A NOTE ON TATE'S CONJECTURES FOR ABELIAN VARIETIES

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ABSTRACT. In this mostly expository note, we explain a proof of Tate's two conjectures [Tat65] for algebraic cycles of arbitrary codimension on certain products of elliptic curves and abelian surfaces over number fields.

1. Statement. Let X be a smooth projective variety over a finitely generated field F. Let $\mathrm{Ch}^r(X)$ be the Chow group of codimension r algebraic cycles of X defined over F modulo rational equivalence. Let $\overline{F}$ be a separable algebraic closure of F and $\Gamma_F := \mathrm{Gal}(\overline{F}/F)$. Tate [Tat65, Conjecture 1] made the following far-reaching conjecture (often known as *the Tate conjecture*), relating algebraic cycles and Γ_F-invariants of the ℓ-adic cohomology of X.

Conjecture 1.1 (Tate I)**.** *For any $1 \leq r \leq \dim X$ and for any prime $\ell \neq \mathrm{char}(F)$, the ℓ-adic cycle class map*

$$\mathrm{Ch}^r(X) \otimes \mathbb{Q}_\ell \to \mathrm{H}^{2r}(X_{\overline{F}}, \mathbb{Q}_\ell(r))^{\Gamma_F}$$

is surjective.

Let $\mathrm{Ch}^r_{\mathrm{hom}}(X)$ be the quotient group of $\mathrm{Ch}^r(X)$ modulo ℓ-adic homological equivalence. It is further conjectured (and known when $\mathrm{char}(F) = 0$) that $\mathrm{Ch}^r_{\mathrm{hom}}(X)$ is independent of ℓ, and the ℓ-adic cycle class map is injective on $\mathrm{Ch}^r_{\mathrm{hom}}(X) \otimes \mathbb{Q}_\ell$ (see [Tat65, p.97]). In particular, when $\mathrm{char}(F) = 0$, Tate I implies an isomorphism $\mathrm{Ch}^r_{\mathrm{hom}}(X) \otimes \mathbb{Q}_\ell \simeq \mathrm{H}^{2r}(X_{\overline{F}}, \mathbb{Q}_\ell(r))^{\Gamma_F}$ and thus

$$\mathrm{rank}\, \mathrm{Ch}^r_{\mathrm{hom}}(X) = \dim \mathrm{H}^{2r}(X_{\overline{F}}, \mathbb{Q}_\ell(r))^{\Gamma_F} \tag{1.1.1}$$

for any prime ℓ.

Tate [Tat65, Conjecture 2] further made a conjecture relating algebraic cycles to poles of zeta functions (often known as *the strong Tate conjecture*). When F is a number field, we denote by $L(\mathrm{H}^{2r}(X)(r), s)$ the (incomplete) L-function associated to the compatible system $\{\mathrm{H}^{2r}(X_{\overline{F}}, \overline{\mathbb{Q}}_\ell(r))\}$ of Γ_F-representations, which converges absolutely for $\mathrm{Re}(s) > 1$. Then [Tat65, Conjecture 2] specializes to the following.

Conjecture 1.2 (Tate II)**.** *Assume that F is a number field. Then for any $1 \leq r \leq \dim X$,*

$$\mathrm{rank}\, \mathrm{Ch}^r_{\mathrm{hom}}(X) = -\mathrm{ord}_{s=1} L(\mathrm{H}^{2r}(X)(r), s).$$

Tate I for divisors ($r = 1$) is known for various X, including abelian varieties over any finitely generated fields ([Fal83, Zar75, Tat66]). Much less is known when $r > 1$. We refer to the surveys [Tot17, Mil07, Tat94, Ram89] for a nice summary of known results. The goal of this short note

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is to present some examples of abelian varieties X over number fields for which Tate's conjectures hold for algebraic cycles in *arbitrary* codimension r.

Theorem 1.3 (Tate I)**.** *Assume that F is finitely generated with* $\mathrm{char}(F) = 0$*. Then Tate I holds for any abelian variety X over F with simple factors all having dimension ≤ 2.*

Theorem 1.4 (Tate II)**.** *Assume that F is a number field. Let E_1, E_2, E_3, E_4 be elliptic curves over F. Let A be an abelian surface over F. Then Tate II holds for the following cases:*
(i) F is totally real or imaginary CM and $X = E_1^{n_1} \times E_2^{n_2}$ for any $n_1 \geq 1, n_2 \geq 0$,
(ii) F is totally real or imaginary CM and $X = E_1^{n_1} \times E_2^{n_2} \times E_3$ for any $n_1 \geq 1$ and $1 \leq n_2 \leq 2$.
(iii) F is totally real or imaginary CM and $X = E_1^{n_1} \times E_2^{n_2} \times E_3 \times E_4$ for any $1 \leq n_1, n_2 \leq 2$.
(iv) F is totally real and $X = A$, $X = A^2$.

Remark 1.5. It is worth mentioning that the special case when $X = E^n$ is a power of an elliptic curve was considered by Tate himself [Tat65, p.106], and played an important role in his formulation of the Sato–Tate conjecture.

Theorem 1.3 (Tate I) can be deduced from recent theorems on the Hodge conjecture and the Mumford–Tate conjecture ([RM08, Lom16]), as mentioned e.g. in [Moo17, p.284]. Theorem 1.4 (Tate II) can be deduced from more recent potential automorphy theorems ([ACC$^+$18, BCGP21]) and known cases of Langlands functionality, and should also be known to the experts. All these ingredients are available in more generality, but to illustrate the ideas we do not aim for maximal generality in the statement of the theorems.

2. Proof of Theorem 1.3 (Tate I). Choose an embedding $F \hookrightarrow \mathbb{C}$ and view F as a subfield of $\mathbb{C}$. Since all simple factors of X have dimension ≤ 2, the Hodge conjecture for $X_{\mathbb{C}}$ holds (in any codimension r) by [RM08, Theorem 3.15]. In fact in this case all Hodge classes on $X_{\mathbb{C}}$ are generated by products of divisor classes. Also by [Lom16, Corollary 1.2], the Mumford–Tate conjecture for X holds.

Now the desired result follows due to the well-known general fact (see e.g. [Can16, §6]) that the Mumford–Tate conjecture for the abelian variety X over F together with the Hodge conjecture for $X_{\mathbb{C}}$ (in codimension r) implies Tate I (Conjecture 1.1) for X (in codimension r). In particular all Tate classes on X are also generated by products of divisor classes.

Remark 2.1. We refer to the references in [RM08], [Lom16] for related previous works on the Hodge and Mumford–Tate conjectures. When X is a product of elliptic curves, the Hodge conjecture was proved in [Mur90] (see also [Lew99, Appendix B, §3]) and the same method should also apply to prove Tate I.

3. Potential automorphy. Let F be a number field. Let $V = \{V_\ell\}$ and $W = \{W_\ell\}$ be compatible systems of semisimple ℓ-adic Γ_F-representations (e.g., in the sense of strictly compatible systems of ℓ-adic representations of Γ_F defined over $\mathbb{Q}$ of [BCGP21, §2.8]). Recall that V is *potentially automorphic* if there exists a finite Galois extension L/F such that the restriction $V|_{\Gamma_L}$ is automorphic (e.g., in the sense of [BCGP21, Definition 9.1.1]). We introduce the following variants of potential automorphy.

Definition 3.1. Let S be a nonempty set of rational primes. Let L/F be a finite Galois extension.

We say that V is *S-strongly automorphic over L*, if for any subextension L'/F of L/F with L/L' solvable, the following conditions are satisfied:

(i) $V|_{\Gamma_{L'}}$ is automorphic.

(ii) Let π be an isobaric automorphic representation on $\mathrm{GL}_n(\mathbb{A}_{L'})$ associated to $V|_{\Gamma_{L'}}$ ($n = \dim V$ and $\mathbb{A}_{L'}$ is the ring of adèles of L'). Write $\pi = \boxplus_{i=1}^k \pi_i$ as an isobaric direct sum of cuspidal automorphic representations on $\mathrm{GL}_{n_i}(\mathbb{A}_{L'})$ ($n = \sum_{i=1}^k n_i$). Write $V|_{\Gamma_{L'}} = \oplus_{i=1}^k V_i$ as the corresponding direct sum decomposition into compatible systems of $\Gamma_{L'}$-representations. Then the ℓ-adic $\Gamma_{L'}$-representation $V_{i,\ell}$ ($i = 1, \dots k$) is irreducible for any $\ell \in S$. (Notice that the irreducibility of $V_{i,\ell}$ is conjectured but not known in general).

We say that V is *S-strongly potentially automorphic*, if V is S-strongly automorphic over L for some finite Galois extension L/F. We say that V is *strongly potentially automorphic*, if V is S-strongly potentially automorphic for some Dirichlet density one set S.

We say that V and W are *jointly S-strongly potentially automorphic*, if V and W are both S-strongly automorphic over L for some finite Galois extension L/F. We say that V and W are *jointly strongly potentially automorphic*, if V and W are jointly S-strongly potentially automorphic for some Dirichlet density one set S.

Lemma 3.2. *Let $V = \{V_\ell\}$ and $W = \{W_\ell\}$ be compatible systems of semisimple ℓ-adic Γ_F-representations. Let S be a nonempty set of rational primes.*

(i) Assume that V is S-strongly potentially automorphic. Then $L(V, s)$ has meromorphic continuation to all of $\mathbb{C}$, and for any $\ell \in S$,

$$\dim V_\ell^{\Gamma_F} = -\operatorname{ord}_{s=1} L(V, s).$$

(ii) Assume that V and W are jointly S-strongly potentially automorphic. Then $L(V \otimes W, s)$ has meromorphic continuation to all of $\mathbb{C}$, and for any $\ell \in S$,

$$\dim(V_\ell \otimes W_\ell)^{\Gamma_F} = -\operatorname{ord}_{s=1} L(V \otimes W, s).$$

(iii) Assume that V has a finite direct sum decomposition $V \simeq \oplus_{i=1}^k V_i \otimes W_i$ into tensor products of compatible systems of Γ_F-representations. Assume that V_i and W_i are jointly S-strongly potentially automorphic for each i. Then $L(V, s)$ has meromorphic continuation to all of $\mathbb{C}$, and for any $\ell \in S$,

$$\dim V_\ell^{\Gamma_F} = -\operatorname{ord}_{s=1} L(V, s).$$

Remark 3.3. Lemma 3.2 should be known to the experts and the proof idea, using Brauer's induction theorem and known properties of automorphic L-functions, is an old one (see e.g. [Tay02, HSBT10, Har09]). Notice that Item (i) also follows as a special case of Item (iii). We keep Item (i) to illustrate the ideas.

Proof. (i) Let L/F be a finite Galois extension such that V is S-strongly automorphic over L. By Brauer's induction theorem, we may find a virtual decomposition

$$\mathbf{1}_{\Gamma_F} = \sum_{j=1}^{k} c_j \mathrm{Ind}_{\Gamma_{L_j}}^{\Gamma_F} \psi_i,$$

where $c_j \in \mathbb{Z}$, $F \subseteq L_j \subseteq L$ with L/L_j solvable, and ψ_j is a 1-dimensional representation of $\mathrm{Gal}(L/L_j)$ ($j = 1, \ldots, k$). Since V is S-strongly automorphic over L, we know that for each j there exists an isobaric direct sum of cuspidal automorphic representations $\pi_{L_j} = \boxplus_{i=1}^{m_i} \pi_{L_j,i}$ of $\mathrm{GL}_n(\mathbb{A}_{L_j})$ and a direct sum decomposition $V|_{\Gamma_{L_j}} = \oplus_{i=1}^{m_j} V_{L_j,i}$ into Γ_{L_j}-representations such that

$$L(V|_{\Gamma_{L_j}}, s) = L(s, \pi_{L_j}) \quad L(V_{L_j,i}, s) = L(s, \pi_{L_j,i}),$$

and each ℓ-adic representation $V_{L_j,i,\ell}$ is irreducible for any $\ell \in S$. Here $L(s, \pi_{L_j})$ is the (incomplete) standard L-function as in [GJ72] and has meromorphic continuation to all of $\mathbb{C}$. Hence

$$L(V \otimes \mathrm{Ind}_{\Gamma_{L_j}}^{\Gamma_F} \psi_j, s) = L(V|_{\Gamma_{L_j}} \otimes \psi_j, s) = \prod_{i=1}^{m_j} L(V_{L_j,i} \otimes \psi_j, s) = \prod_{i=1}^{m_j} L(s, \pi_{L_j,i} \otimes \chi_j),$$

where χ_j is the automorphic character on $\mathrm{GL}_1(\mathbb{A}_{L_j})$ associated to ψ_j. It follows that

$$L(V, s) = L(V \otimes \mathbf{1}_{\Gamma_F}, s) = \prod_{j=1}^{k} \prod_{i=1}^{m_j} L(s, \pi_{L_j,i} \otimes \chi_j)^{c_j}$$

and thus $L(V, s)$ has meromorphic continuation to all of $\mathbb{C}$.

Since $\pi_{L_j,i} \otimes \chi_j$ is cuspidal, by [JS77] we know that $L(s, \pi_{L_j,i} \otimes \chi_j)$ has no zero or pole at $s = 1$, unless $\pi_{L_j,i} \otimes \chi_j$ is the trivial representation in which case it has a simple pole at $s = 1$. Hence $-\operatorname{ord}_{s=1} L(V, s)$ equals the number of trivial representations among $\pi_{L_j,i} \otimes \chi_j$ weighted by c_j, and so we obtain

$$-\operatorname{ord}_{s=1} L(V, s) = \sum_{j=1}^{k} \sum_{i=1}^{m_j} c_j \dim \mathrm{Hom}_{\Gamma_{L_j}}(\mathbf{1}_{\Gamma_{L_j}}, V_{L_j,i,\ell} \otimes \psi_{j,\ell}),$$

for any $\ell \in S$ by the irreducibility of $V_{L_j,i,\ell}$. This evaluates to

$$\sum_{j=1}^{k} c_j \dim \mathrm{Hom}_{\Gamma_{L_j}}(\mathbf{1}_{\Gamma_{L_j}}, V_\ell|_{\Gamma_{L_j}} \otimes \psi_{j,\ell}),$$

which by the Frobenius reciprocity equals

$$\dim \mathrm{Hom}_{\Gamma_F}(\mathbf{1}_{\Gamma_F}, V_\ell) = \dim V_\ell^{\Gamma_F}.$$

(ii) Let L/F be a finite Galois extension such that both V and W are S-strongly automorphic over L. By the same notation and argument in the proof of Item (i), we know that for each j there exists an isobaric direct sum of cuspidal representations $\pi_{L_j} = \boxplus_{i=1}^{m_j} \pi_{L_j,i}$ (resp. $\Pi_{L_j} = \boxplus_{i'=1}^{m'_j} \Pi_{L_j,i'}$), together with a corresponding decomposition into Γ_{L_j}-representations $V|_{\Gamma_{L_j}} \simeq \oplus_{i=1}^{m_j} V_{L_j,i}$ (resp. $W|_{\Gamma_{L_j}} \simeq \oplus_{i'=1}^{m'_j} W_{L_j,i'}$) such that each ℓ-adic representation $V_{L_j,i,\ell}$ (resp. $W_{L_j,i',\ell}$) is irreducible for any $\ell \in S$. It follows that

$$L(V \otimes W, s) = \prod_{j=1}^{k} L(V \otimes W \otimes \mathbf{1}_{\Gamma_F}, s) = \prod_{j=1}^{k} \prod_{i=1}^{m_j} \prod_{i'=1}^{m'_j} L(s, \pi_{L_j,i} \times (\Pi_{L_j,i'} \otimes \chi_j))^{c_j},$$

where $L(s, \pi_{L_j,i} \times (\Pi_{L_j,i'} \otimes \chi_j))$ is the (incomplete) Rankin–Selberg L-function as in [JPSS83], and thus $L(V \otimes W, s)$ has meromorphic continuation to all of $\mathbb{C}$.

Since $\pi_{L_j,i}$ and $\Pi_{L_j,i}\otimes\chi_j$ are cuspidal, we know that $L(s,\pi_{L_j,i}\times(\Pi_{L_j,i}\otimes\chi_j))$ has no zero at $s=1$ by [Sha80] (see also [Mor85, Lemma 3.1], [Sar04, p. 721]). Also by [JS81, (4.6) and (4.11)] (see also [MW89, Appendice], [CPS04, Theorem 2.4]), it has no pole at $s=1$, unless $\pi_{L_j,i}\simeq(\Pi_{L_j,i'}\otimes\chi_j)^\vee$ in which case it has a simple pole at $s=1$. The latter happens if and only if $V_{L_j,i}\simeq(W_{L_j,i'}\otimes\psi_j)^\vee$. Hence

$$-\operatorname{ord}_{s=1}L(V,s)=\sum_{j=1}^{k}\sum_{i=1}^{m_j}\sum_{i'=1}^{m'_j}c_j\dim\operatorname{Hom}_{\Gamma_{L_j}}(\mathbf{1}_{\Gamma_{L_j}},V_{L_j,i,\ell}\otimes W_{L_j,i',\ell}\otimes\psi_{j,\ell})$$

for any $\ell\in S$ by the irreducibility of $V_{L_j,i,\ell}$ and $W_{L_j,i',\ell}$. This evaluates to

$$\sum_{j=1}^{k}c_j\dim\operatorname{Hom}_{\Gamma_{L_j}}(\mathbf{1}_{\Gamma_{L_j}},(V_\ell\otimes W_\ell)|_{\Gamma_{L_j}}\otimes\psi_{j,\ell}),$$

which by the Frobenius reciprocity equals

$$\dim\operatorname{Hom}_{\Gamma_F}(\mathbf{1}_{\Gamma_F},V_\ell\otimes W_\ell)=\dim(V_\ell\otimes W_\ell)^{\Gamma_F}.$$

(iii) It follows directly from Item (ii) and the factorization $L(V,s)=\prod_{i=1}^k L(V_i\otimes W_i,s)$. $\square$

Lemma 3.4. *Assume that F is a number field. Let E_1,E_2,E_3,E_4 be elliptic curves over F. Let A be an abelian surface over F.*

(i) If F is totally real or imaginary CM, then $\{\operatorname{Sym}^{k_1}\mathrm{H}^1(E_{1,\overline{F}},\overline{\mathbb{Q}}_\ell)\}$ and $\{\operatorname{Sym}^{k_2}\mathrm{H}^1(E_{2,\overline{F}},\overline{\mathbb{Q}}_\ell)\}$ are jointly strongly potentially automorphic for any $k_1,k_2\geq 0$.

(ii) If F is totally real or imaginary CM, then $\{\operatorname{Sym}^{k_1}\mathrm{H}^1(E_{1,\overline{F}},\overline{\mathbb{Q}}_\ell)\}$ and $\{\operatorname{Sym}^{k_2}\mathrm{H}^1(E_{2,\overline{F}},\overline{\mathbb{Q}}_\ell)\otimes\operatorname{Sym}^{k_3}\mathrm{H}^1(E_{3,\overline{F}},\overline{\mathbb{Q}}_\ell)\}$ are jointly strongly potentially automorphic for any $k_1\geq 0$, $0\leq k_2\leq 2$, and $0\leq k_3\leq 1$.

(iii) If F is totally real or imaginary CM, then $\{\operatorname{Sym}^{k_1}\mathrm{H}^1(E_{1,\overline{F}},\overline{\mathbb{Q}}_\ell)\otimes\operatorname{Sym}^{k_3}\mathrm{H}^1(E_{3,\overline{F}},\overline{\mathbb{Q}}_\ell)\}$ and $\{\operatorname{Sym}^{k_2}\mathrm{H}^1(E_{2,\overline{F}},\overline{\mathbb{Q}}_\ell)\otimes\operatorname{Sym}^{k_4}\mathrm{H}^1(E_{4,\overline{F}},\overline{\mathbb{Q}}_\ell)\}$ are jointly strongly potentially automorphic for any $0\leq k_1,k_2\leq 2$ and $0\leq k_3,k_4\leq 1$.

(iv) If F is totally real, then $\{\mathrm{H}^{k_1}(A_{\overline{F}},\overline{\mathbb{Q}}_\ell)\}$ and $\{\mathrm{H}^{k_2}(A_{\overline{F}},\overline{\mathbb{Q}}_\ell)\}$ are jointly strongly potentially automorphic for any $0\leq k_1,k_2\leq 4$.

Proof. (i) If one of E_1 or E_2 has CM, say E_1 has CM, then $\{\operatorname{Sym}^{k_1}\mathrm{H}^1(E_{1,\overline{F}},\overline{\mathbb{Q}}_\ell)\}$ is automorphic, as an isobaric direct sum of automorphic characters on $\mathrm{GL}_1(\mathbb{A}_F)$, and possibly automorphic inductions of automorphic characters on $\mathrm{GL}_1(\mathbb{A}_K)$ for a quadratic extension K/F. In particular, we know that $\{\operatorname{Sym}^{k_1}\mathrm{H}^1(E_{1,\overline{F}},\overline{\mathbb{Q}}_\ell)\}|_{\Gamma_L}$ is S-strongly automorphic over any finite Galois extension L/F and any nonempty set S of primes. The result follows if E_2 also has CM. If E_2 has no CM, then $\{\mathrm{H}^1(E_{2,\overline{F}},\overline{\mathbb{Q}}_\ell)\}$ is strongly irreducible in the sense defined before [ACC+18, Lemma 7.1.1] (i.e., for any finite extension F'/F, the representation $\mathrm{H}^1(E_{2,\overline{F}},\overline{\mathbb{Q}}_\ell)|_{\Gamma_{F'}}$ is irreducible for ℓ in a Dirichlet density one set of primes), and we can apply [ACC+18, Corollary 7.1.11] to $\{\operatorname{Sym}^{k_2}\mathrm{H}^1(E_{2,\overline{F}},\overline{\mathbb{Q}}_\ell)\}$ together with [ACC+18, Proposition 6.5.13] to obtain the desired joint S-strong potential automorphy for a Dirichlet density one set S of primes. If neither of E_1 and E_2 has CM, then the desired result follows from the more general [ACC+18, Theorem 7.1.10] together with [ACC+18, Proposition 6.5.13]. (In the case $F = \mathbb{Q}$, we may also directly apply [NT20, Theorem A (non-CM case) and Theorem A.1 (CM case)]).

(ii) By the same argument in Item (i), there are a finite Galois extension L/F and a Dirichlet density one set S of primes such that $\{\mathrm{Sym}^{k_i} \mathrm{H}^1(E_{i,\overline{F}}, \overline{\mathbb{Q}}_\ell)\}$ is S-strongly automorphic over L for any $1 \le i \le 3$. Hence by the functorial products for $\mathrm{GL}(2) \times \mathrm{GL}(2) \to \mathrm{GL}(4)$ ([Ram00, Theorem M]) and $\mathrm{GL}(2) \times \mathrm{GL}(3) \to \mathrm{GL}(6)$ ([KS02, Theorem A]), we know that $\{\mathrm{Sym}^{k_2} \mathrm{H}^1(E_{2,\overline{F}}, \overline{\mathbb{Q}}_\ell) \otimes \mathrm{Sym}^{k_3} \mathrm{H}^1(E_{3,\overline{F}}, \overline{\mathbb{Q}}_\ell)\}$ is also S-strongly automorphic over L for any $0 \le k_2 \le 2$ and $0 \le k_3 \le 1$. The result then follows.

(iii) By the same argument in Item (ii), there are a finite Galois extension L/F and a Dirichlet density one set S of primes such that $\{\mathrm{Sym}^{k_i} \mathrm{H}^1(E_{i,\overline{F}}, \overline{\mathbb{Q}}_\ell) \otimes \mathrm{Sym}^{k_j} \mathrm{H}^1(E_{j,\overline{F}}, \overline{\mathbb{Q}}_\ell)\}$ is S-strongly automorphic over L for any $0 \le k_i \le 2$ and $0 \le k_j \le 1$, which gives the result.

(iv) The result follows from [BCGP21, Theorem 9.3.1] and its proof. □

Remark 3.5. For each item of Lemma 3.4, the proof supplies a Dirichlet density one set S of primes such that the joint S-strong potential automorphy holds. Since compatible systems in Lemma 3.4 come from elliptic curves and abelian surfaces, one should also be able to prove directly that the irreducible conditions required in Definition 3.1 (ii) hold for all primes ℓ, and hence the joint S-strong potential automorphy holds for the set S of all primes. For the purpose of the proof of Theorem 1.4 (Tate II) below, any nonempty S suffices.

4. Proof of Theorem 1.4 (Tate II). Let $1 \le r \le \dim X$. Let $V = \{\mathrm{H}^{2r}(X_{\overline{F}}, \overline{\mathbb{Q}}_\ell(r))\}$. By Theorem 1.3 (Tate I), we know from (1.1.1) that $\operatorname{rank} \mathrm{Ch}^r_{\mathrm{hom}}(X) = \dim V_\ell^{\Gamma_F}$ for any prime ℓ. Thus it remains to show that $\dim V_\ell^{\Gamma_F} = -\operatorname{ord}_{s=1} L(V, s)$ for some prime ℓ.

(i) By the Künneth formula and the decomposition of $\mathrm{H}^1(E_{i,\overline{F}}, \overline{\mathbb{Q}}_\ell)^{\otimes k_i}$ into symmetric powers of $\mathrm{H}^1(E_{i,\overline{F}}, \overline{\mathbb{Q}}_\ell)$ $(i = 1, 2)$, we have an isomorphism of semisimple Γ_F-representations

$$\mathrm{H}^{2r}(X_{\overline{F}}, \overline{\mathbb{Q}}_\ell(r)) \simeq \bigoplus_{\substack{0 \le k_i \le n_i \\ i=1,2}} m_{k_1,k_2} \left(\mathrm{Sym}^{k_1} \mathrm{H}^1(E_{1,\overline{F}}, \overline{\mathbb{Q}}_\ell) \otimes \mathrm{Sym}^{k_2} \mathrm{H}^1(E_{2,\overline{F}}, \overline{\mathbb{Q}}_\ell)\right)\left(\tfrac{k_1+k_2}{2}\right),$$

where $m_{k_1,k_2} \ge 0$ are certain multiplicities (nonzero only if $k_1 + k_2 \le 2r$ is even). The result then follows from Lemma 3.2 (iii) and Lemma 3.4 (i).

(ii) Similarly, set $n_3 = 1$ then we have an isomorphism of semisimple Γ_F-representations

$$\mathrm{H}^{2r}(X_{\overline{F}}, \overline{\mathbb{Q}}_\ell(r)) \simeq \bigoplus_{\substack{0 \le k_i \le n_i \\ 1 \le i \le 3}} m_{k_1,k_2,k_3} \left(\otimes_{1 \le i \le 3} \mathrm{Sym}^{k_i} \mathrm{H}^1(E_{i,\overline{F}}, \overline{\mathbb{Q}}_\ell)\right)\left(\tfrac{k_1+k_2+k_3}{2}\right),$$

where $m_{k_1,k_2,k_3} \ge 0$ are certain multiplicities (nonzero only if $k_1 + k_2 + k_3 \le 2r$ is even). The result then follows from Lemma 3.2 (iii) and Lemma 3.4 (ii).

(iii) Similarly, the result follows from Lemma 3.2 (iii) and Lemma 3.4 (iii).

(iv) For $X = A$, the result follows from Lemma 3.2 (i) and Lemma 3.4 (iv). For $X = A^2$, by the Künneth formula, we have an isomorphism of semisimple Γ_F-representations

$$\mathrm{H}^{2r}(X_{\overline{F}}, \overline{\mathbb{Q}}_\ell(r)) \simeq \bigoplus_{\substack{k_1+k_2=2r \\ 0 \le k_1,k_2 \le 4}} (\mathrm{H}^{k_1}(A_{\overline{F}}, \overline{\mathbb{Q}}_\ell) \otimes \mathrm{H}^{k_2}(A_{\overline{F}}, \overline{\mathbb{Q}}_\ell))(r).$$

The result then follows from Lemma 3.2 (iii) and Lemma 3.4 (iv).

Remark 4.1. When X is an abelian surface of the type $\mathrm{Res}_{K/F} E$, where F is totally real, K/F is a quadratic CM extension and E is an elliptic curve over K, Tate II was proved in [Vir15] using a similar argument. We also refer to [Joh17, Tay20] for more detailed analysis for L-functions of abelian surfaces.

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References.

[ACC+18] Patrick B. Allen, Frank Calegari, Ana Caraiani, Toby Gee, David Helm, Bao V. Le Hung, James Newton, Peter Scholze, Richard Taylor, and Jack A. Thorne. Potential automorphy over CM fields. *arXiv e-prints*, page arXiv:1812.09999, December 2018. 2, 5, 6

[BCGP21] George Boxer, Frank Calegari, Toby Gee, and Vincent Pilloni. Abelian surfaces over totally real fields are potentially modular. *Publ. Math. Inst. Hautes Études Sci.*, 134:153–501, 2021. 2, 6

[Can16] Victoria Cantoral Farfán. A survey around the Hodge, Tate and Mumford-Tate conjectures for abelian varieties. *arXiv e-prints*, page arXiv:1602.08354, February 2016. 2

[CPS04] James W. Cogdell and Ilya I. Piatetski-Shapiro. Remarks on Rankin-Selberg convolutions. In *Contributions to automorphic forms, geometry, and number theory*, pages 255–278. Johns Hopkins Univ. Press, Baltimore, MD, 2004. 5

[Fal83] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. Math.*, 73(3):349–366, 1983. 1

[GJ72] Roger Godement and Hervé Jacquet. *Zeta functions of simple algebras*. Lecture Notes in Mathematics, Vol. 260. Springer-Verlag, Berlin-New York, 1972. 4

[Har09] Michael Harris. Potential automorphy of odd-dimensional symmetric powers of elliptic curves and applications. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, volume 270 of *Progr. Math.*, pages 1–21. Birkhäuser Boston, Boston, MA, 2009. 3

[HSBT10] Michael Harris, Nick Shepherd-Barron, and Richard Taylor. A family of Calabi-Yau varieties and potential automorphy. *Ann. of Math. (2)*, 171(2):779–813, 2010. 3

[Joh17] Christian Johansson. On the Sato-Tate conjecture for non-generic abelian surfaces. *Trans. Amer. Math. Soc.*, 369(9):6303–6325, 2017. With an appendix by Francesc Fité. 7

[JPSS83] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika. Rankin-Selberg convolutions. *Amer. J. Math.*, 105(2):367–464, 1983. 4

[JS81] H. Jacquet and J. A. Shalika. On Euler products and the classification of automorphic representations. I. *Amer. J. Math.*, 103(3):499–558, 1981. 5

[JS77] Hervé Jacquet and Joseph A. Shalika. A non-vanishing theorem for zeta functions of GL_n. *Invent. Math.*, 38(1):1–16, 1976/77. 4

[KS02] Henry H. Kim and Freydoon Shahidi. Functorial products for $\mathrm{GL}_2 \times \mathrm{GL}_3$ and the symmetric cube for GL_2. *Ann. of Math. (2)*, 155(3):837–893, 2002. With an appendix by Colin J. Bushnell and Guy Henniart. 6

[Lew99] James D. Lewis. *A survey of the Hodge conjecture*, volume 10 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, second edition, 1999. Appendix B by B. Brent Gordon. 2

[Lom16] Davide Lombardo. On the ℓ-adic Galois representations attached to nonsimple abelian varieties. *Ann. Inst. Fourier (Grenoble)*, 66(3):1217–1245, 2016. 2

[Mil07] James S. Milne. The Tate conjecture over finite fields (AIM talk). *arXiv e-prints*, page arXiv:0709.3040, September 2007. 1

[Moo17] Ben Moonen. Families of motives and the Mumford-Tate conjecture. *Milan J. Math.*, 85(2):257–307, 2017. 2

[Mor85] Carlos J. Moreno. Analytic proof of the strong multiplicity one theorem. *Amer. J. Math.*, 107(1):163–206, 1985. 5

[Mur90] V. Kumar Murty. Computing the Hodge group of an abelian variety. In *Séminaire de Théorie des Nombres, Paris 1988–1989*, volume 91 of *Progr. Math.*, pages 141–158. Birkhäuser Boston, Boston, MA, 1990. 2

[MW89] C. Mœglin and J.-L. Waldspurger. Le spectre résiduel de $\mathrm{GL}(n)$. *Ann. Sci. École Norm. Sup. (4)*, 22(4):605–674, 1989. 5

[NT20] James Newton and Jack A. Thorne. Symmetric power functoriality for holomorphic modular forms, II. *arXiv e-prints*, page arXiv:2009.07180, September 2020. To appear in Publ. Math. IHÉS. 6

[Ram89] Dinakar Ramakrishnan. Regulators, algebraic cycles, and values of L-functions. In *Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987)*, volume 83 of *Contemp. Math.*, pages 183–310. Amer. Math. Soc., Providence, RI, 1989. 1

[Ram00] Dinakar Ramakrishnan. Modularity of the Rankin-Selberg L-series, and multiplicity one for $\mathrm{SL}(2)$. *Ann. of Math. (2)*, 152(1):45–111, 2000. 6

[RM08] José J. Ramón Marí. On the Hodge conjecture for products of certain surfaces. *Collect. Math.*, 59(1):1–26, 2008. 2

[Sar04] Peter Sarnak. Nonvanishing of L-functions on $\Re(s) = 1$. In *Contributions to automorphic forms, geometry, and number theory*, pages 719–732. Johns Hopkins Univ. Press, Baltimore, MD, 2004. 5

[Sha80] Freydoon Shahidi. On nonvanishing of L-functions. *Bull. Amer. Math. Soc. (N.S.)*, 2(3):462–464, 1980. 5

[Tat65] John T. Tate. Algebraic cycles and poles of zeta functions. In *Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963)*, pages 93–110. Harper & Row, New York, 1965. 1, 2

[Tat66] John Tate. Endomorphisms of abelian varieties over finite fields. *Invent. Math.*, 2:134–144, 1966. 1

[Tat94] John Tate. Conjectures on algebraic cycles in l-adic cohomology. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 71–83. Amer. Math. Soc., Providence, RI, 1994. 1

[Tay02] Richard Taylor. Remarks on a conjecture of Fontaine and Mazur. *J. Inst. Math. Jussieu*, 1(1):125–143, 2002. 3

[Tay20] Noah Taylor. Sato-Tate distributions on Abelian surfaces. *Trans. Amer. Math. Soc.*, 373(5):3541–3559, 2020. 7

[Tot17] Burt Totaro. Recent progress on the Tate conjecture. *Bull. Amer. Math. Soc. (N.S.)*, 54(4):575–590, 2017. 1

[Vir15] Cristian Virdol. Tate conjecture for some abelian surfaces over totally real or CM number fields. *Funct. Approx. Comment. Math.*, 52(1):57–63, 2015. 7

[Zar75] Ju. G. Zarhin. Endomorphisms of Abelian varieties over fields of finite characteristic. *Izv. Akad. Nauk SSSR Ser. Mat.*, 39(2):272–277, 471, 1975. 1

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