V)^{\mathrm{reg}}$, we know that the Jordan canonical form of g over k has only one Jordan block for each eigenvalue, by Proposition 2.6.5. Analyzing each Jordan block one by one, we see that $(\ker U(g))_k$ is equal to $(SU^*)(g)(V)$, and has dimension $i-1$. To check that $(\ker U(g))_k$ is totally isotropic, let $v \in (\ker U(g))_k$. Let $w \in V$ such that $v = (SU^*)(g)w$. Then

$$[v, v] = [v, (SU^*)(g)e] = [v, U^*(g)S(g)w] = [U^*(g^{-1})v, S(g)w] = [U(0)^{-1}g^{1-i}U(g)v, S(g)w] = 0,$$

where the last equality holds because $U(g)v = 0$. The claim is proved.

[1] In contrast, even over the algebraically closed field k, there are two $G(k)$-orbits of n-dimensional totally isotropic k-subspaces of V.

By the claim, $\ker U(g)$ is an element of $(\mathcal{M}_i^g)'$. It then follows from the Cayley–Hamilton theorem that $U \mapsto \ker U(g)$ is the inverse map of (4.3.3). Hence (4.3.3) is a bijection as desired. □

Theorem 4.3.3. *Let $g \in G(\mathbb{F}_q) \cap \mathrm{GL}(V)^{\mathrm{reg}}$. We use the notations in Definition 4.1.5. For each $Q \in \mathsf{SR}$, we simply write m_Q for $m_Q(f_g)$. The following statements hold.*

(1) We have $m_{(\lambda+1)} = 0$, and $m_{(\lambda-1)}$ is zero or odd.

(2) If $\mathrm{tr}(g, J, \mathscr{L}) \neq 0$, then there is a unique element $Q_0 \in \mathsf{SR}$ such that m_{Q_0} is odd. In this case we also know that $Q_0 \neq \lambda \pm 1$. (In particular, by part (2) we have $m_{(\lambda+1)} = m_{(\lambda-1)} = 0$ in this case.)

(3) Assume there is a unique element $Q_0 \in \mathsf{SR}$ such that m_{Q_0} is odd. Assume $Q_0 \neq \lambda \pm 1$. Then

$$\mathrm{tr}(g, J, \mathscr{L}) = \frac{\deg Q_0}{2} \frac{m_{Q_0} + 1}{2} \mathscr{M}(f_g).$$

Proof. Part (1) follows from Proposition 4.2.1 and the fact that $m_{(\lambda+1)}$ must be even in order for $\det g = 1$.

By Proposition 2.6.3, we have $g \in G(\mathbb{F}_q) \cap G^{\mathrm{reg}}$, and so we may apply Theorem 2.8.1 to compute $\mathrm{tr}(g, J, \mathscr{L})$ in the following.

Firstly, assume $1 \leq i \leq n$ and $\mathcal{M}_i^{g,\gamma} \neq \emptyset$ for some $\gamma \in Z_{G_i}(\mathbb{F}_q)$. Here Z_{G_i} denotes the center of G_i. Take $rP_i(\mathbb{F}_q) \in \mathcal{M}_i^{g,\gamma}$. Then $f_{\pi_i(r^{-1}gr)} = (\lambda - j)^{2(n+1-i)}$ for $j = 1$ or -1, and it follows from Lemma 4.3.2 that

$$f_g(\lambda) = (\lambda - j)^{2(n+1-i)} U(\lambda) U^*(\lambda)$$

for some $U(\lambda) \in \mathbb{F}_q[\lambda]$. Then $m_{(\lambda-j)}$ must be positive even, a contradiction with part (1). Hence $\mathcal{M}_i^{g,\gamma} = \emptyset$ for all $1 \leq i \leq n$ and all $\gamma \in Z_{G_i}(\mathbb{F}_q)$.

We now prove part (2) of the theorem. Assume $\mathrm{tr}(g, J, \mathscr{L}) \neq 0$. Then there exist $1 \leq i \leq n$ and $\gamma \in \Gamma_i$ such that $\mathcal{M}_i^{g,\gamma} \neq \emptyset$. By the previous paragraph, we know that $\gamma \notin Z_{G_i}(\mathbb{F}_q)$. Take $rP_i(\mathbb{F}_q) \in \mathcal{M}_i^{g,\gamma}$. Then by Theorem 4.3.1 (1), we have $f_{\pi_i(r^{-1}gr)} = Q^m$, for some $Q \in \mathsf{SR} - \{\lambda \pm 1\}$ and some odd m. Here $Q \neq \lambda \pm 1$ because $\gamma \notin Z_{G_i}$. By Lemma 4.3.2 we have $f_g = Q^m U U^*$ for some $U \in \mathbb{F}_q[\lambda]^{\mathrm{monic}}$. It then follows that Q, which is not $\lambda \pm 1$, is the unique element of SR with m_Q odd. Part (2) is proved.

We now prove part (3). By Lemma 4.1.8 we know $\deg Q_0$ is even. Define

$$\mathscr{I} := \{i; 1 \leq i \leq n, 2(n+1-i)/\deg Q_0 \text{ is an odd integer } \leq m_{Q_0}\}.$$

For $i \in \mathscr{I}$, define $m_i := 2(n+1-i)/\deg Q_0$. Note that $i \mapsto m_i$ is a bijection $\mathscr{I} \to \{1, 3, 5, \cdots, m_{Q_0}\}$. In particular $|\mathscr{I}| = (m_{Q_0}+1)/2$. In the proof of part (2), we saw that if $rP_i(\mathbb{F}_q) \in \mathcal{M}_i^{g,\gamma}$ for some $1 \leq i \leq n$ and some $\gamma \in \Gamma_i$, then

$$i \in \mathscr{I}, \text{ and } f_{\pi_i(r^{-1}gr)} = Q_0^{m_i}. \tag{4.3.4}$$

Conversely, assume $i \in \mathscr{I}$ and assume $rP_i(\mathbb{F}_q) \in \mathcal{M}_i^g$ is such that (4.3.4) holds. Then $\pi_i(r^{-1}gr)_s$ is $G_i(k)$-conjugate to an element of $T_i(\mathbb{F}_q)$, by Theorem 4.3.1 (3). By Theorem 4.3.1 (4) and the Lang–Steinberg theorem, $\pi_i(r^{-1}gr)_s$ is in fact $G_i(\mathbb{F}_q)$-conjugate to an element of $T_i(\mathbb{F}_q)$. Thus $rP_i(\mathbb{F}_q) \in \mathcal{M}_i^{g,\gamma}$ for a unique $\gamma \in \Gamma_i$. In conclusion, we have a bijection

$$\begin{aligned} &\{(i, \gamma, rP_i(\mathbb{F}_q)); 1 \leq i \leq n, \gamma \in \Gamma_i, rP_i(\mathbb{F}_q) \in \mathcal{M}_i^{g,\gamma}\} \xrightarrow{\sim} \\ &\left\{(i, rP_i(\mathbb{F}_q)); i \in \mathscr{I}, rP_i(\mathbb{F}_q) \in \mathcal{M}_i^g, f_{\pi_i(r^{-1}gr)} = Q_0^{m_i}\right\} \\ &(i, \gamma, rP_i(\mathbb{F}_q)) \longmapsto (i, rP_i(\mathbb{F}_q)). \end{aligned} \tag{4.3.5}$$

We also note that if $(i, \gamma, rP_i(\mathbb{F}_q))$ is in the left hand side of (4.3.5), then $f_\gamma = Q_0^{m_i}$, and so by Theorem 4.3.1 (5) we have

$$\mathcal{T}(w_i, \gamma) = \frac{\deg Q_0}{2}. \tag{4.3.6}$$

Now we compute

$$\begin{aligned}
\operatorname{tr}(g, J, \mathscr{L}) &= \sum_{i=1}^{n} \sum_{\gamma\in\Gamma_i} \#\mathcal{M}_i^{g,\gamma} \cdot \mathcal{T}(w_i, \gamma) && \text{(by Theorem 2.8.1)}\\
&= \sum_{i\in\mathscr{I}} \#\left\{ rP_i(\mathbb{F}_q) \in \mathcal{M}_i^g ; f_{\pi_i(r^{-1}gr)} = Q_0^{m_i} \right\} \cdot \frac{\deg Q_0}{2} && \text{(by (4.3.5), (4.3.6))}\\
&= \frac{\deg Q_0}{2} \sum_{i\in\mathscr{I}} \#\left\{ U \in \mathbb{F}_q[\lambda]^{\text{monic}} ; UU^* = f_g/Q_0^{m_i} \right\} && \text{(by Lemma 4.3.2)}\\
&= \frac{\deg Q_0}{2} \left|\mathscr{I}\right| \mathscr{M}(f_g) && \text{(by Lemma 4.1.6)}\\
&= \frac{\deg Q_0}{2} \frac{m_{Q_0}+1}{2} \mathscr{M}(f_g). && \square
\end{aligned}$$

4.4. **The odd special orthogonal group.** In this subsection we consider case (2) in §3.

We fix a non-degenerate $2n+1$-dimensional quadratic space $(\mathbb{V}, [\cdot,\cdot])$ over $\mathbb{F}_q$, with $n \geq 0$. Let $\mathbb{G} = \mathrm{SO}(\mathbb{V}, [\cdot,\cdot])$. Let $V := \mathbb{V} \otimes_{\mathbb{F}_q} k$. By the classification of quadratic forms over $\mathbb{F}_q$ (see [Kit93, §1.3]), there exists an $\mathbb{F}_q$-basis $e_1, \cdots, e_{2n+1}$ of $\mathbb{V}$, satisfying

$$[e_\alpha, e_\beta] = \delta_{2n+2,\alpha+\beta}, \quad \forall \alpha, \beta \neq n+1,$$

$$[e_{n+1}, e_{n+1}] \in \mathbb{F}_q^\times.$$

For each $1 \leq i \leq n+1$, we define

$$\mathbb{V}_i := \operatorname{span}_{\mathbb{F}_q}(e_i, e_{i+1}, \cdots, e_{2n+2-i}) \subset \mathbb{V}, \qquad \mathbb{W}_i := \operatorname{span}_{\mathbb{F}_q}(e_1, \cdots, e_i) \subset \mathbb{V}.$$

We define $V := \mathbb{V} \otimes k$, $V_i := \mathbb{V}_i \otimes k$, $W_i := \mathbb{W}_i \otimes k$.

Let $G = \mathbb{G}_k$. Let $B \subset G$ be the stabilizer of the flag $W_1 \subset W_2 \subset \cdots \subset W_n$ inside V. Then B is a σ-stable Borel subgroup of G. Let T be the intersection of G with the diagonal torus in $\mathrm{GL}(V)$ under the basis $e_1, \cdots, e_{2n+1}$. Then T is the maximal torus of G contained in B.

We number the simple roots of (G, B, T) according to Bourbaki [Bou68]. We consider the σ-unbranched datum $(J = \mathbb{S} - \{s_n\}, \mathscr{L} = (s_1, \cdots, s_n))$. Following the notation of §2.4 and §2.5, we have $i_{\max} = n+1$, and for $1 \leq i \leq n+1$ we have

$$\mathbb{P}_i = \operatorname{Stab}_{\mathbb{G}}(\mathbb{W}_{i-1}), \qquad \mathbb{L}_i = \mathbb{L}_i^\natural = \mathrm{GL}(\mathbb{W}_{i-1}) \times \mathrm{SO}(\mathbb{V}_i)$$

$$\mathbb{G}_i = \mathrm{SO}(\mathbb{V}_i) = \mathrm{SO}_{2(n+1-i)+1}, \qquad \mathbb{H}_i = \mathrm{GL}(\mathbb{W}_{i-1}) = \mathrm{GL}_{i-1}\,.$$

Here by convention $\mathbb{W}_0 = 0$ and $\mathrm{GL}_0 = \{1\}$. As in §2.5, we have natural projections $\pi_i : \mathbb{P}_i \to \mathbb{G}_i$ and $\pi_i' : \mathbb{P}_i \to \mathbb{H}_i$.

For any $h \in G_i(k)$, we denote by $f_h \in k[\lambda]$ the characteristic polynomial of h acting on V_i, which has degree $2(n+1-i)+1$. Thus if $h \in G_i(\mathbb{F}_q)$, then f_h is self-reciprocal in $\mathbb{F}_q[\lambda]$. Similarly, for any $h \in H_i(k)$, we denote by $f_h(\lambda) \in k[\lambda]$ the characteristic polynomial of h acting on W_i, which has degree $i-1$.

Theorem 4.4.1. *We fix $1 \leq i \leq n$. Write n' for $n+1-i$. Thus $\mathbb{G}_i = \mathrm{SO}_{2n'+1}$, with $n' \geq 1$. We have the following statements about $T_i(\mathbb{F}_q)$.*

(1) If $\gamma \in T_i(\mathbb{F}_q)$, then $f_\gamma(\lambda) = Q(\lambda)^m(\lambda-1)$ for some $Q \in \mathsf{SR}$, and some positive integer m. Moreover, either $Q(\lambda) = \lambda \pm 1$, or m is odd.

(2) Let $Q \in \mathsf{SR}$. Assume m is an odd integer such that $m \deg Q = 2n'$. (In particular $Q(\lambda) \neq \lambda \pm 1$ for degree reasons). Then there exists $\gamma \in T_i(\mathbb{F}_q)$ with $f_\gamma(\lambda) = Q(\lambda)^m(\lambda-1)$.

(3) *Let Q and m be as in part (2). Let $\gamma \in G_i(k)$ be a semi-simple element such that $f_\gamma(\lambda) = Q(\lambda)^m(\lambda-1)$. Then γ is $G_i(k)$-conjugate to an element of $T_i(\mathbb{F}_q)$.*

(4) *For any $\gamma \in T_i(\mathbb{F}_q)$ such that $(\lambda+1)$ does not divide $f_\gamma(\lambda)$, the centralizer $G_{i,\gamma}$ is connected.*

(5) *Let $\gamma \in T_i(\mathbb{F}_q)$. Write $f_\gamma(\lambda) = Q(\lambda)^m(\lambda-1)$ as in part (1). Assume $Q(\lambda) \neq \lambda \pm 1$. Then $\mathcal{T}(w_i, \gamma) = \deg Q$.*

Proof. On T_i we have coordinates

$$(k^\times)^{n'} \xrightarrow{\sim} T_i, \qquad (\lambda_1, \cdots, \lambda_{n'}) \longmapsto \gamma(\lambda_1, \cdots, \lambda_{n'}),$$

such that the eigenvalues (with multiplicities) of $\gamma(\lambda_1, \cdots, \lambda_{n'})$ acting on V_i are $\lambda_1, \cdots, \lambda_{n'}, \lambda_1^{-1}, \cdots, \lambda_{n'}^{-1}, 1$, and such that

$$\gamma(\lambda_1, \cdots, \lambda_{n'})^\sigma = \gamma((\lambda_{n'}^{-1})^\sigma, \lambda_1^\sigma, \lambda_2^\sigma, \cdots, \lambda_{n'-1}^\sigma). \tag{4.4.1}$$

Observing that (4.4.1) has the same form as (4.3.1), parts (1) (2) (3) are proved in exactly the same way as parts (1) (2) (3) of Theorem 4.3.1. (In fact the proof of part (3) here is even easier, due to the fact that the Weyl group W_i in the current case is larger.)

The proof of part (4) is also similar to the proof of Theorem 4.3.1 (4). In fact, fix a $\mathbb{Z}$-basis $\epsilon_1, \cdots, \epsilon_{n'}$ of $X^*(T_i)$, such that the roots are $\pm\epsilon_\alpha, \pm\epsilon_\alpha \pm \epsilon_\beta,\ \alpha \neq \beta$. Using the same notation as the proof of Theorem 4.3.1 (4), we can again reduce to the case $A(x) = \emptyset$. Then the new feature is that $\#B(x)$ need not be even. However, since -1 is not an eigenvalue by assumption, we know that $\epsilon_\alpha(\gamma) = 1$ for all $\alpha \in B(x)$. Then x is the product of the reflections associated to the roots ϵ_α, for $\alpha \in B(x)$.

The proof of part (5) is again similar to the proof of Theorem 4.3.1 (5), the only difference being that here all $\deg Q$ admissible tuples Λ' show up in the counting, as opposed to only $(\deg Q)/2$ of them. This is due to the fact that the Weyl group W_i is larger in the current case. ☐

Lemma 4.4.2. *Let $g \in G(\mathbb{F}_q) \cap \mathrm{GL}(V)^{\mathrm{reg}}$. For each $1 \leq i \leq n+1$, let $\mathcal{M}_i^g$ be as in §2.8. We have a bijection*

$$\mathcal{M}_i^g \longrightarrow \left\{U \in \mathbb{F}_q[\lambda]^{\mathrm{monic}}; \deg U = i-1, UU^* \text{ divides } f_g \text{ in } \mathbb{F}_q[\lambda]\right\}, \quad rP_i(\mathbb{F}_q) \longmapsto f_{\pi_i'(r^{-1}gr)}.$$

Proof. The proof is identical to the proof of Lemma 4.3.2, based on the fact that all $(i-1)$-dimensional totally isotropic $\mathbb{F}_q$-subspaces of $\mathbb{V}$ are in the same $G(\mathbb{F}_q)$-orbit. ☐

Theorem 4.4.3. *Let $g \in G(\mathbb{F}_q) \cap \mathrm{GL}(V)^{\mathrm{reg}}$. We use the notations in Definition 4.1.5. For each $Q \in \mathsf{SR}$, we simply write m_Q for $m_Q(f_g)$. The following statements hold.*

(1) *We have $m_{(\lambda+1)} = 0$, and $m_{(\lambda-1)}$ is odd.*

(2) *If $\mathrm{tr}(g, J, \mathscr{L}) \neq 0$, then inside $\mathsf{SR} - \{\lambda - 1\}$ there is at most one element Q_0 with m_{Q_0} odd.*

(3) *Assume there exists a unique $Q_0 \in \mathsf{SR} - \{\lambda-1\}$ such that m_{Q_0} is odd. Then*

$$\mathrm{tr}(g, J, \mathscr{L}) = \deg Q_0 \frac{m_{Q_0}+1}{2} \mathscr{M}(f_g).$$

(4) *Assume there is no element $Q_0 \in \mathsf{SR} - \{\lambda-1\}$ such that m_{Q_0} is odd. Then*

$$\mathrm{tr}(g, J, \mathscr{L}) = \frac{m_{(\lambda-1)}+1}{2} \mathscr{M}(f_g).$$

Proof. Part (1) follows from Proposition 4.2.1, the fact that $\lambda - 1$ always divides f_g, and the fact that $m_{(\lambda+1)}$ must be even in order for $\det g = 1$.

By Proposition 2.6.3, we have $g \in G(\mathbb{F}_q) \cap G^{\mathrm{reg}}$, and so we may apply Theorem 2.8.1 to compute $\mathrm{tr}(g, J, \mathscr{L})$ in the following.

We prove part (2). Assume $\mathrm{tr}(g, J, \mathscr{L}) \neq 0$. Then there exist $1 \leq i \leq n+1$ and $\gamma \in \Gamma_i$ such that $\mathcal{M}_i^{g,\gamma} \neq \emptyset$. Take $rP_i(\mathbb{F}_q) \in \mathcal{M}_i^{g,\gamma}$. If $i = n+1$, then $f_{\pi_i(r^{-1}gr)} = \lambda - 1$. If $1 \leq i \leq n$, then by Theorem 4.4.1 (1), we have $f_{\pi_i(r^{-1}gr)} = Q(\lambda)^m(\lambda-1)$, for some $Q \in \mathsf{SR}$ and some integer $m > 0$. To simplify notation we set $Q := 1$ and $m := 0$ when $i = n+1$. Then in all cases $f_{\pi_i(r^{-1}gr)} = Q(\lambda)^m(\lambda-1)$. By Lemma 4.4.2 we have

$$f_g(\lambda) = Q(\lambda)^m(\lambda-1)U(\lambda)U^*(\lambda) \tag{4.4.2}$$

for some $U \in \mathbb{F}_q[\lambda]^{\mathrm{monic}}$. Now if $Q(\lambda) = \lambda - 1$ or $m = 0$, then it follows from (4.4.2) that $\lambda - 1$ is the only element of SR whose multiplicity in f is odd. On the other hand, if $Q(\lambda) \neq \lambda - 1$ and $m > 0$, then $Q(\lambda) \neq \lambda \pm 1$ by part (1), and we know that m is odd by Theorem 4.4.1 (1). In this case, we conclude from (4.4.2) that m_Q is odd, and that Q is the unique element of $\mathsf{SR} - \{\lambda - 1\}$ whose multiplicity in f is odd. Part (2) is proved.

We remark that the above analysis shows that under the sole assumption that $\mathsf{SR} - \{\lambda - 1\}$ has an element Q with m_Q odd, we have

$$\mathcal{M}_{n+1}^{g,\gamma} = \emptyset, \quad \forall \gamma \in \Gamma_{n+1} \tag{4.4.3}$$

(where Γ_{n+1} in fact has only one element, the identity).

We now prove part (3). Under the hypothesis of part (3), the assertion (4.4.3) holds. Since $Q_0 \neq \lambda \pm 1$, by Lemma 4.1.8 we know that $\deg Q_0$ is even. Define

$$\mathscr{I} := \{i; 1 \leq i \leq n, 2(n+1-i)/\deg Q_0 \text{ is an odd integer } \leq m_{Q_0}\}.$$

For $i \in \mathscr{I}$, define $m_i := 2(n+1-i)/\deg Q_0$. Note that $i \mapsto m_i$ is a bijection $\mathscr{I} \to \{1, 3, 5, \cdots, m_{Q_0}\}$. In particular $|\mathscr{I}| = (m_{Q_0}+1)/2$. Similar to the bijection (4.3.5), we obtain a bijection

$$\begin{aligned} &\{(i, \gamma, rP_i(\mathbb{F}_q)); 1 \leq i \leq n, \gamma \in \Gamma_i, rP_i(\mathbb{F}_q) \in \mathcal{M}_i^{g,\gamma}\} \xrightarrow{\sim} \\ &\left\{(i, rP_i(\mathbb{F}_q)); i \in \mathscr{I}, rP_i(\mathbb{F}_q) \in \mathcal{M}_i^g, f_{\pi_i(r^{-1}gr)} = Q_0^{m_i} \cdot (\lambda - 1)\right\} \\ &(i, \gamma, rP_i(\mathbb{F}_q)) \longmapsto (i, rP_i(\mathbb{F}_q)), \end{aligned} \tag{4.4.4}$$

based on parts (3) (4) of Theorem 4.4.1 (part (4) being applicable because $m_{(\lambda+1)} = 0$). We also note that if $(i, \gamma, rP_i(\mathbb{F}_q))$ is in the left hand side of (4.3.5), then $f_\gamma(\lambda) = Q_0(\lambda)^{m_i}(\lambda - 1)$, and so by Theorem 4.4.1 (5) we have

$$\mathcal{T}(w_i, \gamma) = \deg Q_0. \tag{4.4.5}$$

Now we compute

$$\begin{aligned} \mathrm{tr}(g, J, \mathscr{L}) &= \sum_{i=1}^{n} \sum_{\gamma \in \Gamma_i} \# \mathcal{M}_i^{g,\gamma} \cdot \mathcal{T}(w_i, \gamma) && \text{(by Theorem 2.8.1, and (4.4.3))} \\ &= \sum_{i \in \mathscr{I}} \# \left\{rP_i(\mathbb{F}_q) \in \mathcal{M}_i^g; f_{\pi_i(r^{-1}gr)} = Q_0^{m_i} \cdot (\lambda - 1)\right\} \cdot \deg Q_0 && \text{(by (4.4.4), (4.4.5))} \\ &= \deg Q_0 \sum_{i \in \mathscr{I}} \# \left\{U \in \mathbb{F}_q[\lambda]^{\mathrm{monic}}; UU^* = \frac{f_g}{Q_0^{m_i}(\lambda - 1)}\right\} && \text{(by Lemma 4.4.2)} \\ &= \deg Q_0 \, |\mathscr{I}| \, \mathscr{M}(\frac{f_g}{\lambda - 1}) && \text{(by Lemma 4.1.6 applied to } \frac{f_g}{\lambda-1} \text{ and } Q_0) \\ &= \deg Q_0 \frac{m_{Q_0}+1}{2} \mathscr{M}(f_g). \end{aligned}$$

In the second last step Lemma 4.1.6 is applicable because Q_0 is the unique element of SR such that $m_{Q_0}(f_g/(\lambda - 1))$ is odd, which follows from the definition of Q_0 and part (1). Part (3) is proved.

Finally we prove part (4). By the proof of part (2), we know that for any $1 \le i \le n+1$, we have $\mathcal{M}_i^{g,\gamma} \neq \emptyset$ only if $f_\gamma(\lambda) = (\lambda-1)^{2(n+1-i)+1}$. The last condition is equivalent to $\gamma = \mathrm{id} \in T_i$.

Define

$$\mathscr{I} = \left\{ i \in \mathbb{Z}; n+1 - \frac{m_{(\lambda-1)}-1}{2} \le i \le n+1 \right\}.$$

Now assume $rP_i(\mathbb{F}_q) \in \mathcal{M}_i^{g,\mathrm{id}}$ for some $1 \le i \le n+1$. Then we have

$$f_{\pi_i(r^{-1}gr)}(\lambda) = (\lambda-1)^{2(n+1-i)+1}. \tag{4.4.6}$$

In particular, $2(n+1-i)+1 \le m_{\lambda-1}$, and so $i \in \mathscr{I}$. Conversely, assume $i \in \mathscr{I}$, and $rP_i(\mathbb{F}_q) \in \mathcal{M}_i^g$ such that (4.4.6) holds. Then $rP_i(\mathbb{F}_q) \in \mathcal{M}_i^{g,\mathrm{id}}$ because the only semi-simple element of G_i whose characteristic polynomial equals $(\lambda-1)^{2(n+1-i)+1}$ is the identity. Therefore similar to the proof of part (3), we have

$$\mathrm{tr}(g, J, \mathscr{L}) = \sum_{i \in \mathscr{I}} \mathcal{T}(w_i, \mathrm{id}) \cdot \# \left\{ U \in \mathbb{F}_q[\lambda]^{\mathrm{monic}}; UU^* = f_g/(\lambda-1)^{2(n+1-i)+1} \right\} = \sum_{i \in \mathscr{I}} \mathcal{T}(w_i, \mathrm{id}) \mathscr{M}(f_g).$$

By Definition 2.7.2, we have $\mathcal{T}(w_i, \mathrm{id}) = 1$ for each $i \in \mathscr{I}$. Hence

$$\mathrm{tr}(g, J, \mathscr{L}) = |\mathscr{I}| \, \mathscr{M}(f_g) = \frac{m_{(\lambda-1)}+1}{2} \mathscr{M}(f_g). \qquad \square$$

4.5. **The symplectic group.** In this subsection we consider case (3) in §3.

We fix a $2n$-dimensional symplectic space $(\mathbb{V}, [\cdot,\cdot])$ over $\mathbb{F}_q$, with $n \ge 0$. Let $\mathbb{G} = \mathrm{Sp}(\mathbb{V}, [\cdot,\cdot])$. We fix an $\mathbb{F}_q$-basis $e_1, \cdots, e_{2n}$ of $\mathbb{V}$, satisfying

$$[e_\alpha, e_\beta] = \delta_{2n+1, \alpha+\beta}, \ \forall 1 \le \alpha \le \beta \le 2n.$$

For each $1 \le i \le n+1$, we define

$$\mathbb{V}_i := \mathrm{span}_{\mathbb{F}_q}(e_i, e_{i+1}, \cdots, e_{2n+1-i}) \subset \mathbb{V}, \qquad \mathbb{W}_i := \mathrm{span}_{\mathbb{F}_q}(e_1, \cdots, e_i) \subset \mathbb{V}.$$

We define $V := \mathbb{V} \otimes k$, $V_i := \mathbb{V}_i \otimes k$, $W_i := \mathbb{W}_i \otimes k$.

Let $G = \mathbb{G}_k$. Let $B \subset G$ be the stabilizer of the flag $W_1 \subset W_2 \subset \cdots \subset W_n$ inside V. Then B is a σ-stable Borel subgroup of G. Let T be the intersection of G with the diagonal torus in $\mathrm{GL}(V)$ under the basis $e_1, \cdots, e_{2n}$. Then T is the maximal torus of G contained in B.

We number the simple roots of (G, B, T) according to Bourbaki [Bou68]. We consider the σ-unbranched datum $(J = \mathbb{S} - \{s_n\}, \mathscr{L} = (s_1, \cdots, s_n))$. Following the notation of §2.4 and §2.5, we have $i_{\max} = n+1$, and for $1 \le i \le n+1$ we have

$$\mathbb{P}_i = \mathrm{Stab}_{\mathbb{G}}(\mathbb{W}_{i-1}), \qquad \mathbb{L}_i = \mathbb{L}_i^\natural = \mathrm{GL}(\mathbb{W}_{i-1}) \times \mathrm{Sp}(\mathbb{V}_i)$$

$$\mathbb{G}_i = \mathrm{Sp}(\mathbb{V}_i) = \mathrm{Sp}_{2(n+1-i)}, \qquad \mathbb{H}_i = \mathrm{GL}(\mathbb{W}_{i-1}) = \mathrm{GL}_{i-1}.$$

Here by convention $\mathbb{W}_0 = 0$ and $\mathrm{GL}_0 = \{1\}$. As in §2.5, we have natural projections $\pi_i : \mathbb{P}_i \to \mathbb{G}_i$ and $\pi_i' : \mathbb{P}_i \to \mathbb{H}_i$.

For any $h \in G_i(k)$, we denote by $f_h \in k[\lambda]$ the characteristic polynomial of h acting on V_i, which has degree $2(n+1-i)$. Thus if $h \in G_i(\mathbb{F}_q)$, then f_h is self-reciprocal in $\mathbb{F}_q[\lambda]$. Similarly, for any $h \in H_i(k)$, we denote by $f_h(\lambda) \in k[\lambda]$ the characteristic polynomial of h acting on W_i, which has degree $i-1$.

Theorem 4.5.1. *We fix $1 \le i \le n$. Write n' for $n+1-i$. Thus $\mathbb{G}_i = \mathrm{Sp}_{2n'}$, with $n' \ge 1$. We have the following statements about $T_i(\mathbb{F}_q)$.*

(1) If $\gamma \in T_i(\mathbb{F}_q)$, then $f_\gamma = Q^m$ for some irreducible, self-reciprocal $Q \in \mathbb{F}_q[\lambda]$, and some positive integer m. Moreover, either $Q(\lambda) = \lambda \pm 1$, or m is odd.

(2) Let $Q \in \mathbb{F}_q[\lambda]$ be an irreducible, self-reciprocal polynomial. Assume m is an odd integer such that $m \deg Q = 2n'$. (In particular $Q(\lambda) \neq \lambda \pm 1$). Then there exists $\gamma \in T_i(\mathbb{F}_q)$ with $f_\gamma = Q^m$.

(3) Let Q and m be as in part (2). Let $\gamma \in G_i(k)$ be a semi-simple element such that $f_\gamma = Q^m$. Then γ is $G_i(k)$-conjugate to an element of $T_i(\mathbb{F}_q)$.

(4) Let $\gamma \in T_i(\mathbb{F}_q)$. Write $f_\gamma = Q^m$ as in part (1). Assume $Q(\lambda) \neq \lambda \pm 1$. Then $\mathcal{T}(w_i, \gamma) = \deg Q$.

Proof. Since the root datum of G_i is dual to that of an odd special orthogonal group, the torus T_i has a similar description as the torus T_i in Theorem 4.4.1. Thus the proof of the theorem is identical to the proof of Theorem 4.4.1. □

Remark 4.5.2. In Theorem 4.5.1 we do not state the analogue of Theorem 4.3.1 (4) and Theorem 4.4.1 (4). This is because G being simply connected, the centralizer in G of any semi-simple element is automatically connected, see §2.7.4.

Lemma 4.5.3. *Let $g \in G(\mathbb{F}_q) \cap \mathrm{GL}(V)^{\mathrm{reg}}$. For each $1 \leq i \leq n+1$, let $\mathcal{M}_i^g$ be as in §2.8. We have a bijection*

$$\mathcal{M}_i^g \longrightarrow \left\{U \in \mathbb{F}_q[\lambda]^{\mathrm{monic}}; \deg U = i-1, UU^* \text{ divides } f_g \text{ in } \mathbb{F}_q[\lambda]\right\}, \quad rP_i(\mathbb{F}_q) \longmapsto f_{\pi_i'(r^{-1}gr)}.$$

Proof. The proof is identical to the proof of Lemma 4.3.2, based on the fact that all $(i-1)$-dimensional totally isotropic $\mathbb{F}_q$-subspaces of $\mathbb{V}$ are in the same $G(\mathbb{F}_q)$-orbit. □

Theorem 4.5.4. *Let $g \in G(\mathbb{F}_q) \cap \mathrm{GL}(V)^{\mathrm{reg}}$. We use the notations in Definition 4.1.5. For each $Q \in \mathsf{SR}$, we simply write m_Q for $m_Q(f_g)$. The following statements hold.*

(1) Assume $\mathrm{tr}(g, J, \mathscr{L}) \neq 0$. Then inside SR there is at most one element Q_0 with m_{Q_0} odd. Moreover, if such Q_0 exists, then $Q_0 \neq \lambda \pm 1$.

(2) Assume there exists a unique $Q_0 \in \mathsf{SR}$ such that m_{Q_0} is odd. Assume $Q_0 \neq \lambda \pm 1$. Then

$$\mathrm{tr}(g, J, \mathscr{L}) = \deg Q_0 \frac{m_{Q_0} + 1}{2} \mathscr{M}(f_g).$$

(3) Assume there is no element $Q_0 \in \mathsf{SR}$ such that m_{Q_0} is odd. Then

$$\mathrm{tr}(g, J, \mathscr{L}) = \Big(\frac{m_{(\lambda-1)}}{2} + 1 + \frac{m_{(\lambda+1)}}{2}\Big) \mathscr{M}(f_g).$$

Proof. By Proposition 2.6.3, we have $g \in G(\mathbb{F}_q) \cap G^{\mathrm{reg}}$, and so we may apply Theorem 2.8.1 to compute $\mathrm{tr}(g, J, \mathscr{L})$ in the following.

We prove part (1). Assume $\mathrm{tr}(g, J, \mathscr{L}) \neq 0$. Then there exist $1 \leq i \leq n+1$ and $\gamma \in \Gamma_i$ such that $\mathcal{M}_i^{g,\gamma} \neq \emptyset$. Take $rP_i(\mathbb{F}_q) \in \mathcal{M}_i^{g,\gamma}$. If $i = n+1$, then $f_{\pi_i(r^{-1}gr)} = 1$. If $1 \leq i \leq n$, then by Theorem 4.5.1 (1), we have $f_{\pi_i(r^{-1}gr)} = Q^m$, for some $Q \in \mathsf{SR}$ and some integer $m > 0$. To simplify notation we set $Q := 1$ and $m := 0$ when $i = n+1$. Then in all cases $f_{\pi_i(r^{-1}gr)} = Q^m$. By Lemma 4.5.3 we have $f_g = Q^m UU^*$ for some $U \in \mathbb{F}_q[\lambda]^{\mathrm{monic}}$. It immediately follows that inside SR there is at most one element whose multiplicity in f_g is odd. Moreover, if such an element exists, denoted by Q_0, then Q in the current discussion must equal to Q_0, and m must be odd. (In particular, $i \leq n$.) In this case, we show that $Q_0 \neq \lambda \pm 1$. In fact, if $Q_0 = \lambda \pm 1$, then m is even because $Q^m = Q_0^m$ has even degree. This contradicts with our previous assertion that m must be odd. Part (1) is proved.

We remark that the above analysis also shows that under the sole assumption that SR has an element Q with m_Q odd, we have

$$\mathcal{M}_{n+1}^{g,\gamma} = \emptyset, \quad \forall \gamma \in \Gamma_{n+1} \tag{4.5.1}$$

(where Γ_{n+1} in fact has only one element, the identity).

We now prove part (2). Under the hypothesis of part (2), the assertion (4.5.1) holds. Since $Q_0 \neq \lambda \pm 1$, by Lemma 4.1.8 we know that $\deg Q_0$ is even. Define

$$\mathscr{I} := \{i; 1 \leq i \leq n, 2(n+1-i)/\deg Q_0 \text{ is an odd integer } \leq m_{Q_0}\}.$$

For $i \in \mathscr{I}$, define $m_i := 2(n+1-i)/\deg Q_0$. Note that $i \mapsto m_i$ is a bijection $\mathscr{I} \to \{1, 3, 5, \cdots, m_{Q_0}\}$. In particular $|\mathscr{I}| = (m_{Q_0}+1)/2$. Similar to the bijection (4.3.5), we obtain a bijection

$$\begin{aligned}(4.5.2)\qquad &\{(i, \gamma, rP_i(\mathbb{F}_q)); 1 \leq i \leq n, \gamma \in \Gamma_i, rP_i(\mathbb{F}_q) \in \mathcal{M}_i^{g,\gamma}\} \xrightarrow{\sim} \\ &\left\{(i, rP_i(\mathbb{F}_q)); i \in \mathscr{I}, rP_i(\mathbb{F}_q) \in \mathcal{M}_i^g, f_{\pi_i(r^{-1}gr)} = Q_0^{m_i}\right\} \\ &(i, \gamma, rP_i(\mathbb{F}_q)) \longmapsto (i, rP_i(\mathbb{F}_q))\end{aligned}$$

based on Theorem 4.5.1 (3) and Remark 4.5.2. We also note that if $(i, \gamma, rP_i(\mathbb{F}_q))$ is in the left hand side of (4.3.5), then $f_\gamma = Q_0^{m_i}$, and so by Theorem 4.5.1 (4) we have

$$(4.5.3)\qquad \mathcal{T}(w_i, \gamma) = \deg Q_0.$$

Now we compute

$$\begin{aligned}\operatorname{tr}(g, J, \mathscr{L}) &= \sum_{i=1}^{n} \sum_{\gamma \in \Gamma_i} \#\mathcal{M}_i^{g,\gamma} \cdot \mathcal{T}(w_i, \gamma) && \text{(by Theorem 2.8.1, and (4.5.1))} \\ &= \sum_{i \in \mathscr{I}} \#\left\{rP_i(\mathbb{F}_q) \in \mathcal{M}_i^g; f_{\pi_i(r^{-1}gr)} = Q_0^{m_i}\right\} \cdot \deg Q_0 && \text{(by (4.5.2), (4.5.3))} \\ &= \deg Q_0 \sum_{i \in \mathscr{I}} \#\left\{U \in \mathbb{F}_q[\lambda]^{\text{monic}}; UU^* = f_g/Q_0^{m_i}\right\}. && \text{(by Lemma 4.5.3)} \\ &= \deg Q_0 \, |\mathscr{I}| \, \mathscr{M}(f_g) && \text{(by Lemma 4.1.6)} \\ &= \deg Q_0 \frac{m_{Q_0}+1}{2} \mathscr{M}(f_g).\end{aligned}$$

Part (2) is proved.

Finally we prove part (3). We claim that for each $1 \leq i \leq n+1$, we have $\mathcal{M}_i^{g,\gamma} \neq \emptyset$ for some $\gamma \in \Gamma_i$ only if $f_\gamma(\lambda) = (\lambda \pm 1)^{2(n+1-i)}$. In fact, assume this is not the case. Take $rP_i(\mathbb{F}_q) \in \mathcal{M}_i^{g,\gamma}$. Then by Theorem 4.5.1 (1), we have $f_{\pi_i(r^{-1}gr)} = Q^m$, for some $Q \in \mathsf{SR}$ and some odd integer m. By Lemma 4.5.3 we have $f_g = Q^m UU^*$ for some $U \in \mathbb{F}_q[\lambda]^{\text{monic}}$, contradicting with the assumption that there is no element in SR with odd multiplicity in f_g. The claim is proved.

Define

$$\mathscr{I} = \left\{i \in \mathbb{Z}; n+1-\frac{m_{(\lambda-1)}}{2} \leq i \leq n+1\right\}, \quad \mathscr{J} = \left\{i \in \mathbb{Z}; n+1-\frac{m_{(\lambda+1)}}{2} \leq i \leq n\right\}.$$

Now assume $rP_i(\mathbb{F}_q) \in \mathcal{M}_i^{g,\gamma}$ for some $1 \leq i \leq n+1$ and some $\gamma \in \Gamma_i$. Then by the previous claim either of the following two statements holds:

- $i \in \mathscr{I}$, and $f_{\pi_i(r^{-1}gr)}(\lambda) = (\lambda-1)^{2(n+1-i)}$.
- $i \in \mathscr{J}$, and $f_{\pi_i(r^{-1}gr)}(\lambda) = (\lambda+1)^{2(n+1-i)}$.

Moreover, in the above two cases, the image of γ in $\mathrm{GL}(V_i)$ is id and $-$id respectively. Conversely, if $i \in \mathscr{I}$ and if $rP_i(\mathbb{F}_q) \in \mathcal{M}_i^g$ is such that $f_{\pi_i(r^{-1}gr)}(\lambda) = (\lambda-1)^{2(n+1-i)}$, then $rP_i(\mathbb{F}_q) \in \mathcal{M}_i^{g,\mathrm{id}}$. Similarly, if $i \in \mathscr{J}$ and if $rP_i(\mathbb{F}_q) \in \mathcal{M}_i^g$ is such that $f_{\pi_i(r^{-1}gr)}(\lambda) = (\lambda+1)^{2(n+1-i)}$, then $rP_i(\mathbb{F}_q) \in \mathcal{M}_i^{g,-\mathrm{id}}$. Therefore as in the proof of part (2), we have

$$\operatorname{tr}(g, J, \mathscr{L}) = \sum_{i\in\mathscr{I}} \mathcal{T}(w_i, \mathrm{id}) \cdot \# \left\{U \in \mathbb{F}_q[\lambda]^{\mathrm{monic}}; UU^* = f_g/(\lambda-1)^{2(n+1-i)}\right\}$$
$$+ \sum_{i\in\mathscr{J}} \mathcal{T}(w_i, -\mathrm{id}) \cdot \# \left\{U \in \mathbb{F}_q[\lambda]^{\mathrm{monic}}; UU^* = f_g/(\lambda+1)^{2(n+1-i)}\right\}.$$

Let $i \in \mathscr{I}$. By the obvious analogue of Lemma 4.1.6 applied to $f_g/(\lambda-1)^{2(n+1-i)}$ and $Q_0 = 1$, we have

$$\# \left\{U \in \mathbb{F}_q[\lambda]^{\mathrm{monic}}; UU^* = f_g/(\lambda-1)^{2(n+1-i)}\right\} = \mathscr{M}(f_g/(\lambda-1)^{2(n+1-i)}),$$

which is equal to $\mathscr{M}(f_g)$. Similarly, for $i \in \mathscr{J}$, we have

$$\# \left\{U \in \mathbb{F}_q[\lambda]^{\mathrm{monic}}; UU^* = f_g/(\lambda+1)^{2(n+1-i)}\right\} = \mathscr{M}(f_g).$$

On the other hand by Definition 2.7.2 we have $\mathcal{T}(w_i, \mathrm{id}) = 1$ for all $i \in \mathscr{I}$ and $\mathcal{T}(w_i, -\mathrm{id}) = 1$ for all $i \in \mathscr{J}$. Therefore

$$\operatorname{tr}(g, J, \mathscr{L}) = (|\mathscr{I}| + |\mathscr{J}|)\mathscr{M}(f_g) = \Big(\frac{m_{(\lambda-1)}}{2} + 1 + \frac{m_{(\lambda+1)}}{2}\Big)\mathscr{M}(f_g). \qquad \square$$

4.6. **The odd unitary group.** In this subsection we consider case (4) in §3.

We fix a $(2n+1)$-dimensional Hermitian space $(\mathbb{V}, [\cdot,\cdot])$ over $\mathbb{F}_{q^2}$ (for the quadratic extension $\mathbb{F}_{q^2}/\mathbb{F}_q$), with $n \geq 0$. Let $\mathbb{G} = \mathrm{U}(\mathbb{V}, [\cdot,\cdot])$. By [PR94, Proposition 2.15], the Witt index of $(\mathbb{V}, [\cdot,\cdot])$ is equal to the $\mathbb{F}_q$-rank of $\mathbb{G}$, which we know is n. Also the norm map $\mathbb{F}_{q^2}^\times \to \mathbb{F}_q^\times$ is surjective. Hence there exists an $\mathbb{F}_{q^2}$-basis $e_1, \cdots, e_{2n+1}$ of $\mathbb{V}$, satisfying

$$[e_\alpha, e_\beta] = \delta_{2n+2,\alpha+\beta}.$$

For each $1 \leq i \leq n+1$, we define

$$\mathbb{V}_i := \operatorname{span}_{\mathbb{F}_{q^2}}(e_i, e_{i+1}, \cdots, e_{2n+2-i}) \subset \mathbb{V}, \qquad \mathbb{W}_i := \operatorname{span}_{\mathbb{F}_{q^2}}(e_1, \cdots, e_i) \subset \mathbb{V}.$$

We fix an embedding $\mathbb{F}_{q^2} \to k$, viewed as the identity, and we let $V := \mathbb{V} \otimes_{\mathbb{F}_{q^2}} k$. For each $1 \leq i \leq n+1$ we also let $V_i := \mathbb{V}_i \otimes_{\mathbb{F}_{q^2}} k \subset V$, and $W_i := \mathbb{W}_i \otimes_{\mathbb{F}_{q^2}} k \subset V$.

Let $G = \mathbb{G}_k$. The action of G on $\mathbb{V} \otimes_{\mathbb{F}_q} k \cong V \oplus (\mathbb{V} \otimes_{\mathbb{F}_{q^2},\sigma} k)$ preserves the subspace V, and this induces a k-isomorphism $G \cong \mathrm{GL}(V)$. Let $B \subset G$ (resp. $T \subset G$) be the upper triangular subgroup (resp. diagonal subgroup) under the basis $e_1, \cdots, e_{2n+1}$. Then B is a σ-stable Borel subgroup of G, and T is the maximal torus of G contained in B.

We number the simple roots of (G, B, T) according to Bourbaki [Bou68]. We consider the σ-unbranched datum $(J = \mathbb{S} - \{s_n\}, \mathscr{L} = (s_1, \cdots, s_n))$. Following the notation of §2.4 and §2.5, we have $i_{\max} = n+1$, and for $1 \leq i \leq n+1$ we have

$$\mathbb{P}_i = \mathrm{Stab}_{\mathbb{G}}(\mathbb{W}_{i-1}), \qquad \mathbb{L}_i = \mathbb{L}_i^\natural = \mathrm{GL}_{\mathbb{F}_{q^2}}(\mathbb{W}_{i-1}) \times \mathrm{U}(\mathbb{V}_i)$$
$$\mathbb{G}_i = \mathrm{U}(\mathbb{V}_i) = \mathrm{U}_{2(n+1-i)+1}, \qquad \mathbb{H}_i = \mathrm{GL}_{\mathbb{F}_{q^2}}(\mathbb{W}_{i-1}) = \mathrm{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathrm{GL}_{i-1}.$$

Here by convention $\mathbb{W}_0 = 0$ and $\mathrm{GL}_0 = \{1\}$. As in §2.5, we have natural projections $\pi_i : \mathbb{L}_i \to \mathbb{G}_i$ and $\pi_i' : \mathbb{L}_i \to \mathbb{H}_i$.

For any $h \in G_i(k) \cong \mathrm{GL}_{2n+1-i+1}(k)$, we denote by $f_h \in k[\lambda]$ the characteristic polynomial of h, of degree $2n+1-i+1$. When $h \in G_i(\mathbb{F}_q)$, we know that f_h is self-reciprocal in $\mathbb{F}_{q^2}[\lambda]$. Similarly, for any $h \in H_i(\mathbb{F}_q) = \mathrm{GL}_{\mathbb{F}_{q^2}}(\mathbb{W}_i)$, we denote by $f_h(\lambda) \in \mathbb{F}_{q^2}[\lambda]$ the characteristic polynomial of h acting on $\mathbb{W}_i$, which has degree $i-1$.

Theorem 4.6.1. *We fix $1 \leq i \leq n+1$. Write n' for $n+1-i$. Thus $\mathbb{G}_i = \mathrm{U}_{2n'+1}$. We have the following statements about $T_i(\mathbb{F}_q)$.*

(1) If $\gamma \in T_i(\mathbb{F}_q)$, then the $f_\gamma = Q^m$ for some $Q \in \mathsf{SR}_2$, and some positive integer m.

(2) Let $Q \in \mathsf{SR}_2$. Assume m is an integer such that $m \deg Q = 2n' + 1$. Then there exists $\gamma \in T_i(\mathbb{F}_q)$ with $f_\gamma = Q^m$.

(3) Let Q and m be as in part (2). Let $\gamma \in G_i(\mathbb{F}_q)$ be a semi-simple element such that $f_\gamma = Q^m$. Then γ is $G_i(\mathbb{F}_q)$-conjugate to an element of $T_i(\mathbb{F}_q)$.

(4) Let $\gamma \in T_i(\mathbb{F}_q)$. Write $f_\gamma = Q^m$ as in part (1). Then $\mathcal{T}(w_i, \gamma) = \deg Q$.

Proof. On T_i we have coordinates

$$(k^\times)^{2n'+1} \xrightarrow{\sim} T_i, \qquad (\lambda_1, \cdots, \lambda_{2n'+1}) \longmapsto \gamma(\lambda_1, \cdots, \lambda_{2n'+1}),$$

such that the eigenvalues (with multiplicities) of $\gamma(\lambda_1, \cdots, \lambda_{n'})$ acting on V_i are $\lambda_1, \cdots, \lambda_{2n'+1}$, and such that

$$\gamma(\lambda_1, \cdots, \lambda_{2n'+1})^\sigma = \gamma(\lambda_{n'+1}^{-q}, \lambda_{n'+2}^{-q}, \cdots, \lambda_{2n'+1}^{-q}, \lambda_1^{-q}, \cdots, \lambda_{n'}^{-q}). \tag{4.6.1}$$

In particular, we have

$$\gamma(\lambda_1, \cdots, \lambda_{2n'+1})^{\sigma^2} = \gamma(\lambda_{2n'+1}^{\sigma^2}, \lambda_1^{\sigma^2}, \cdots, \lambda_{2n'}^{\sigma^2}). \tag{4.6.2}$$

(1) Let $(\lambda_1, \cdots, \lambda_{2n'+1})$ be the coordinates of γ. Since $\gamma^\sigma = \gamma$, it follows from (4.6.2) that all eigenvalues of γ are in one σ^2-orbit. Hence f_γ has a unique monic irreducible factor $Q \in \mathbb{F}_{q^2}[\lambda]$. Since f_γ is self-reciprocal, so is Q.

(2) Let $d = \deg Q$. Then d is odd by hypothesis. Let $\Lambda = (\lambda_1, \cdots, \lambda_d)$ be an admissible enumeration of the roots of Q, in the sense of Definition 4.1.10. Then $\gamma := \gamma(\Lambda, \cdots, \Lambda)$ (with m appearances of Λ) is an element of $T_i(k)$. We now show that $\gamma \in T_i(\mathbb{F}_q)$.

If $d = 1$, then $\lambda_1^{-q} = \lambda_1$, and it is clear that $\gamma \in T_i(\mathbb{F}_q)$ by (4.6.1). Hence assume $d \geq 3$. By (4.6.1), we need only show that $\lambda_\alpha^{-q} = \lambda_{\alpha+n'+1}$, where the subscripts are in $\mathbb{Z}/d\mathbb{Z}$, for all $\alpha \in \mathbb{Z}/d\mathbb{Z}$. By Lemma 4.1.11 (2), it suffices to show that $n' + 1 \equiv (d+1)/2 \mod d$. Since d is odd, the last congruence is equivalent to $2n' + 2 \equiv d + 1 \mod d$. But the last congruence is true because $2n' + 1 = md$. We have proved that $\gamma \in T_i(\mathbb{F}_q)$. By construction, $f_\gamma = Q^m$. Part (2) is proved.

(3) Firstly, as G_i is isomorphic to $\mathrm{GL}(V_i) = \mathrm{GL}_{2n'+1}$ over k, we know that two semi-simple elements in $G_i(k)$ are conjugate if and only if they have the same characteristic polynomial. Secondly, since G_i has simply connected derived subgroup, by the Lang–Steinberg theorem we know that any two semi-simple elements in $G_i(\mathbb{F}_q)$ are $G_i(\mathbb{F}_q)$-conjugate if and only if they are $G_i(k)$-conjugate (cf. §2.7.4 and the proof of Lemma 2.7.5). The assertion now follows from part (2).

(4) Let $d = \deg Q$. Since G_i has simply connected derived subgroup, we may use Lemma 2.7.5 to compute $\mathcal{T}(w_i, \gamma)$. We have

$$\mathcal{T}(w_i, \gamma) = \#\{\gamma' \in T_i(\mathbb{F}_q); \gamma' = {}^x\gamma \text{ for some } x \in W_i\}.$$

By (4.6.2), it is clear that any $\gamma' \in T_i(\mathbb{F}_q)$ with characteristic polynomial Q^m must be of the form $\gamma' = \gamma(\Lambda', \cdots, \Lambda')$, for some admissible enumeration Λ' of the d roots of Q. There are d such admissible enumerations (Lemma 4.1.11), and all of them correspond to elements in $T_i(\mathbb{F}_q)$ by the proof of part (2). Moreover, it is clear that these d resulting elements of $T_i(\mathbb{F}_q)$ are in the same W_i-orbit. Hence $\mathcal{T}(w_i, \gamma) = d$. □

Lemma 4.6.2. *Let $g \in G(\mathbb{F}_q) \cap G^{\mathrm{reg}}$. For each $1 \leq i \leq n+1$, let $\mathcal{M}_i^g$ be as in §2.8. We have a bijection*

$$\mathcal{M}_i^g \longrightarrow \left\{U \in \mathbb{F}_{q^2}[\lambda]^{\mathrm{monic}}; \deg U = i - 1, UU^* \text{ divides } f_g \text{ in } \mathbb{F}_{q^2}[\lambda]\right\}, \quad rP_i(\mathbb{F}_q) \longmapsto f_{\pi_i'(r^{-1}gr)}.$$

Proof. The proof is completely analogous to Lemma 4.3.2, based on the fact that all $(i-1)$-dimensional totally isotropic $\mathbb{F}_{q^2}$-subspaces of $\mathbb{V}$ are in the same $G(\mathbb{F}_q)$-orbit. □

Theorem 4.6.3. *Let $g \in G(\mathbb{F}_q) \cap G^{\mathrm{reg}}$. We use the notations in Definition 4.1.5. For each $Q \in \mathsf{SR}_2$, we simply write m_Q for $m_Q(f_g)$. The following statements hold.*

(1) If $\mathrm{tr}(g, J, \mathscr{L}) \neq 0$, then there is a unique element $Q_0 \in \mathsf{SR}_2$ such that m_{Q_0} is odd.

(2) Assume there is a unique element $Q_0 \in \mathsf{SR}_2$ such that m_{Q_0} is odd. Then

$$\mathrm{tr}(g, J, \mathscr{L}) = \deg Q_0 \frac{m_{Q_0} + 1}{2} \mathscr{M}(f_g).$$

Proof. We apply Theorem 2.8.1 to compute $\mathrm{tr}(g, J, \mathscr{L})$ in the following.

We prove part (1). Assume $\mathrm{tr}(g, J, \mathscr{L}) \neq 0$. Then there exist $1 \leq i \leq n+1$ and $\gamma \in \Gamma_i$ such that $\mathcal{M}_i^{g,\gamma} \neq \emptyset$. Take $rP_i(\mathbb{F}_q) \in \mathcal{M}_i^{g,\gamma}$. Then by Theorem 4.6.1 (1), we have $f_{\pi_i(r^{-1}gr)} = Q^m$, for some $Q \in \mathsf{SR}_2$ and some positive integer m. In particular m is odd because Q^m has odd degree. By Lemma 4.6.2, we have $f_g = Q^m UU^*$ for some $U \in \mathbb{F}_{q^2}[\lambda]^{\mathrm{monic}}$. It then follows that Q is the unique element of SR_2 such that m_Q is odd. Part (1) is proved.

We now prove part (2). Since f_g has odd degree, it immediately follows from the hypothesis that $\deg Q_0$ is odd. Define

$$\mathscr{I} := \{i; 1 \leq i \leq n+1, \frac{2(n+1-i)+1}{\deg Q_0} \text{ is a (necessarily odd) integer } \leq m_{Q_0}\}.$$

For $i \in \mathscr{I}$, define $m_i := [2(n+1-i)+1]/\deg Q_0$. Note that $i \mapsto m_i$ is a bijection $\mathscr{I} \to \{1, 3, 5, \cdots, m_{Q_0}\}$. In particular $|\mathscr{I}| = (m_{Q_0}+1)/2$. Similar to the bijection (4.3.5), we obtain a bijection

$$\begin{aligned} &\{(i, \gamma, rP_i(\mathbb{F}_q)); 1 \leq i \leq n+1, \gamma \in \Gamma_i, rP_i(\mathbb{F}_q) \in \mathcal{M}_i^{g,\gamma}\} \xrightarrow{\sim} \\ &\left\{(i, rP_i(\mathbb{F}_q)); i \in \mathscr{I}, rP_i(\mathbb{F}_q) \in \mathcal{M}_i^{g}, f_{\pi_i(r^{-1}gr)} = Q_0^{m_i}\right\} \\ &(i, \gamma, rP_i(\mathbb{F}_q)) \longmapsto (i, rP_i(\mathbb{F}_q)) \end{aligned} \tag{4.6.3}$$

based on Theorem 4.6.1 (3). We also note that if $(i, \gamma, rP_i(\mathbb{F}_q))$ is in the left hand side of (4.6.3), then $f_\gamma = Q_0^{m_i}$, and so by Theorem 4.6.1 (4) we have

$$\mathcal{T}(w_i, \gamma) = \deg Q_0. \tag{4.6.4}$$

The rest of the proof is identical to the proof of Theorem 4.3.3 (3), based on (4.6.3), (4.6.4), and Lemma 4.6.2. ☐

5. Application to arithmetic intersection

In this section we apply Theorem 4.6.3 to prove the arithmetic fundamental lemma in the minuscule case, generalizing the main result of [RTZ13] and [LZ17]. We also apply Theorem 4.3.3 to compute certain arithmetic intersection in GSpin Rapoport–Zink spaces, generalizing the main result of [LZ18].

5.1. **The arithmetic fundamental lemma in the minuscule case.** We follow the notation of [RTZ13] and [LZ17]. Let p be an odd prime. Let F be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$ and a uniformizer π. As usual we denote $k := \overline{\mathbb{F}}_q$. Let E/F be a quadratic unramified extension. Let $\breve{E}$ be the completion of the maximal unramified extension of E. Let $S = \mathrm{Spf}\, \mathcal{O}_{\breve{E}}$. Fix an integer $n \geq 2$. Let $\mathcal{N}_n$ be the *unitary Rapoport–Zink space of signature* $(1, n-1)$, which is a formal scheme over S parameterizing deformations up to quasi-isogeny of height 0 of unitary $\mathcal{O}_F$-modules of signature $(1, n-1)$. For details on $\mathcal{N}_n$ see [KR11], [Mih16], and [Cho18].

Let C_n be a non-split Hermitian space of dimension n, for the quadratic extension E/F. Here non-split means that the discriminant has odd valuation. We identify C_n with the space of special quasi-homomorphisms for the framing object in the moduli problem of $\mathcal{N}_n$, see [KR11] for $F = \mathbb{Q}_p$ (cf. [LZ17, §2.2,

§2.3]), and [Cho18] for general F. Similarly, we form $\mathcal{N}_{n-1}$ and C_{n-1}. We identify C_{n-1} with the orthogonal complement in C_n of a fixed vector $u \in C_n$ of norm 1, thus $C_n = C_{n-1} \oplus Eu$. We have a natural closed immersion

$$\delta : \mathcal{N}_{n-1} \hookrightarrow \mathcal{N}_n.$$

In fact δ identifies $\mathcal{N}_{n-1}$ with the special divisor in $\mathcal{N}_n$ associated to u, see [KR11] for $F = \mathbb{Q}_p$, and see [Cho18] for general F.

The unitary group $J(F) := \mathrm{U}(C_n)(F)$ acts on $\mathcal{N}_n$. Let $g \in J(F)$. Define

$$L(g) := \mathcal{O}_E \cdot u + \mathcal{O}_E \cdot gu + \cdots + \mathcal{O}_E \cdot g^{n-1}u \subset C_n.$$

Throughout we make two assumptions on g. Firstly, we assume that g is *regular semi-simple minuscule*, in the sense that $L(g)$ is a full-rank $\mathcal{O}_E$-lattice in C_n satisfying

$$\pi L(g)^\vee \subset L(g) \subset L(g)^\vee.$$

Secondly, we assume that g has non-empty fixed points in $\mathcal{N}_n(k)$. By [RTZ13, §5], our second assumption implies that both $L(g)$ and $L(g)^\vee$ are stable under g.

Define $\mathbb{V} := L(g)^\vee/L(g)$. This is an odd-dimensional vector space over $\mathbb{F}_{q^2}$, with a natural structure of a Hermitian space, see [LZ17, §2.4]. Let $\mathcal{V} := \mathcal{V}(L(g)^\vee)$ be the smooth projective generalized Deligne–Lusztig variety associated to the vertex lattice $L(g)^\vee$ as in [Vol10] and [VW11]. (These references assume $F = \mathbb{Q}_p$, but see [Cho18] for general F.) The finite group $\mathrm{U}(\mathbb{V})(\mathbb{F}_q)$ naturally acts on $\mathcal{V}$. Let $\mathbb{G} = \mathrm{U}(\mathbb{V})$, $G = \mathbb{G}_k$, and let $(J, \mathscr{L})$ be the σ-unbranched datum for $\mathbb{G}$ specified in §4.6.

Lemma 5.1.1. *The variety $\mathcal{V}$ is $\mathbb{G}(\mathbb{F}_q)$-equivariantly isomorphic to $\overline{X_{J,w_1}}$.*

Proof. Since $G_1 = P_1 = G$, by Proposition 2.5.1 we have an isomorphism

$$X_{w_1} \xrightarrow{\sim} X_{J,w_1} \subset G/P_J, \qquad gB \longmapsto gP_J$$

where X_{w_1} is the classical Deligne–Lusztig variety associated to w_1 in the full flag variety G/B. The lemma then follows from [Vol10, Theorem 2.15], which asserts that $\mathcal{V}$ is also the closure in G/P_J of the image of X_{w_1}. (Again, the reference [Vol10] assumes $F = \mathbb{Q}_p$ and $\mathbb{F}_q = \mathbb{F}_p$, but the result [Vol10, Theorem 2.15] easily generalizes.) □

The action of g on $\mathbb{V}$ defines an element $\bar{g} \in \mathbb{G}(\mathbb{F}_q)$. We also know that $\bar{g}$ is regular, because $\mathbb{V}$ is a cyclic $\mathbb{F}_{q^2}[\bar{g}]$-module. Let $f = f_{\bar{g}} \in \mathbb{F}_{q^2}[\lambda]$ be the characteristic polynomial of $\bar{g}$. Thus f is self-reciprocal. We use the notations in Definition 4.1.5.

Theorem 5.1.2. *As before, assume $g \in J(F)$ is regular semi-simple minuscule, such that $\mathcal{N}_n^g \neq \emptyset$. The following statements hold.*

(1) The formal scheme $\delta(\mathcal{N}_{n-1}) \cap \mathcal{N}_n^g$ over S is a k-scheme.

(2) The k-scheme $\delta(\mathcal{N}_{n-1}) \cap \mathcal{N}_n^g$ is non-empty if and only if there is a unique element $Q_0 \in \mathsf{SR}_2$ with $m_{Q_0}(f)$ odd. In this case, $\delta(\mathcal{N}_{n-1}) \cap \mathcal{N}_n^g$ has finitely many k-points, and is in particular Artinian, and moreover $\mathrm{Int}(g)$ is equal to the total k-length of $\delta(\mathcal{N}_{n-1}) \cap \mathcal{N}_n^g$.

(3) Assume there is a unique element $Q_0 \in \mathsf{SR}_2$ with $m_{Q_0}(f)$ odd. Then the total k-length of $\delta(\mathcal{N}_{n-1}) \cap \mathcal{N}_n^g$ is equal to

$$\deg(Q_0)\frac{m_{Q_0}(f) + 1}{2}\mathscr{M}_2(f).$$

Proof. We temporarily assume that $F = \mathbb{Q}_p$. Then part (1) follows from [LZ17, Proposition 4.1.2] (cf. [RTZ13, (9.6), Theorem 9.4]). Part (2) is proved in [RTZ13, Proposition 8.1 (1)] and [RTZ13, Proposition 4.2 (iii)].

For part (3), we first apply [LZ17, Proposition 4.1.2] to identify $\delta(\mathcal{N}_{n-1}) \cap \mathcal{N}_n^g$ with $\mathcal{V}^{\bar{g}}$, the scheme theoretic fixed points of $\mathcal{V}$ under $\bar{g} \in \mathbb{G}(\mathbb{F}_q)$. By part (2), $\mathcal{V}^{\bar{g}}$ is an Artinian scheme. Since $\mathcal{V}$ is smooth over k and since $\mathcal{V}^{\bar{g}}$ is Artinian, it is well known (see for instance [Ser00, p. 111]) that the intersection multiplicities of the graph of identity and the graph of $\bar{g}$ in $\mathcal{V} \times_k \mathcal{V}$ are simply given by the lengths of the local rings of $\mathcal{V}^{\bar{g}}$, as the higher Tor terms vanish. It then follows from the Lefschetz fixed point formula [GD77, Corollaire 3.7] that the k-length of $\mathcal{V}^{\bar{g}}$ is equal to $\mathrm{tr}(\bar{g}, \mathbf{H}^*(\mathcal{V}))$. By Lemma 5.1.1, the last number is equal to $\mathrm{tr}(\bar{g}, J, \mathscr{L})$. Hence part (3) follows from Theorem 4.6.3 and the fact that $\bar{g}$ is regular. We have proved the theorem assuming $F = \mathbb{Q}_p$.

We now explain the proof when F is an arbitrary finite extension of $\mathbb{Q}_p$. In fact, the reason that the references [RTZ13] and [LZ17] assumed $F = \mathbb{Q}_p$ was because two ingredients needed in the arguments depended on this assumption. The first is the theory of special cycles considered in [KR11], and the second is the Bruhat–Tits stratification of the reduced subscheme of $\mathcal{N}_n$ into generalized Deligne–Lusztig varieties, worked out in [Vol10] and [VW11]. Both of these ingredients have now been generalized to arbitrary F in [Cho18]. Based on this, all the previous arguments carry over.[2] □

Remark 5.1.3. Theorem 5.1.2 (3) was previously proved in [RTZ13] and [LZ17], under the assumption that $F = \mathbb{Q}_p$ with $p > (m_{Q_0} + 1)/2$. This assumption is removed in Theorem 5.1.2. On the other hand, under the same assumption on p the papers [RTZ13] and [LZ17] determine each local ring of $\delta(\mathcal{N}_{n-1}) \cap \mathcal{N}_n^g$. This is a result not revealed by the methods of the current paper.

Corollary 5.1.4. *The minuscule case of the arithmetic fundamental lemma conjecture* [RTZ13, Conjecture 7.4] *(cf.* [RSZ17b, §1]*) holds.*

Proof. It follows from the formula for the arithmetic intersection number $\mathrm{Int}(g)$ in Theorem 5.1.2 (2–3) and the explicit computation of the analytic side in [RTZ13, Proposition 8.2]. □

5.2. **Arithmetic intersection on GSpin Rapoport–Zink spaces.** We follow the notation of [LZ18]. Let p be an odd prime, and fix an integer $n \geq 4$. Let RZ (resp. $\mathrm{RZ}^\flat$) be the GSpin Rapoport–Zink space associated to a self-dual quadratic $\mathbb{Z}_p$-lattice of rank n (resp. $n-1$). We have a natural closed immersion

$$\delta : \mathrm{RZ}^\flat \longrightarrow \mathrm{RZ}$$

of formal schemes over $\mathrm{Spf}\, W(k)$. These are specific Hodge-type Rapoport–Zink spaces introduced by Howard–Pappas [HP17]. Associated to the precise data used to define $\mathrm{RZ}^\flat$ and RZ, we have a pair of quadratic spaces $V_K^{\flat,\Phi}$ and V_K^Φ over $\mathbb{Q}_p$, and $V_K^{\flat,\Phi}$ can be identified with the orthogonal complement in V_K^Φ of a fixed vector $x_n \in V_K^\Phi$ whose norm is 1. (The triple $(V_K^{\flat,\Phi}, V_K^\Phi, x_n)$ is analogous to the triple (C_{n-1}, C_n, u) in §5.1.)

The group $J(\mathbb{Q}_p) = \mathrm{GSpin}(V_K^\Phi)(\mathbb{Q}_p)$ acts on RZ. As in [HP17, §4.3], RZ is the disjoint union of open and closed formal subschemes $\mathrm{RZ}^{(l)}$, indexed by $l \in \mathbb{Z}$. The action of any $g \in J(\mathbb{Q}_p)$ on RZ maps each $\mathrm{RZ}^{(l)}$ isomorphically to $\mathrm{RZ}^{(l+l_g)}$, where l_g is the p-adic valuation of the spinor norm of g in $\mathbb{Q}_p^\times$. We view p as an

[2] It should be pointed out that in [LZ17, §2.6], for a vertex lattice Λ the notation $\mathcal{N}_\Lambda$ denotes the special cycle in $\mathcal{N}_n$ associated to $\Lambda^\vee$. Thus a priori $\mathcal{N}_\Lambda$ is a formal scheme over S, but it is a theorem ([RTZ13, Theorems 9.4, 10.1], see also [LZ17, Corollary 3.2.3]) that $\mathcal{N}_\Lambda$ is in fact a reduced scheme over k. This result plays a key role in [RTZ13] and [LZ17], and its proof depends on Grothendieck–Messing theory. In contrast, in [VW11] and [Cho18] the notation $\mathcal{N}_\Lambda$ is by definition a scheme over characteristic p. Thus the two notations agree only a posteriori.

element of $J(\mathbb{Q}_p)$ by viewing it as an scalar in the GSpin group. Thus p maps each $\mathrm{RZ}^{(l)}$ isomorphically to $\mathrm{RZ}^{(l+2)}$.

Let $g \in J(\mathbb{Q}_p)$. Define

$$L(g) := \mathbb{Z}_p \cdot x_n + \mathbb{Z}_p \cdot gx_n + \cdots \mathbb{Z}_p \cdot g^{n-1}x_n \subset V_K^\Phi.$$

Here g acts on V_K^Φ via the natural map $\mathrm{GSpin}(V_K^\Phi) \to \mathrm{SO}(V_K^\Phi)$. Throughout we make two assumptions on g. Firstly, we assume that g is *regular semi-simple minuscule*, in the sense that $L(g)$ is a full-rank $\mathbb{Z}_p$-lattice in V_K^Φ satisfying

$$pL(g)^\vee \subset L(g) \subset L(g)^\vee.$$

Secondly, we assume that g has non-empty fixed points in $\mathrm{RZ}(k)$. By [LZ18, §3.6], our second assumption implies that both $L(g)$ and $L(g)^\vee$ are stable under g. It also directly follows from our second assumption that $l_g = 0$. In particular g stabilizes each $\mathrm{RZ}^{(l)}$.

Define $\mathbb{V} := L(g)^\vee/L(g)$. This is an even-dimensional, non-zero vector space over $\mathbb{F}_p$, with a natural structure of a non-split quadratic space, see [LZ18, §2.7]. Let $S = S_{L(g)^\vee}$ be the smooth projective k-variety associated to the vertex lattice $L(g)^\vee$ as in [HP17, §5.3]. The finite group $\mathrm{O}(\mathbb{V})(\mathbb{F}_p)$ naturally acts on S. By [HP17, Proposition 5.3.2] and its proof, we know that S has two connected components S^+, S^-, that the action of $\mathrm{SO}(\mathbb{V})(\mathbb{F}_p)$ on S stabilizes each of S^+, S^-, and that any element of $\mathrm{O}(\mathbb{V})(\mathbb{F}_p) - \mathrm{SO}(\mathbb{V})(\mathbb{F}_p)$ interchanges S^+, S^-. Let $\mathbb{G} = \mathrm{SO}(\mathbb{V})$, $G = \mathbb{G}_k$, and let $(J, \mathscr{L})$ be the σ-unbranched datum for $\mathbb{G}$ specified in §4.3. For definiteness, we fix the convention so that our w_1 corresponds to the Weyl group element w^- in [HP14, §3.2].[3]

Lemma 5.2.1. *The variety S^- is $\mathbb{G}(\mathbb{F}_q)$-equivariantly isomorphic to $\overline{X_{J,w_1}}$.*

Proof. Since $G_1 = P_1 = G$, by Proposition 2.5.1 we have an isomorphism

$$X_{w_1} \xrightarrow{\sim} X_{J,w_1} \subset G/P_J, \qquad gB \longmapsto gP_J$$

where X_{w_1} is the classical Deligne–Lusztig variety associated to w_1 in the full flag variety G/B. The claim then follows from [HP14, Proposition 3.8], which asserts that S^- (denoted by $\mathscr{X}^-$ in *loc. cit.*) is also the closure of the image of X_{w_1} in G/P_J. □

The action of g on $\mathbb{V}$ defines an element $\bar{g} \in \mathrm{O}(\mathbb{V})(\mathbb{F}_p)$. The following result is implicitly assumed in [LZ18], but is not explicitly stated and proved there. We give two proofs here, for the sake of completeness.

Lemma 5.2.2. *The element $\bar{g} \in \mathrm{O}(\mathbb{V})(\mathbb{F}_p)$ lies in $\mathrm{SO}(\mathbb{V})(\mathbb{F}_p)$.*

Proof. **First proof.** Let $S = S_{L(g)^\vee}$ be as before. By [HP17, Theorem 6.3.1], we have an isomorphism $p^{\mathbb{Z}}\backslash \mathrm{RZ}^{\mathrm{red}}_{L(g)^\vee} \xrightarrow{\sim} S$, where $\mathrm{RZ}^{\mathrm{red}}_{L(g)^\vee}$ is a certain g-stable subscheme of RZ. It is easy to see that this isomorphism intertwines the action of g on the left and the action of $\bar{g}$ on the right, for example by checking the statement on k-points. Since g stabilizes each $\mathrm{RZ}^{(l)}$, by [HP17, Corollary 6.3.2] we know that g stabilizes each of the two connected components of $p^{\mathbb{Z}}\backslash \mathrm{RZ}^{\mathrm{red}}_\Lambda$. Therefore $\bar{g}$ stabilizes each of the two connected components of S. By the proof of [HP17, Proposition 5.3.2], any element of $\mathrm{O}(\mathbb{V})(\mathbb{F}_p) - \mathrm{SO}(\mathbb{V})(\mathbb{F}_p)$ interchanges the two connected components of S. It then follows that $\bar{g} \in \mathrm{SO}(\mathbb{V})$.

Second proof. The result follows from Lemma 5.2.3 in the following, applied to $W := V_K^\Phi$, $L := L(g)$, and $h :=$ the image of g under $\mathrm{GSpin}(V_K^\Phi)(\mathbb{Q}_p) \to \mathrm{SO}(V_K^\Phi)(\mathbb{Q}_p)$. The hypothesis on the spinor norm of h is satisfied because $l_g = 0$. □

[3]This is harmless because up to outer automorphism of G, our w_1 corresponds to either w^- or w^+ in [HP14, §3.2]. All the arguments below are the same in the two cases.

Lemma 5.2.3. *Let $(W, [\cdot,\cdot])$ be a quadratic space over $\mathbb{Q}_p$. Let $h \in \mathrm{O}(W)(\mathbb{Q}_p)$ be an element whose spinor norm (see* [Kit93, §1.6]*) in $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times,2}$ has even valuation. Let L be a full-rank lattice in W satisfying $pL^\vee \subset L \subset L^\vee$. Assume L is stable under h. Then the induced action $\bar{h}$ of h on the $\mathbb{F}_p$-vector space $L^\vee/L$ has determinant* 1*.*

Proof. Since h stabilizes L, by [Kit93, Theorem 5.3.3] we have $h = \tau_1 \cdots \tau_m$, where each $\tau_j \in \mathrm{O}(W)(\mathbb{Q}_p)$ is the reflection associated to an anisotropic vector $v_j \in L$ (namely $\tau_j(x) = x - 2[x, v_j][v_j, v_j]^{-1}v_j, \forall x \in W$), such that τ_j also stabilizes L. By rescaling, we may and shall assume that each $v_j \in L - pL$. We now fix $1 \le j \le m$.

Since τ_j stabilizes L, we have $[x, v_j] \in [v_j, v_j]\mathbb{Z}_p$ for all $x \in L$, or equivalently that

$$v_j \in [v_j, v_j]L^\vee. \tag{5.2.1}$$

Since $pL^\vee \subset L \subset L^\vee$ and $v_j \in L - pL$, it follows from (5.2.1) that $[v_j, v_j]$ has valuation 0 or 1. If $[v_j, v_j]$ has valuation 0, then τ_j maps each $x \in L^\vee$ into $x + \mathbb{Z}_p v_j \subset x + L$, and so the image of τ_j in $\mathrm{GL}(L^\vee/L)$ is trivial. Assume $[v_j, v_j]$ has valuation 1. Then $v_j \in pL^\vee$ by (5.2.1), and so $v_j = pw_j$ for some $w_j \in L^\vee - L$. In this case we have

$$\tau_j(x) = x - 2\frac{p[x, w_j]}{p[w_j, w_j]}w_j, \quad \forall x \in L. \tag{5.2.2}$$

Now the map

$$L^\vee \times L^\vee \longrightarrow \mathbb{F}_p, \quad (x, y) \longmapsto p[x, y] \mod p$$

is well defined and descends to a non-degenerate bi-linear pairing on the $\mathbb{F}_p$-vector space $L^\vee/L$ (cf. [HP17, §5.3.1]). Noting that $p[w_j, w_j] = p^{-1}[v_j, v_j]$ is by assumption in $\mathbb{Z}_p^\times$, we see from (5.2.2) that the image of τ_j in $\mathrm{GL}(L^\vee/L)$ is given by the reflection associated to an anisotropic vector in $L^\vee/L$, namely the image of w_j.

In conclusion, the image of h in $\mathrm{GL}(L^\vee/L)$ is the product of m' reflections, where m' is the number of the v_j's such that $[v_j, v_j] \in p\mathbb{Z}_p^\times$, whereas the $m - m'$ other v_j's satisfy $[v_j, v_j] \in \mathbb{Z}_p^\times$. Since the spinor norm of h has even valuation, we know that m' is even. The lemma follows. □

By Lemma 5.2.2 we have $\bar{g} \in \mathrm{SO}(\mathbb{V})(\mathbb{F}_p)$. We also know that the image of $\bar{g}$ in $\mathrm{GL}(\mathbb{V})$ is regular, because $\mathbb{V}$ is a cyclic $\mathbb{F}_p[\bar{g}]$-module. Let $f = f_{\bar{g}} \in \mathbb{F}_p[\lambda]$ be the characteristic polynomial of $\bar{g}$. Thus f is self-reciprocal. We use the notations in Definition 4.1.5.

Theorem 5.2.4. *As before, assume $g \in J(\mathbb{Q}_p)$ is regular semi-simple minuscule, such that $\mathrm{RZ}^g \neq \emptyset$. The following statements hold.*

(1) The formal scheme $\delta(\mathrm{RZ}^\flat) \cap \mathrm{RZ}^g$ over $\mathrm{Spf}\, W(k)$ is a k-scheme.

(2) The k-scheme $\delta(\mathrm{RZ}^\flat) \cap \mathrm{RZ}^g$ is non-empty if and only if there is a unique element $Q_0 \in \mathsf{SR}$ with $m_{Q_0}(f)$ odd. Moreover, when this is the case $p^{\mathbb{Z}}\backslash(\delta(\mathrm{RZ}^\flat) \cap \mathrm{RZ}^g)$ has finitely many k-points, and is in particular Artinian.

(3) Assume there is a unique element $Q_0 \in \mathsf{SR}$ with $m_{Q_0}(f)$ odd. Then the total k-length of $p^{\mathbb{Z}}\backslash(\delta(\mathrm{RZ}^\flat) \cap \mathrm{RZ}^g)$ is equal to

$$\deg(Q_0)\frac{m_{Q_0}(f) + 1}{2}\mathscr{M}(f).$$

Proof. Part (1) follows from [LZ18, Corollary 5.1.2], and part (2) is proved in [LZ18, Theorem 3.6.4].

For part (3), we first apply [LZ18] to identify $p^{\mathbb{Z}}\backslash(\delta(\mathrm{RZ}^\flat) \cap \mathrm{RZ}^g)$ with $S^{\bar{g}}$, the scheme theoretic fixed points of S under $\bar{g}$. Since $\bar{g}$ is in $\mathrm{SO}(\mathbb{V})(\mathbb{F}_p)$ (Lemma 5.2.2), it stabilizes S^+ and S^-. Hence $S^{\bar{g}} = (S^+)^{\bar{g}} \sqcup (S^-)^{\bar{g}}$. By the same arguments as in the proof of Theorem 5.1.2 (3), the k-length of $S^{\bar{g}}$ is equal to $\mathrm{tr}(\bar{g}, \mathbf{H}^*(S)) = \mathrm{tr}(\bar{g}, \mathbf{H}^*(S^+)) + \mathrm{tr}(\bar{g}, \mathbf{H}^*(S^-))$.

By Lemma 5.2.1 and by the fact that $\bar{g}$ is regular in $\mathrm{GL}(\mathbb{V})$, we know that $\mathrm{tr}(\bar{g}, \mathbf{H}^*(S^-))$ is given by the formula in Theorem 4.3.3 (3). Fix $g_0 \in \mathrm{O}(\mathbb{V})(\mathbb{F}_p) - \mathrm{SO}(\mathbb{V})(\mathbb{F}_p)$. Then under the natural action of $\mathrm{O}(\mathbb{V})(\mathbb{F}_q)$ on S, the element g_0 interchanges S^+ and S^-, by the proof of [HP17, Proposition 5.3.2]. Hence we have $\mathrm{tr}(\bar{g}, \mathbf{H}^*(S^+)) = \mathrm{tr}(g_0\bar{g}g_0^{-1}, \mathbf{H}^*(S^-))$. Since the formula in Theorem 4.3.3 (3) only depends on the characteristic polynomial, and since $\bar{g}$ and $g_0\bar{g}g_0^{-1}$ are elements of $\mathrm{SO}(\mathbb{V})(\mathbb{F}_p)$ which are both regular in $\mathrm{GL}(\mathbb{V})$ and have the same characteristic polynomial, we have $\mathrm{tr}(\bar{g}, \mathbf{H}^*(S^+)) = \mathrm{tr}(\bar{g}, \mathbf{H}^*(S^-))$. It follows that $\mathrm{tr}(\bar{g}, \mathbf{H}^*(S))$ is equal to twice the formula in Theorem 4.3.3 (3). The proof of part (3) is finished. □

Remark 5.2.5. Theorem 5.2.4 (3) was previously proved in [LZ18], under the assumption that $p > (m_{Q_0} + 1)/2$. This assumption is removed in Theorem 5.2.4. On the other hand, under the same assumption on p the paper [LZ18] determines each local ring of $\delta(\mathrm{RZ}^\flat) \cap \mathrm{RZ}^g$. This is a result not revealed by the methods of the current paper.

Remark 5.2.6. We correct two mistakes in [LZ17] and [LZ18]. Firstly, in both the papers the definition of the reciprocal of a polynomial should be normalized so as to be monic, as in §4.1. This mistake does not affect the correctness of any of the proofs. Secondly, in [LZ18, Theorem A (2), Theorem 3.6.4], the product should be over pairs of non-self-reciprocal irreducible monic factors, as in Theorem 5.2.4 and Definition 4.1.5, as opposed to over single non-self-reciprocal irreducible monic factors. To correct the proof of [LZ18, Theorem 3.6.4], one interprets the symbol $\prod_{R(T)\neq R^*(T)}$ in the proof as the product over such pairs $\{R(T), R^*(T)\}$ rather than over such $R(T)$'s.

References

[Bou68] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968.

[Car93] R. W. Carter, *Finite groups of Lie type*, Wiley Classics Library, John Wiley & Sons, Ltd., Chichester, 1993, Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication.

[Cho18] S. Cho, *The basic locus of the unitary Shimura variety with parahoric level structure, and special cycles*, ArXiv e-prints (July 2018), 1807.09997.

[DL76] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) **103**(1), 103–161 (1976).

[DM91] F. Digne and J. Michel, *Representations of finite groups of Lie type*, volume 21 of *London Mathematical Society Student Texts*, Cambridge University Press, Cambridge, 1991.

[GD77] A. Grothendieck and P. Deligne, La classe de cohomologie associée à un cycle, in *Cohomologie étale*, volume 569 of *Lecture Notes in Math.*, pages 129–153, Springer, Berlin, 1977.

[GGP12] W. T. Gan, B. H. Gross and D. Prasad, *Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups*, Astérisque (346), 1–109 (2012), Sur les conjectures de Gross et Prasad. I.

[GH15] U. Görtz and X. He, *Basic loci of Coxeter type in Shimura varieties*, Camb. J. Math. **3**(3), 323–353 (2015).

[He07a] X. He, *The G-stable pieces of the wonderful compactification*, Trans. Amer. Math. Soc. **359**(7), 3005–3024 (2007).

[He07b] X. He, *Minimal length elements in some double cosets of Coxeter groups*, Adv. Math. **215**(2), 469–503 (2007).

[He09] X. He, G-stable pieces and partial flag varieties, in *Representation theory*, volume 478 of *Contemp. Math.*, pages 61–70, Amer. Math. Soc., Providence, RI, 2009.

[HP14] B. Howard and G. Pappas, *On the supersingular locus of the* $\mathrm{GU}(2,2)$ *Shimura variety*, Algebra Number Theory **8**(7), 1659–1699 (2014).

[HP17] B. Howard and G. Pappas, *Rapoport–Zink spaces for spinor groups*, Compos. Math. **153**(5), 1050–1118 (2017).

[Kit93] Y. Kitaoka, *Arithmetic of quadratic forms*, volume 106 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, 1993.

[Kot82] R. E. Kottwitz, *Rational conjugacy classes in reductive groups*, Duke Math. J. **49**(4), 785–806 (1982).

[KR11] S. Kudla and M. Rapoport, *Special cycles on unitary Shimura varieties I. Unramified local theory*, Invent. Math. **184**(3), 629–682 (2011).

[Liu18] Y. Liu, Fourier-Jacobi cycles and arithmetic relative trace formula, `https://gauss.math.yale.edu/~yl2269/FJcycle.pdf`, August 2018.

[Lus07] G. Lusztig, *A class of perverse sheaves on a partial flag manifold*, Represent. Theory **11**, 122–171 (2007).

[LZ17] C. Li and Y. Zhu, *Remarks on the arithmetic fundamental lemma*, Algebra Number Theory **11**(10), 2425–2445 (2017).

[LZ18] C. Li and Y. Zhu, *Arithmetic intersection on GSpin Rapoport-Zink spaces*, Compos. Math. **154**(7), 1407–1440 (2018).

[Mih16] A. Mihatsch, *Relative unitary RZ-spaces and the Arithmetic Fundamental Lemma*, arXiv e-prints , arXiv:1611.06520 (November 2016), 1611.06520.

[PR94] V. Platonov and A. Rapinchuk, *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*, Academic Press, Inc., Boston, MA, 1994, Translated from the 1991 Russian original by Rachel Rowen.

[RSZ17a] M. Rapoport, B. Smithling and W. Zhang, *Arithmetic diagonal cycles on unitary Shimura varieties*, ArXiv e-prints (October 2017), 1710.06962.

[RSZ17b] M. Rapoport, B. Smithling and W. Zhang, *On the arithmetic transfer conjecture for exotic smooth formal moduli spaces*, Duke Math. J. **166**(12), 2183–2336 (2017).

[RTZ13] M. Rapoport, U. Terstiege and W. Zhang, *On the arithmetic fundamental lemma in the minuscule case*, Compos. Math. **149**(10), 1631–1666 (2013).

[Ser00] J.-P. Serre, *Local algebra*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000, Translated from the French by CheeWhye Chin and revised by the author.

[Spr74] T. A. Springer, *Regular elements of finite reflection groups*, Invent. Math. **25**, 159–198 (1974).

[Ste65] R. Steinberg, *Regular elements of semisimple algebraic groups*, Inst. Hautes Études Sci. Publ. Math. (25), 49–80 (1965).

[Ste68] R. Steinberg, *Endomorphisms of linear algebraic groups*, Memoirs of the American Mathematical Society, No. 80, American Mathematical Society, Providence, R.I., 1968.

[Vol10] I. Vollaard, *The supersingular locus of the Shimura variety for* $\mathrm{GU}(1,s)$, Canad. J. Math. **62**(3), 668–720 (2010).

[VW11] I. Vollaard and T. Wedhorn, *The supersingular locus of the Shimura variety of* $\mathrm{GU}(1,n-1)$ *II*, Invent. Math. **184**(3), 591–627 (2011).

[Zha12] W. Zhang, *On arithmetic fundamental lemmas*, Invent. Math. **188**(1), 197–252 (2012).

[Zha19] W. Zhang, *Weil representation and arithmetic fundamental lemma*, preprint (2019).

E-mail address: `xuhuahe@math.umd.edu`

Mathematics Building, University of Maryland, College Park, MD 20742-4015

E-mail address: `chaoli@math.columbia.edu`

Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027

E-mail address: `yihang@math.columbia.edu`

Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027