# 2-SELMER GROUPS, 2-CLASS GROUPS AND RATIONAL POINTS ON ELLIPTIC CURVES

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Abstract. Let $E : y^2 = F(x)$ be an elliptic curve over $\mathbb{Q}$ defined by a monic irreducible integral cubic polynomial $F(x)$ with negative square-free discriminant $-D$. We determine its 2-Selmer rank in terms of the 2-rank of the class group of the cubic field $L = \mathbb{Q}[x]/F(x)$.

When the 2-rank of the class group of $L$ is at most 1 and the root number of $E$ is $-1$, the Birch and Swinnerton-Dyer conjecture predicts that $E(\mathbb{Q})$ should have rank 1. We construct a canonical point in $E(\mathbb{Q})$ using a new Heegner point construction. We naturally conjecture it to be of infinite order. We verify this conjecture explicitly for the case $D = 11$, and propose an approach towards the general case based on a mod 2 congruence between elliptic curves and Artin representations.

## Contents



## 1. Introduction

1.1. **$p$-Selmer rank one conjecture.** Given an elliptic curve $E$ defined over $\mathbb{Q}$, the rational points $E(\mathbb{Q})$ form a finitely generated abelian group by the Mordell–Weil theorem. It is a central question in number theory to understand the rank of $E(\mathbb{Q})$, known as the *algebraic rank*

$$r_{\mathrm{alg}}(E/\mathbb{Q}) := \mathrm{rk}\, E(\mathbb{Q}).$$

Let $p$ be a prime number. Recall that we have the *p-descent* exact sequence (see [Sil09, X.4])

$$0 \to E(\mathbb{Q})/pE(\mathbb{Q}) \to \mathrm{Sel}_p(E/\mathbb{Q}) \to Ш(E/\mathbb{Q})[p] \to 0, \tag{1.1}$$

where $\mathrm{Sel}_p(E/\mathbb{Q})$ is the *p-Selmer group* and $Ш(E/\mathbb{Q})$ is the *Tate–Shafarevich group.* We define the *p-Selmer rank* of $E/\mathbb{Q}$ to be

$$s_p(E/\mathbb{Q}) := \dim_{\mathbb{F}_p} \mathrm{Sel}_p(E/\mathbb{Q}) - \dim_{\mathbb{F}_p} E(\mathbb{Q})[p].$$

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It follows from the $p$-descent sequence (1.1) that

$$r_{\mathrm{alg}}(E/\mathbb{Q}) = s_p(E/\mathbb{Q}) - \dim_{\mathbb{F}_p} Ш(E/\mathbb{Q})[p].$$

Due to the Cassels–Tate pairing, the finiteness of the $p$-primary part $Ш(E/\mathbb{Q})[p^\infty]$ would imply that $Ш(E/\mathbb{Q})[p]$ has even $\mathbb{F}_p$-dimension, hence $s_p(E/\mathbb{Q})$ and $r_{\mathrm{alg}}(E/\mathbb{Q})$ have the same parity. In particular, the finiteness of $Ш(E/\mathbb{Q})[p^\infty]$ would imply the following conjecture, which we call the *p-Selmer rank one conjecture*.

**Conjecture 1.2.** *If $s_p(E/\mathbb{Q}) = 1$, then $r_{\mathrm{alg}}(E/\mathbb{Q}) = 1$.*

Conjecture 1.2, despite looking simple, was only recently proved for $p \geq 5$ under certain assumptions ([Zha14], [Ski14], [SZ14], [Wan14], [CW15], [Ven16]). These known cases of Conjecture 1.2 played a key role in the recent breakthrough of Bhargava–Skinner–Zhang ([BSZ14]) that a majority of elliptic curves over $\mathbb{Q}$ satisfy the rank part of the Birch and Swinnerton-Dyer conjecture.

On the other hand, very little about Conjecture 1.2 is known for $p = 2$, though the 2-Selmer group is the easiest to compute in practice and provides as of present the best tool for computing $E(\mathbb{Q})$. More recently there has also been growing interest in studying the 2-Selmer group and its variation in families (e.g., Klagsbrun–Mazur–Rubin [KMR13],[KMR14]), in view of its connection with Hilbert's tenth problem ([MR10]) and Goldfeld's conjecture ([KL16], [Smi17]).

The theme of this article is explore Conjecture 1.2 for a large class of elliptic curves. We now turn to our main results.

1.3. **2-Selmer groups and 2-class groups.** Let $F(x) \in \mathbb{Z}[x]$ be an irreducible monic cubic polynomial with negative and square-free discriminant $-D$. Let $E$ be given by the Weierstrass equation $y^2 = F(x)$. Our first main result is to determine the 2-Selmer rank $s_2(E/\mathbb{Q})$ in terms of the 2-part of the ideal class group of the cubic field $L = \mathbb{Q}(x)/F(x)$ and the global root number $\varepsilon(E/\mathbb{Q})$ of $E/\mathbb{Q}$.

**Theorem 1.4** (Theorem 2.18)**.** *Let $\mathrm{Cl}(L)$ be the ideal class group of the cubic field $L = \mathbb{Q}[x]/F(x)$. Let $k = \dim_{\mathbb{F}_2} \mathrm{Cl}(L)[2]$ be its 2-rank. Then*

$$s_2(E/\mathbb{Q}) = k \text{ or } k+1,$$

*depending on whether the root number $\varepsilon(E/\mathbb{Q}) = (-1)^k$ or $(-1)^{k+1}$.*

1.5. **Heegner points on elliptic curves of conductor $4D$.** Theorem 1.4 has the following immediate consequence (applied to $k = 0$ or 1, only the upper bound on $s_2(E/\mathbb{Q})$ is needed).

**Corollary 1.6.** *Assume $\varepsilon(E/\mathbb{Q}) = -1$. If the 2-rank of $\mathrm{Cl}(L)$ is at most 1, then $s_2(E/\mathbb{Q}) = 1$. In this case, Conjecture 1.2 implies that $r_{\mathrm{alg}}(E/\mathbb{Q}) = 1$.*

This consequence naturally raises two challenges:

(1) to construct a rational point $P \in E(\mathbb{Q})$ when $\varepsilon(E/\mathbb{Q}) = -1$,
(2) to verify the constructed point $P$ is of infinite order when the 2-rank of $\mathrm{Cl}(L)$ is at most 1.

In §3 we complete (1) under an additional assumption that $E$ has Kodaira type IV at 2. This assumption allows us to pin down the conductor of $E$ to be the minimal $N = 4D$, and interestingly also forces the root number $\varepsilon(E/\mathbb{Q}(i))$ to be $-1$. In this case $E$ admits a parametrization by a Shimura curve $X$ associated to the quaternion order of reduced discriminant $4D$,

$$\mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}ij, \quad i^2 = -1, j^2 = D, ij = -ji. \tag{1.2}$$

Our second main result is a construction of a canonical point $P \in E(\mathbb{Q}(i))$ (Theorem 3.23), using Heegner points on $X$ associated to the imaginary quadratic field $\mathbb{Q}(i)$ with the following desired property as in (1).

**Theorem 1.7** (Theorem 3.21)**.** *$P$ lies in $E(\mathbb{Q})$ if and only if $\varepsilon(E/\mathbb{Q}) = -1$.*

1.8. **A conjecture on the canonical point $P$.** In view of the Gross-Zagier formula ([GZ86], [YZZ13]), we naturally conjecture as in (2):

**Conjecture 1.9** (Conjecture 4.1)**.** *Assume $\varepsilon(E/\mathbb{Q}) = -1$. If the 2-rank of $\mathrm{Cl}(L)$ is at most 1, then $P \in E(\mathbb{Q})$ is of infinite order.*

Conjecture 1.9 seems quite difficult to us. The known general approach of showing that a Heegner point on $E$ is of infinite order is usually by showing the $L$-function $L(E, s)$ vanishes to order 1 at $s = 1$, either via explicit forms of the Gross–Zagier formula, or via its $p$-adic variants (e.g., the BDP formula [BDP13], [LZZ14]) and Iwasawa theory. However, the input of Conjecture 1.9 is purely algebraic — in terms of the 2-class group of cubic fields, and a more direct link with Heegner points or $L$-functions seems missing at the moment.

Nevertheless, we provide a piece of evidence of Conjecture 1.9 by verifying it for the case $D = 11$ in Example 4.2. We end this article by proposing a potential approach towards Conjecture 1.9, based on an unusual mod 2 congruence between elliptic curves and Artin representations (§5).

1.10. **Novelty of the methods and remarks on the proofs.**

1.10.1. *2-Selmer groups and 2-class groups.* The idea that there should be a connection between the 2-Selmer group and the 2-class group of the cubic field $L$ has been, of course, known for a long time. For example, the classical work of Brumer and Kramer [BK77, 7.1] gives a general upper bound on the 2-Selmer rank in terms of the 2-class group and other arithmetic invariants of $E$. See also Schaefer [Sch96] (and references therein) for the connection between the $p$-Selmer group and the $p$-class group of the $p$-torsion field for general $p$.

Thus the novelty of Theorem 1.4 lies in making this connection *explicit* and *sharp* for the large class of elliptic curves under consideration.

1.10.2. *Heegner points.* Since quadratic twisting $E : y^2 = F(x)$ by a quadratic field $K$ does not preserve the square-freeness of the discriminant of $F(x)$ unless $K = \mathbb{Q}(i)$, we are naturally led to construct Heegner points over $\mathbb{Q}(i)$. However, the elliptic curve $E$ necessarily has *additive reduction* at 2 and 2 is *ramified* in $\mathbb{Q}(i)$, which forbids the classical construction of Heegner points associated to Eichler orders. See recent works Kohen–Pacetti [KP15], Cai–Chen–Liu [CCL16] and Longo–Rotger–de Vera-Piquero [LRd16] addressing different aspects of this issue.

The order (1.2) we naturally consider is in fact *not* a classical Eichler order, and thus leads to a new construction of Heegner points. The associated Shimura curve $X$ is an example of more general Shimura curves of "level $p^2$" associated to a *non-maximal* order at a prime $p$ *ramified* in the quaternion algebra studied in [Li15, Chap 5], and also an example of Hijikata-Pizer-Shemanske curves more recently studied in [LRd16]. Though our new construction of Heegner points easily generalizes to general prime $p$, we stick to $p = 2$ for simplicity, in connection with the concrete problem at hand.

1.10.3. *A canonical rational point.* For classical modular curves $X_0(N)$ (or Shimura curves of Eichler level), there is essentially one modular parameterization $X_0(N) \to E$, which is given by the associated newform $f_E$ of level $N$. More precisely, the group of homomorphisms $\mathrm{Hom}_{\mathbb{Q}}(J_0(N), E)$ defined over $\mathbb{Q}$ has rank 1. Another new feature of our work is that this 1-dimensionality no longer holds for the Shimura curve $X$. In fact, we show that $\mathrm{Hom}_{\mathbb{Q}}(J_X, E)$ has exactly rank 2 (Proposition 3.13 (1)), so there is *no* canonical choice of a modular parametrization $J_X \to E$.

Nevertheless, we are able to construct a *canonical* point in $E(\mathbb{Q}(i))$ by utilizing the entire rank 2 space of homomorphisms. The observation is that the Shimura curve $X$ admits extra automorphisms by the symmetric group $S_3$ and we are able to determine the $S_3$-action on $\mathrm{Hom}_{\mathbb{Q}}(J_X, E)$ (Proposition 3.13 (3)). To do so, we use the key fact that there is a *unique* admissible representation of $\mathrm{PGL}_2(\mathbb{Q}_2)$ has conductor 2 (§3.10 (5)), which allows us to make the Jacquet–Langlands correspondence completely explicit. Notice that this uniqueness is quite special and fails for admissible representations of conductor 2 of $\mathrm{PGL}_2(\mathbb{Q}_p)$ when $p > 2$.

1.10.4. *The case $D = 11$.* The computation of Heegner points on Shimura curves is more difficult than its counterpart on modular curves. When $D = 11$, to compute the Heegner points on $X$ we utilize the results of Elkies for Heegner points on a classical Shimura curve $Y$ (of Eichler level), and a degree 3 map $X \to Y$. The same method also allows us to verify Conjecture 1.9 for some other small values of $D$. Unfortunately, when $D$ is large ($X$ has large genus), this computation becomes infeasible.

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## 2. 2-Selmer groups and 2-class groups

Let $E/\mathbb{Q}$ be an elliptic curve. We impose the following assumption in this §2.

**Assumption 2.1.** Suppose $E$ has equation $y^2 = F(x)$, where $F(x) = x^3 + a_2x^2 + a_4x + a_6$ is an integral polynomial which

(1) is irreducible, and
(2) has negative and square-free discriminant $-D$.

2.2. **Properties of the cubic field $L$.** Let $L = \mathbb{Q}[x]/F(x)$. Then $L$ has the following elementary properties:

(1) Since $F(x)$ is irreducible and has negative discriminant, we know that $F(x)$ has Galois group $S_3$. The field $L = \mathbb{Q}[x]/F(x)$ is a complex cubic field, i.e., $L_\infty := L \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R} \times \mathbb{C}$.
(2) Since $\operatorname{disc} F(x) = -D$ is square-free, we know that $D \equiv 3 \pmod 4$, $L$ has discriminant $d_L = \operatorname{disc} F(x) = -D$, and $L$ has ring of integers $A = \mathbb{Z}[x]/F(x)$.

(3) Since $L$ has a unique real embedding, we know that the unit group $A^\times$ has rank one by Dirichlet's unit theorem. Let $u_L$ be a fundamental unit. Then $A^\times = u_L^{\mathbb{Z}} \times \{\pm 1\}$. After possibly replacing $u_L$ by $-u_L$, we may assume $u_L > 0$. After possibly replacing $u_L$ by $u_L^{-1}$, we may further assume $u_L > 1$.

2.3. **Properties of the elliptic curve $E$.** The elliptic curve $E : y^2 = F(x)$ has the following elementary properties:

(1) Since $F(x)$ is irreducible, we have $E(\mathbb{Q})[2] = 0$.
(2) The 2-torsion field $\mathbb{Q}(E[2])$, as the Galois closure of $L$, is an $S_3$-extension over $\mathbb{Q}$.
(3) $E$ has discriminant $\Delta = -2^4 D$. Since no 12th power divides $\Delta$, the equation $y^2 = F(x)$ is minimal. It follows that $E$ has bad reduction precisely at $p \mid 2D$.
(4) For $p \mid D$, since $D$ is square-free, by comparing the power of $p$ appearing on both sides of

$$c_4^3 - c_6^2 = -2^{10} \cdot 3^3 \cdot D,$$

we see that $p \nmid c_4$ (even for $p = 3$). Hence $E$ has multiplicative reduction of type $I_1$ at $p \mid D$. In particular, the component group of the Néron model of $E/\mathbb{Q}_p$ is trivial.
(5) We compute that $c_4 = 16(a_2^2 - 3a_4)$, so $2 \mid c_4$ and $E$ has additive reduction at 2.
(6) Therefore the conductor of $E$ is of the form $N = 2^{2+\delta} D$ for some $\delta \geq 0$. The order of $N$ at 2 is determined by the Ogg–Saito formula ([Sil94, 11.1])

$$\mathrm{ord}_2(N) = \mathrm{ord}_2(\Delta) + 1 - m = 5 - m, \tag{2.1}$$

where $m$ is the size of the component group of the Néron model of $E/\mathbb{Q}_2$.

Our main goal in this §2 is to relate the 2-Selmer group of $E/\mathbb{Q}$ and the 2-part of the ideal class group of the cubic field $L$, under Assumption 2.1.

2.4. **2-Selmer groups.** Recall that for an elliptic curve $E/\mathbb{Q}$, we have the global and local Kummer exact sequences, fitting into the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E(\mathbb{Q})/2E(\mathbb{Q}) & \xrightarrow{\delta} & H^1(\mathbb{Q}, E[2]) & \longrightarrow & H^1(\mathbb{Q}, E)[2] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow {\scriptstyle \prod_v \mathrm{res}_v} & & \downarrow & & \\ 0 & \longrightarrow & \prod_v E(\mathbb{Q}_v)/2E(\mathbb{Q}_v) & \xrightarrow{\prod_v \delta_v} & \prod_v H^1(\mathbb{Q}_v, E[2]) & \longrightarrow & \prod_v H^1(\mathbb{Q}_v, E)[2] & \longrightarrow & 0. \end{array}$$

Here the vertical maps are given by the product of restriction maps over all places $v$ of $\mathbb{Q}$.

**Definition 2.5.** The *2-Selmer group*

$$\mathrm{Sel}_2(E/\mathbb{Q}) \subseteq H^1(\mathbb{Q}, E[2])$$

consists of cohomology classes whose restriction at $v$ lies in the image of the local Kummer map $\delta_v$ for every $v$:

$$\mathrm{Sel}_2(E/\mathbb{Q}) = \{c \in H^1(\mathbb{Q}, E[2]) : \mathrm{res}_v(c) \in \mathrm{im}(\delta_v)\}.$$

Colloquially, the 2-Selmer group is cut out by the *local conditions*

$$\mathrm{im}(\delta_v) \subseteq H^1(\mathbb{Q}_v, E[2])$$

coming from local points for all $v$. By definition, we have an injection

$$E(\mathbb{Q})/2E(\mathbb{Q}) \hookrightarrow \mathrm{Sel}_2(E/\mathbb{Q}).$$

The 2-Selmer group, due to its local nature, is easier to understand and the 2-Selmer rank $s_2(E/\mathbb{Q})$ provides an upper bound for the rank of $E(\mathbb{Q})$.

2.6. **Kummer maps.** Our goal is this subsection to give an explicit description of the global and local Kummer maps in terms of the cubic field $L$.

**Lemma 2.7.** *We have an exact sequence of finite group schemes over* $\mathbb{Q}$,

$$1 \to E[2] \to \mathrm{Res}_{L/\mathbb{Q}}\,\mu_2 \xrightarrow{\mathbb{N}} \mu_2 \to 1,$$

*where* $\mathbb{N}$ *is induced by the norm map from* $L$ *to* $\mathbb{Q}$. *We have an isomorphism*

$$H^1(\mathbb{Q}, E[2]) \cong (L^\times/(L^\times)^2)_{\mathbb{N}=\square}. \tag{2.2}$$

*Here* $(L^\times/(L^\times)^2)_{\mathbb{N}=\square}$ *consists of all classes in* $L^\times/(L^\times)^2$ *with square norms to* $\mathbb{Q}^\times$.

*Proof.* It suffices to check that we have an exact sequence of $G_\mathbb{Q}$-modules on the level of $\overline{\mathbb{Q}}$-points. Suppose $F(x)$ has the three roots $x_1, x_2, x_3 \in \overline{\mathbb{Q}}$. Then the $\overline{\mathbb{Q}}$-points of $E[2]$ consist of $P_i = (x_i, 0)$, $(i = 1, 2, 3)$ and $\infty$. The Galois group $G_\mathbb{Q}$ acts trivially on $\infty$ and permutes the three points $P_i$ via its $\mathrm{Gal}(\mathbb{Q}(E[2])/\mathbb{Q}) \cong S_3$-action on $x_i$. Also, the group of $\overline{\mathbb{Q}}$-points of $\mathrm{Res}_{L/\mathbb{Q}}\,\mu_2$ is isomorphic to $\mu_2 \times \mu_2 \times \mu_2$, where the three factors are indexed by the three roots $x_i$ and the Galois action permutes the three factors in the same way. The norm map simply multiplies the three factors and respects the $G_\mathbb{Q}$-action. One sees that the map

$$\infty \mapsto (1, 1, 1), \quad P_1 \mapsto (1, -1, -1), \quad P_2 \mapsto (-1, 1, -1), \quad P_3 \mapsto (-1, -1, 1)$$

gives an injective homomorphism of $G_\mathbb{Q}$-modules $E[2] \to \mathrm{Res}_{L/\mathbb{Q}}\,\mu_2$. Moreover, its image is exactly the kernel of the norm map. This finishes the proof of the first part.

Taking the long exact sequence in Galois cohomology, we obtain

$$H^0(\mathbb{Q}, \mathrm{Res}_{L/\mathbb{Q}}\,\mu_2) \xrightarrow{\mathbb{N}} H^0(\mathbb{Q}, \mu_2) \to H^1(\mathbb{Q}, E[2]) \to H^1(\mathbb{Q}, \mathrm{Res}_{L/\mathbb{Q}}\,\mu_2) \xrightarrow{\mathbb{N}} H^1(\mathbb{Q}, \mu_2).$$

Since $H^0(\mathbb{Q}, \mathrm{Res}_{L/\mathbb{Q}}\,\mu_2) = \mu_2(L) = \{\pm 1\}$, $H^0(\mathbb{Q}, \mu_2) = \mu_2(\mathbb{Q}) = \{\pm 1\}$ and $L/\mathbb{Q}$ has odd degree, we know that the first map is surjective. By Kummer theory, we know that

$$H^1(\mathbb{Q}, \mathrm{Res}_{L/\mathbb{Q}}\,\mu_2) \cong L^\times/(L^\times)^2, \quad H^1(\mathbb{Q}, \mu_2) \cong \mathbb{Q}^\times/(\mathbb{Q}^\times)^2.$$

Therefore

$$H^1(\mathbb{Q}, E[2]) = \ker(\mathbb{N} : L^\times/(L^\times)^2 \to \mathbb{Q}^\times/(\mathbb{Q}^\times)^2) = (L^\times/(L^\times)^2)_{\mathbb{N}=\square}.$$

This finishes the proof of the second part. □

**Proposition 2.8.** *Under the isomorphism (2.2), the global Kummer map* $\delta$ *can be described as*

$$\delta : E(\mathbb{Q})/2E(\mathbb{Q}) \to (L^\times/(L^\times)^2)_{\mathbb{N}=\square}, \quad P \mapsto x(P) - \beta,$$

*where* $x(P)$ *is the* $x$*-coordinate of* $P$ *and* $\beta$ *is the image of* $x$ *in* $L = \mathbb{Q}[x]/F(x)$.

*Proof.* Let $e_2 : E[2] \times E[2] \to \mu_2$ be the Weil pairing. Then we see that the homomorphism $E[2] \to \mathrm{Res}_{L/\mathbb{Q}}\,\mu_2$ in Lemma 2.7 is also given by

$$P \mapsto (e_2(P, P_1), e_2(P, P_2), e_2(P, P_3)).$$

The rational function $f_i = x - x_i$ has the divisor $(f_i) = 2P_i - 2\infty$ $(i = 1, 2, 3)$ and there exists some rational function $g_i$ (over $\overline{\mathbb{Q}}$) such that $f_i \circ [2] = g_i^2$ (see [Sil09, III.8]). The Weil pairing

$e_2$ is then given by

$$e_2(P,P_i) = \frac{g_i(X+P)}{g_i(X)},$$

where $X \in E(\overline{\mathbb{Q}})$ is any point such that $g(X+P)$ and $g(X)$ are both defined and nonzero. For $P \in E(\mathbb{Q})$, we choose $Q \in E(\overline{\mathbb{Q}})$ such that $[2]Q = P$. Then $\delta(P)$ corresponds to the cocycle $\{\sigma \mapsto Q^\sigma - Q\} \in H^1(\mathbb{Q}, E[2])$. Taking $P = Q^\sigma - Q$ and $X = Q$, we know that

$$e_2(Q^\sigma - Q, P_i) = \frac{g_i(Q)^\sigma}{g_i(Q)}. \tag{2.3}$$

By the identification $H^1(\mathbb{Q}, \mathrm{Res}_{L/\mathbb{Q}}\,\mu_2) \cong (L^\times/(L^\times)^2)_{\mathbb{N}=\square}$ coming from Kummer theory, Equation (2.3) implies that

$$\delta(P) \equiv g_i(Q)^2 \mod (L^\times)^2$$

under the embedding $L \hookrightarrow \overline{\mathbb{Q}}$ associated to $x_i$. But by the construction of $g_i$, we have $g_i(Q)^2 = f_i(P) = x(P) - x_i$. Hence

$$\delta(P) \equiv x(P) - x_i \mod (L^\times)^2$$

under the embedding $L \hookrightarrow \overline{\mathbb{Q}}$ associated to $x_i$, which finishes the proof. □

Base changing to $\mathbb{Q}_v$ in Lemma 2.7 and Proposition 2.8 , we obtain the analogous explicit description of the local Kummer maps $\delta_v$.

**Proposition 2.9.** *The local Kummer maps for $E$ are given by*

$$\delta_v : E(\mathbb{Q}_v)/2E(\mathbb{Q}_v) \to H^1(\mathbb{Q}_v, E[2]) \cong (L_v^\times/(L_v^\times)^2)_{\mathbb{N}=\square}, \quad P \mapsto x(P) - \beta,$$

*where $\beta$ is the image of $x$ in $L_v = \mathbb{Q}_v[x]/F(x)$.*

*Remark* 2.10. Even though $E(\mathbb{Q})[2] = 0$, it is possible that $E(\mathbb{Q}_v)[2] \neq 0$. For a nonzero point $P \in E(\mathbb{Q}_v)[2]$, the expression $x(P) - \beta$ does not lie in $L_v^\times$ and it should be interpreted using the group structure: write $P = P_1 - P_2$ as the difference of two points $P_1, P_2 \in E(\mathbb{Q}_v)$ which are not 2-torsion, then $\delta_v(P) = (x(P_1) - \beta)/(x(P_2) - \beta)$.

2.11. **Local conditions.** In this subsection, we explain how Assumption 2.1 allows us to determine explicitly the local condition $\mathrm{im}(\delta_v)$ for each place $v$.

**Lemma 2.12.** *Let $p$ be a prime. Then the valuation of $\delta_p(P)$ is even for any $P \in E(\mathbb{Q}_p)$, namely,*

$$\delta_p(P) \in (A_p^\times/(A_p^\times)^2)_{\mathbb{N}=\square},$$

*where $A = \mathbb{Z}[x]/F(x)$ is the ring of integers of $L$ and $A_p = A \otimes \mathbb{Z}_p$.*

*Proof.* Let $P \in E(\mathbb{Q}_p)$ and write $x = x(P)$ for short.

First consider the case $p \nmid D$. So $p$ is unramified in $L$. There are three cases:

(1) $L_p \cong \mathbb{Q}_{p^3}$ is the unramified cubic extension of $\mathbb{Q}_p$. From $y^2 = F(x)$, we know that $2\,\mathrm{ord}_p(y) = 3\,\mathrm{ord}_p(x-\beta)$, hence $\mathrm{ord}_p(x-\beta)$ is even.
(2) $L_p \cong \mathbb{Q}_{p^2} \times \mathbb{Q}_p$, where $\mathbb{Q}_{p^2}$ is the unramified quadratic extension of $\mathbb{Q}_p$. Write $\beta = (\gamma, c)$, then $\gamma \not\equiv c \pmod p$, $\mathrm{ord}_p(\gamma) = 0$ and $\mathrm{ord}_p(c) \geq 0$. From $y^2 = F(x)$, we know that $2\,\mathrm{ord}_p(y) = \mathrm{ord}_p(x-\gamma) + \mathrm{ord}_p(x-c)$. There are two cases:
   - If $\mathrm{ord}_p(x) < 0$, then $\mathrm{ord}_p(x-\gamma) = \mathrm{ord}_p(x-c) = \mathrm{ord}_p(x)$. Therefore $2\,\mathrm{ord}_p(y) = 3\,\mathrm{ord}_p(x)$, hence $\mathrm{ord}_p(x-\gamma) = \mathrm{ord}_p(x-c) = \mathrm{ord}_p(x)$ are all even.
   - If $\mathrm{ord}_p(x) \geq 0$, then $\mathrm{ord}_p(x-\gamma) = 0$. So $\mathrm{ord}_p(x-c) = 2\,\mathrm{ord}_p(y)$ is even.

(3) $L_p \cong \mathbb{Q}_p \times \mathbb{Q}_p \times \mathbb{Q}_p$. Write $\beta = (c_1, c_2, c_3)$. Then $c_i \not\equiv c_j \pmod p$ whenever $i \neq j$. Similarly, there are two cases:
   - If $\mathrm{ord}_p(x) < 0$, then $\mathrm{ord}_p(x - c_i) = \mathrm{ord}_p(x)$. Therefore $2\,\mathrm{ord}_p(y) = 3\,\mathrm{ord}_p(x)$, hence $\mathrm{ord}_p(x - c_i) = \mathrm{ord}_p(x)$ are all even.
   - If $\mathrm{ord}_p(x) \geq 0$, then $\mathrm{ord}_p(x - c_i) \geq 0$. Since $c_i \not\equiv c_j \pmod p$, at least two of the $\mathrm{ord}_p(x - c_i)$'s are zeros. Thus $2\,\mathrm{ord}_p(y) = \mathrm{ord}_p(x - c_1) + \mathrm{ord}_p(x - c_2) + \mathrm{ord}_p(x - c_3)$ implies the third one must have even valuation as well.

Now consider the case $p \mid D$. So $p$ is ramified in $L$. Since $D$ is square-free, we know that $L_p \cong K_p \times \mathbb{Q}_p$, where $K_p$ is a ramified quadratic extension of $\mathbb{Q}_p$. Denote by $\mathfrak{p}$ the prime for $K_p$ and write $\beta = (\gamma, c)$. Then $\gamma \not\equiv c \pmod p$, $\mathrm{ord}_p(\gamma) > 0$ and $\mathrm{ord}_p(c) \geq 0$. From $y^2 = F(x)$, we know that $2\,\mathrm{ord}_p(y) = \mathrm{ord}_{\mathfrak{p}}(x - \gamma) + \mathrm{ord}_p(x - c)$. We argue similarly:

- If $\mathrm{ord}_p(x) < 0$, then $\mathrm{ord}_{\mathfrak{p}}(x - \gamma) = 2\,\mathrm{ord}_p(x)$ and $\mathrm{ord}_p(x - c) = \mathrm{ord}_p(x)$. Hence $\mathrm{ord}_p(x)$ is even, therefore both $\mathrm{ord}_{\mathfrak{p}}(x - \gamma)$ and $\mathrm{ord}_p(x - c)$ are even.
- If $\mathrm{ord}_p(x) \geq 0$, then $\mathrm{ord}_{\mathfrak{p}}(x - \gamma) \geq 0$ and $\mathrm{ord}_p(x - c) \geq 0$. Since $\gamma \not\equiv c \pmod p$, at least one of $\mathrm{ord}_{\mathfrak{p}}(x - \gamma)$ and $\mathrm{ord}_p(x - c)$ is zero. Hence both of them are even. □

*Remark* 2.13. When $p \neq 2$, Lemma 2.12 can be proved by a more "pure thought" argument: since the component group of Néron model of $E/\mathbb{Q}_p$ is trivial (2.3 (4)), the local condition at $p$ corresponds to the unramified cohomology $H^1_{\mathrm{ur}}(\mathbb{Q}_p, E[2])$ ([GP12, Lemma 6]), which consists of the units $(A_p^\times/(A_p^\times)^2)_{\mathbb{N}=\square}$ under the isomorphism (2.2). Here we preferred the above direct computational proof using the explicit description of $\delta_p$ in Proposition 2.9, which depends on less machinery and also treats the case $p = 2$.

**Proposition 2.14.** *We have*

*(1) For $v = \infty$, both $E(\mathbb{R})/2E(\mathbb{R})$ and $(L_\infty^\times/(L_\infty^\times)^2)_{\mathbb{N}=\square}$ are trivial. In particular, the local condition* $\mathrm{im}(\delta_\infty)$ *is trivial.*

*(2) For $v = p > 2$, the local condition* $\mathrm{im}(\delta_p) = (A_p^\times/(A_p^\times)^2)_{\mathbb{N}=\square}$.

*(3) For $v = p = 2$, the local condition* $\mathrm{im}(\delta_2)$ *has index 2 in* $(A_2^\times/(A_2^\times)^2)_{\mathbb{N}=\square}$ *and contains all units* $\equiv 1 \pmod 4$.

*Proof.* For $v = \infty$, since $L_\infty \cong \mathbb{R} \times \mathbb{C}$, we know that

$$(L_\infty^\times/(L_\infty^\times)^2)_{\mathbb{N}=\square} = (\mathbb{R}^\times/(\mathbb{R}^\times)^2)_{\mathbb{N}=\square} = \{1\}.$$

For $v = p$, we know from Lemma 2.12 that $\mathrm{im}(\delta_p) \subseteq (A_p^\times/(A_p^\times)^2)_{\mathbb{N}=\square}$. Since the norm map is surjective on the units, we know that

$$\#(A_p^\times/(A_p^\times)^2)_{\mathbb{N}=\square} = \frac{\#A_p^\times/(A_p^\times)^2}{\#\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2}.$$

Notice that $\#\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$ is $2^2$ or 2 depending on whether $p = 2$ or not. If $L_p$ is a product of $k$ fields ($k = 1, 2, 3$), then $\#A_p^\times/(A_p^\times)^2$ is $2^{k+3}$ or $2^k$ depending on whether $p = 2$ or not. It follows that

$$\#(A_p^\times/(A_p^\times)^2)_{\mathbb{N}=\square} = \begin{cases} 2^{k-1}, & p \neq 2, \\ 2^{k+1}, & p = 2. \end{cases}$$

Since $E(\mathbb{Q}_p)$ has a finite index subgroup isomorphic to $\mathbb{Z}_p$, we know that

$$\#E(\mathbb{Q}_p)/2E(\mathbb{Q}_p) = \begin{cases} \#E(\mathbb{Q}_p)[2], & p \neq 2, \\ 2 \cdot \#E(\mathbb{Q}_p)[2], & p = 2. \end{cases}$$

All claims except the last one then follow because $\#E(\mathbb{Q}_p)[2]$ is the same as 1 plus the number of $\mathbb{Q}_p$-rational solutions of $F(x)=0$ , which is $2^{k-1}$ in all cases. To see the last claim that $\mathrm{im}(\delta_2)$ contains all the units $\equiv 1 \pmod 4$, let us consider a point $P \in \hat{E}(2\mathbb{Z}_2)$, where $\hat{E}$ is the formal group of $E$ over $\mathbb{Q}_2$ given by the minimal equation. For a general elliptic curve $E$ with minimal equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

the general formula ([Sil94, IV.1]) reads

$$x(P) = z^{-2} - a_1z^{-1} - a_2 - a_3z + O(z^2),$$

where $z = -x/y$ is the parameter for the formal group. In our case $a_1 = 0$, therefore

(2.4)

$$\delta_2(P) = x(P) - \beta = z^{-2} - a_2 - \beta + O(z) \equiv 1-(a_2+\beta)z^2 + O(z^3) \equiv 1-(a_2+\beta)z^2 \mod (L_2^\times)^2,$$

where the last equality is because $z \in 2\mathbb{Z}_2$ and the units $\equiv 1 \pmod 8$ are squares. Since $F(x)$ mod 2 cannot be a product of three distinct linear factors in $\mathbb{F}_2[x]$, we know that 2 does not split in $L$. Now one can then check directly that $\mathrm{im}(\delta_2)$ contains the units $\equiv 1 \pmod 4$ with square norm in the remaining two cases:

- If $L_2 = \mathbb{Q}_8$, then all units $\equiv 1 \pmod 4$ with square norms are actually squares. So the claim that $\mathrm{im}(\sigma_2)$ contains all such units is trivial.
- If $L_2 = \mathbb{Q}_4 \times \mathbb{Q}_2$, then any class $(\alpha, a) \equiv 1 \pmod 4$ with square norm is represented by $(\alpha, \mathbb{N}\alpha)$ with $\alpha \equiv 1 \pmod 4$. There is a unique nontrivial class of units $\alpha \equiv 1 \pmod 4$ modulo squares, represented by $1 + 4\gamma$ if we write $\beta = (\gamma, c)$. It follows from (2.4) that $\mathrm{im}(\delta_2)$ contains this class. □

2.15. **2-class groups of cubic fields.** We now combine all the local conditions to give both upper and lower bounds for the 2-Selmer group.

**Lemma 2.16.** *Let*

$$M_1 = \{\alpha \in L^\times/(L^\times)^2 : L(\sqrt{\alpha})/L \textit{ is unramified}\},$$

*and*

$$M_2 = \{\alpha \in L^\times/(L^\times)^2 : \alpha > 0, (\alpha) = I^2, I \subseteq L \textit{ a fractional ideal}\}$$

*be subgroups of* $(L^\times/(L^\times)^2)_{\mathbb{N}=\square}$. *Then under the isomorphism (2.2), we have*

$$M_1 \subseteq \mathrm{Sel}_2(E/\mathbb{Q}) \subseteq M_2.$$

*Proof.* Elements of $M_2$ clearly have square norms since $\mathbb{N}\alpha = \mathbb{N}(I)^2$. If $\alpha \in \mathrm{Sel}_2(E/\mathbb{Q})$, then by Proposition 2.14, $\alpha > 0$ and $\alpha$ has even valuation at all finite places. The latter implies that there exists a fractional ideal $I$ such that $(\alpha) = I^2$. Thus $\alpha \in M_2$.

Let $\alpha \in L^\times/(L^\times)^2$. For $p$ odd, $L_p(\sqrt{\alpha})/L_p$ is unramified if and only if $\alpha$ has even valuation. For $p = 2$, $L_2(\sqrt{\alpha})/L_2$ is unramified if and only if $\alpha$ has even valuation and is represented by a unit $\equiv 1 \pmod 4$. From this description we see that $M_1 \subseteq M_2$ and elements $M_1$ have square norm. It also follows from this description and Proposition 2.14 that $M_1 \subseteq \mathrm{Sel}_2(E/\mathbb{Q})$. □

Class field theory supplies information about the two groups $M_1$ and $M_2$.

**Lemma 2.17.** *Let* $\mathrm{Cl}(L)$ *be the ideal class group of the cubic field* $L = \mathbb{Q}[x]/F(x)$. *Let* $k = \dim_{\mathbb{F}_2} \mathrm{Cl}(L)[2]$. *Then* $M_1$ *(resp.* $M_2$*) is an elementary 2-group of size* $2^k$ *(resp.* $2^{k+1}$*).*

*Proof.* Kummer theory tells us that

$$M_1 \cong \mathrm{Hom}(\mathrm{Gal}(M/L), \mu_2),$$

where $M$ is obtained by adjoining the square roots of all $\alpha \in M_1$ to $L$, which is the maximal unramified extension of $L$ of exponent 2. By class field theory, we have

$$\mathrm{Gal}(M/L) \cong \mathrm{Cl}(L)[2].$$

Since $\#(\mathrm{Cl}(L)/2\,\mathrm{Cl}(L)) = \#\,\mathrm{Cl}(L)[2]$, $M_1$ is an elementary 2-group of size $2^k$.

Suppose $\alpha \in M_2$ such that $(\alpha) = I^2$. Then the assignment $\alpha \mapsto I$ gives a well defined map

$$M_2 \to \mathrm{Cl}(L)[2].$$

This map is clearly surjective and its kernel is given by the positive units

$$\{\alpha \in A^\times/(A^\times)^2 : \alpha > 0\} = u_L^{\mathbb{Z}}/u_L^{2\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z}.$$

Since $\#\,\mathrm{Cl}(L)[2] = 2^k$, we know that $M_2$ is an elementary 2-group of size $2^{k+1}$. □

Now we are ready to prove the main theorem of this §2.

**Theorem 2.18.** *Let* $\mathrm{Cl}(L)$ *be the ideal class group of the cubic field* $L = \mathbb{Q}[x]/F(x)$*. Let* $k = \dim_{\mathbb{F}_2} \mathrm{Cl}(L)/2\,\mathrm{Cl}(L)$*. Then*

$$s_2(E/\mathbb{Q}) = k \text{ or } k+1,$$

*depending on whether the root number* $\varepsilon(E/\mathbb{Q}) = (-1)^k$ *or* $(-1)^{k+1}$*.*

*Proof.* It follows from Lemma 2.16 and 2.17 that $s_2(E/\mathbb{Q}) = k$ or $k+1$. By [Mon96, Theorem 1.5], the parity of $s_2(E/\mathbb{Q})$ is determined by the root number $\varepsilon(E/\mathbb{Q})$, that is,

$$(-1)^{s_2(E/\mathbb{Q})} = \varepsilon(E/\mathbb{Q}).$$

The desired result then follows. □

2.19. **Examples.** We end this section with several explicit examples illustrating Theorem 2.18.

**Example 2.20.** Consider the elliptic curve (in Cremona's labeling)

$$E = 1132a1 : y^2 = F(x) = x^3 + x^2 - 5x + 4.$$

The polynomial $F(x)$ is irreducible and has discriminant $-D = -283$ and thus Assumption 2.1 holds. The elliptic curve $E$ has discriminant $\Delta = -2^4 \cdot 283$, conductor $N = 2^2 \cdot 283$. The cubic field $L = \mathbb{Q}[x]/F(x)$ has discriminant $d_L = -283$ and class number 2. We remark that $L$ has the *smallest* discriminant among all class number 2 cubic fields. Theorem 2.18 predicts that $s_2(E/\mathbb{Q}) = 1$ or 2 according to the root number. In fact, $\varepsilon(E/\mathbb{Q}) = +1$, $r_{\mathrm{alg}}(E/\mathbb{Q}) = s_2(E/\mathbb{Q}) = 2$ and the two points $(-1, 3)$ and $(1, -1)$ generate $E(\mathbb{Q})$.

**Example 2.21.** Consider the elliptic curve

$$E = 26284a1 : y^2 = F(x) = x^3 + x^2 - 9x + 16.$$

The polynomial $F(x)$ is irreducible and has discriminant $-D = -6571$ and thus Assumption 2.1 holds. The elliptic curve $E$ has discriminant $\Delta = -2^4 \cdot 6571$, conductor $N = 2^2 \cdot 6571$. The cubic field $L = \mathbb{Q}[x]/F(x)$ has discriminant $d_L = -6571$ and class group $\mathrm{Cl}(L) \cong (\mathbb{Z}/2\mathbb{Z})^2$. We remark that $L$ has the *smallest* discriminant among all cubic fields with class group $(\mathbb{Z}/2\mathbb{Z})^2$.

Theorem 2.18 predicts that $s_2(E/\mathbb{Q}) = 2$ or $3$ according to the root number. In fact, $\varepsilon(E/\mathbb{Q}) = -1$, $r_{\mathrm{alg}}(E/\mathbb{Q}) = s_2(E/\mathbb{Q}) = 3$ and the three points $(-4, 2)$, $(-3, 5)$, $(-1, 5)$ generate $E(\mathbb{Q})$.

**Example 2.22.** Consider the elliptic curve

$$E : y^2 = F(x) = x^3 - 49x + 169.$$

The polynomial $F(x)$ is irreducible and has discriminant $-D = -37 \cdot 8123$ and thus Assumption 2.1 holds. The elliptic curve $E$ has discriminant $\Delta = -2^4 \cdot 37 \cdot 8123$, conductor $N = 1202204 = 2^2 \cdot 37 \cdot 8123$. The cubic field $L = \mathbb{Q}[x]/F(x)$ has discriminant $d_L = -37 \cdot 8123$ and class group $\mathrm{Cl}(L) \cong (\mathbb{Z}/2\mathbb{Z})^3$. We remark that $L$ has the *smallest* discriminant among all cubic fields with class group $(\mathbb{Z}/2\mathbb{Z})^3$. Theorem 2.18 predicts that $s_2(E/\mathbb{Q}) = 3$ or $4$ according to the root number. In fact, $\varepsilon(E/\mathbb{Q}) = +1$, $r_{\mathrm{alg}}(E/\mathbb{Q}) = s_2(E/\mathbb{Q}) = 4$ and the four points $(-8, 7)$, $(-7, 13)$, $(-5, 17)$, $(-3, 17)$ generate $E(\mathbb{Q})$.

## 3. Heegner points on elliptic curves of conductor $4D$

Throughout this §3, we will impose the following assumption.

**Assumption 3.1.** Suppose $E$ has equation $y^2 = F(x)$, where $F(x) = x^3 + a_2x^2 + a_4x + a_6$ is an integral polynomial which

(1) is irreducible, and
(2) has negative and square-free discriminant $-D$.

We further assume that

(3) $E/\mathbb{Q}_2$ has Kodaira type IV.

*Remark* 3.2. The additional assumption (3) that $E/\mathbb{Q}_2$ has Kodaira type IV means that the special fiber of the minimal regular model of $E/\mathbb{Q}_2$ consists of three $\mathbb{P}^1$'s intersecting at a triple point. It implies that $m = 3$ in Equation (2.1). Hence $\mathrm{ord}_2(N) = 2$ and $E$ has the minimal possible conductor $N = 4D$. It also implies that the $j$-invariant of $E$ has positive 2-adic valuation ([Sil94, Table 4.1]). In particular, $E$ has potentially good reduction at 2.

**Example 3.3.** If we assume

$$\begin{cases} a_6 \equiv 1 & \pmod 4, \\ a_4^2 \equiv 4a_2 & \pmod 8, \end{cases}$$

then it follows from Tate's algorithm [Sil94, IV.9] that $E$ has Kodaira type IV over $\mathbb{Q}_2$. In fact, making a change of variable $y' = y + 1$, $x' = x$, we obtain the following equation

$$y'^2 + 2y' = x'^3 + a_2x'^2 + a_4x' + (a_6 - 1),$$

satisfying $2 \mid a_3', a_4', a_6'$. We compute that

$$b_2' = 4a_2, \quad b_4' = 2a_4, \quad b_6' = 4a_6, \quad b_8' = 4a_2a_6 - a_4^2, \quad c_4' = 16(a_2^2 - 3a_4).$$

So

$$2 \mid c_4', \quad 2^2 \mid a_6', \quad 2^3 \mid b_8', \quad 2^3 \nmid b_6',$$

and Tate's algorithm outputs the Kodaira type IV. Moreover, the component group of the Néron model over $\mathbb{Q}_2$ is either $\mathbb{Z}/3\mathbb{Z}$ or $\mu_3$. It is $\mathbb{Z}/3\mathbb{Z}$ if and only if $x^2 + x - (a_6 - 1)/4 \equiv 0 \pmod 2$ has a solution, if and only if $a_6 \equiv 1 \pmod 8$.

3.4. **Root numbers.** As we will see, the additional assumption (3) also pins down the root number of $E$ over the quadratic field $\mathbb{Q}(i)$ to be $-1$ (Proposition 3.7).

**Lemma 3.5.** *The field $\mathbb{Q}_2(E[3])$ generated by 3-torsion points of $E$ is the tamely ramified $S_3$-extension $\mathbb{Q}_2(\zeta_3, \sqrt[3]{\Delta})$.*

*Proof.* The assumption of Kodaira type IV at 2 allows us to compute $\mathbb{Q}_2(E[3])$ as follows. Since the component group of the Néron model of $E/\mathbb{Q}_2$ is either $\mathbb{Z}/3\mathbb{Z}$ or $\mu_3$, we know that $E$ has a subgroup $\mathbb{Z}/3\mathbb{Z}$ over the unramified quadratic extension $M = \mathbb{Q}_2(\zeta_3)$. Let $G = \mathrm{Gal}(M(E[3])/M) \subseteq \mathrm{GL}_2(\mathbb{F}_3)$. Then by the Weil pairing, the inertia subgroup $I \subseteq G$ acts as a subgroup of $\{(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix})\} \subseteq \mathrm{GL}_2(\mathbb{F}_3)$, hence $I$ is either trivial or of order 3. But $\mathbb{Q}_2(E[3])$ always contains $\sqrt[3]{\Delta}$ (see [Ser72, p. 305]), so $\mathbb{Q}_2(E[3])/\mathbb{Q}_2$ is ramified and $I$ is of order 3. Notice that $\mathbb{Q}_2(\zeta_3, \sqrt[3]{\Delta})$, as the Galois closure of $\mathbb{Q}_2(\sqrt[3]{\Delta})$, has Galois group $S_3$, so we know that $G$ cannot be cyclic of order 3 or 6. Since $I$ is normal in $G$ and $G/I$ is cyclic, it follows that $G \cong S_3$ by inspection on possible subgroups of $\mathrm{GL}_2(\mathbb{F}_3)$. Therefore $\mathbb{Q}_2(E[3]) = \mathbb{Q}_2(\zeta_3, \sqrt[3]{\Delta})$. □

*Remark* 3.6. It also follows from this lemma directly that $\mathrm{ord}_2(N) = 2$, which reproves the Ogg–Saito formula in this case.

**Proposition 3.7.** *The root number $\varepsilon(E/\mathbb{Q}(i))$ of $E/\mathbb{Q}(i)$ is $-1$.*

*Proof.* Recall that the root number is the product of local root numbers over all places $v$

$$\varepsilon(E/\mathbb{Q}(i)) = \prod_v \varepsilon_v(E/\mathbb{Q}(i)).$$

We compute all the local root numbers (cf. [Dok13, 3.4]) as follows.

(1) The local root number of an elliptic curve is always $-1$ at an infinite place.
(2) For $\mathfrak{p} \nmid 2D$, the elliptic curve $E$ has good reduction at $\mathfrak{p}$ and thus $\varepsilon_\mathfrak{p}(E/\mathbb{Q}(i)) = +1$.
(3) For $\mathfrak{p} \mid D$, the elliptic curve $E$ has multiplicative reduction at $\mathfrak{p}$.
   - When $\mathfrak{p}$ lies above $p \equiv 3 \bmod 4$, $\mathbb{Q}(i)_\mathfrak{p}$ is the unramified quadratic extension of $\mathbb{Q}_p$ and thus $E$ has split multiplicative reduction at $\mathfrak{p}$. Therefore $\varepsilon_\mathfrak{p}(E/\mathbb{Q}(i)) = -1$. Because $D \equiv 3 \pmod 4$ is square-free, there are an odd number of primes $\mathfrak{p} \mid D$ lying above $p \equiv 3 \pmod 4$. Hence the product over all $\mathfrak{p}$ above $p \equiv 3 \pmod 4$ is $-1$.
   - When $\mathfrak{p}$ lies above $p \equiv 1 \pmod 4$, $E$ may have split or nonsplit multiplicative reduction at $\mathfrak{p}$. But since $p = \mathfrak{p}\mathfrak{p}'$ splits as two primes in $\mathbb{Q}(i)$ and $\varepsilon_\mathfrak{p}(E/\mathbb{Q}(i)) = \varepsilon_{\mathfrak{p}'}(E/\mathbb{Q}(i))$, we know that the product over all $\mathfrak{p}$ above $p \equiv 1 \pmod 4$ is $+1$.
(4) When $\mathfrak{p} \mid 2$, we know the 3-torsion points $E[3]$ generate a $S_3$-extension over the wildly ramified quadratic extension $\mathbb{Q}(i)_\mathfrak{p} = \mathbb{Q}_2(i)$ by Lemma 3.5. The local root number is $-1$ in this case by [DD08, Remark 5].

Combining all the local results gives the desired root number $\varepsilon(E/\mathbb{Q}(i)) = -1$. □

Now let

$$E^* : y^2 = F^*(x) = x^3 - a_2x^2 + a_4x - a_6$$

be the quadratic twist of $E$ by $\mathbb{Q}(i)$. By Proposition 3.7, we have

$$\varepsilon(E/\mathbb{Q}) \cdot \varepsilon(E^*/\mathbb{Q}) = \varepsilon(E/\mathbb{Q}(i)) = -1.$$

It follows that the functional equations for $L(E/\mathbb{Q}, s)$ and $L(E^*/\mathbb{Q}, s)$ have different signs $\varepsilon(E/\mathbb{Q})$ and $\varepsilon(E^*/\mathbb{Q})$. We denote by $E^\pm = E$ or $E^*$ so that $\varepsilon(E^\pm/\mathbb{Q}) = \pm 1$.

Notice that $E^*$ also satisfies Assumption 2.1. Moreover, the cubic field defined by $\mathbb{Q}[x]/F^*(x)$ is isomorphic to $L = \mathbb{Q}[x]/F(x)$. When the 2-rank of $\mathrm{Cl}(L)$ is at most 1, the 2-Selmer rank $s_2(E^-/\mathbb{Q}) = 1$ by Theorem 2.18. The 2-Selmer rank one conjecture (Conjecture 1.2) then predicts that

**Conjecture 3.8.** *If the 2-rank of* $\mathrm{Cl}(L)$ *is at most 1, then* $r_{\mathrm{alg}}(E^-/\mathbb{Q}) = 1$.

We pursue a canonical construction of a point (conjecturally) of infinite order using Shimura curves in the sequel.

*Remark* 3.9. Our construction below works well for any elliptic curve $E/\mathbb{Q}$ with conductor $N = 4D$ and $\varepsilon(E/\mathbb{Q}(i)) = -1$. We assume (3) of Assumption 3.1 only for concreteness.

3.10. **Explicit Jacquet–Langlands correspondence.** By the modularity theorem, there is an automorphic representation $\pi = \bigotimes_v \pi_v$ of $\mathrm{GL}_2(\mathbb{A})$ associated to the elliptic curve $E$, where $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$. It can be characterized as follows:

(1) $\pi$ has trivial central character.
(2) $\pi_\infty$ is a holomorphic discrete series with Harish-Chandra parameter $\frac{1}{2}$ (corresponding to weight 2 modular forms).
(3) For $p \nmid 2D$, $\pi_p$ is unramified. Its Satake parameter has characteristic polynomial $X^2 - a_pX + p$, where $a_p = p + 1 - \#E(\mathbb{F}_p)$.
(4) For $p \mid D$, $\pi_p$ is the Steinberg representation or the unramified quadratic twist of the Steinberg representation, depending on whether $\varepsilon_p(E/\mathbb{Q}) = -1$ or $\varepsilon_p(E/\mathbb{Q}) = +1$.
(5) For $p = 2$, $\pi_2$ has conductor 2 (since $\mathrm{ord}_2(N) = 2$). It cannot be a tamely ramified principal series, since there are no tamely ramified characters of $\mathbb{Q}_2^\times$! Therefore $\pi_2$ is a depth zero supercuspidal representation, which is compactly induced from $\mathrm{PGL}_2(\mathbb{Z}_2)$ using the unique discrete series representation of $\mathrm{PGL}_2(\mathbb{F}_2) \cong S_3$, the sign character $S_3 \to \{\pm 1\}$.

Let $B = (-1, D)_{\mathbb{Q}}$ be the rational quaternion algebra

$$\mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}ij, \quad i^2 = -1, j^2 = D, ij = -ji.$$

Then $B$ is split at $\infty$ and is ramified at primes in

$$\Sigma = \{2\} \cup \{p \mid D : p \equiv 3 \pmod 4\}.$$

Notice that $\Sigma$ has even cardinality since $D \equiv 3 \pmod 4$. Let

$$R = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}ij \subseteq B. \tag{3.1}$$

Then $R$ is an order of reduced discriminant $4D$. We now give a description of the local orders $R_p = R \otimes \mathbb{Z}_p$ and the normalizers of $R_p^\times$.

**Proposition 3.11.** *Let* $W_p = N_{B_p^\times}(R_p^\times)/R_p^\times \mathbb{Q}_p^\times$, *where* $N_G(H)$ *denotes the normalizer of* $H$ *in* $G$. *Then*

$$W_p \cong \begin{cases} \mathbb{Z}/2\mathbb{Z}, & p \mid D, \\ S_3, & p = 2, \\ \{1\}, & \textit{otherwise}. \end{cases}$$

*Proof.* We have the following cases:

- For $p \nmid 2D$, $B_p$ is isomorphic to the matrix algebra $M_2(\mathbb{Q}_p)$ and $R_p$ is a maximal order $M_2(\mathbb{Z}_p)$ (up to conjugation). Hence $N_{B_p^\times}(R_p^\times) = \mathrm{GL}_2(\mathbb{Z}_p) \cdot \mathbb{Q}_p^\times = R_p^\times \cdot \mathbb{Q}_p^\times$ and $W_p$ is trivial.

- For $p \mid 2D$, $p \notin \Sigma$, $B_p$ is isomorphic to $M_2(\mathbb{Q}_p)$ and $R_p$ is an order of reduced discriminant $p\mathbb{Z}_p$, which is $\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$ (up to conjugation). Therefore $R_p^\times$ is the standard Iwahori subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$ and $N_{B_p^\times}(R_p^\times)/R_p^\times\mathbb{Q}_p^\times \cong \mathbb{Z}/2\mathbb{Z}$ generated by the element $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$.
- For odd $p \in \Sigma$, $B_p$ is a quaternion algebra over $\mathbb{Q}_p$ and $R_p$ is the maximal order of $B_p$. Hence $N_{B_p^\times}(R_p^\times)/R_p^\times\mathbb{Q}_p^\times \cong \mathbb{Z}/2\mathbb{Z}$ is generated by a uniformizer of $B_p$.
- For $p = 2$, since $R_2$ has reduced discriminant $4\mathbb{Z}_2$, it is the unique index 2 suborder of the maximal order $\mathcal{O}_2$ of $B_2$,
$$R_2 = \{x \in \mathcal{O}_2 : (x \bmod \mathfrak{m}_2) \in \mathbb{F}_2 \subseteq \mathbb{F}_4 = \mathcal{O}_2/\mathfrak{m}_2\},$$
where $\mathfrak{m}_2$ is the maximal ideal of $\mathcal{O}_2$. Then $N_{B_2^\times}(R_2^\times)/R_2^\times\mathbb{Q}_2^\times \cong B_2^\times/R_2^\times\mathbb{Q}_2^\times \cong S_3$, which is generated by the order 2 class of a uniformizer and the cyclic quotient $\mathcal{O}_2^\times/R_2^\times \cong \mathbb{F}_4^\times/\mathbb{F}_2^\times \cong \mathbb{Z}/3\mathbb{Z}$. □

*Remark* 3.12. Since $R_p$ is a local Eichler order of reduced discriminant $p$ for any $p \neq 2$ (for background on Eichler orders, see [AB04, 1.2]) and Eichler orders are determined by its localizations ([AB04, 1.51]), we know that there is a unique Eichler order $S$ such that
$$S_p = R_p \text{ for } p \neq 2, \quad S_2 = \mathcal{O}_2.$$
Then $S$ is the unique Eichler order of reduced discriminant $2D$ containing $R$ and $R \subseteq S$ is the unique index 2 suborder, given by
$$R = \{x \in S : (x \bmod \mathfrak{m}_2) \in \mathbb{F}_2 \subseteq \mathbb{F}_4 = \mathcal{O}_2/\mathfrak{m}_2\}.$$

The Jacquet–Langlands correspondence associates to $\pi$ an automorphic representation
$$\sigma = \sigma_f \otimes \sigma_\infty = \bigotimes_v \sigma_v$$
of $B^\times(\mathbb{A})$ of the same conductor $4D$. We can characterize it as follows:

(1) $\sigma$ has trivial central character.
(2) $\sigma_\infty \cong \pi_\infty$ is a holomorphic discrete series with Harish-Chandra parameter $\frac{1}{2}$ (corresponding to weight 2 modular forms).
(3) For $p \nmid 2D$, we have $B_p^\times \cong \mathrm{GL}_2(\mathbb{Q}_p)$ and $\sigma_p \cong \pi_p$ is unramified. Its Satake parameter has characteristic polynomial $X^2 - a_pX + p$, where $a_p = p + 1 - \#E(\mathbb{F}_p)$.
(4) For $p \mid 2D$, $p \notin \Sigma$, we also have $B_v^\times \cong \mathrm{GL}_2(\mathbb{Q}_v)$ and $\sigma_v \cong \pi_v$. Since $\sigma_p$ has conductor 1 and $R_p^\times$ is the standard Iwahori subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$, the fixed space $\sigma_p^{R_p^\times}$ is 1-dimensional. The group $W_p$ acts on $\delta_p^{R_p^\times}$ via the sign or the trivial character depending on whether $\pi_p$ is the Steinberg representation or the unramified quadratic twist of the Steinberg representation.
(5) For odd $p \in \Sigma$, since $\sigma_p$ has conductor 1, the fixed space $\sigma_p^{R_p^\times}$ is a 1-dimensional representation of $W_p$. It is either the trivial or the sign character depending on whether $\pi_p$ is the Steinberg representation or the unramified quadratic twist of the Steinberg representation.
(6) For $p = 2$, since $\sigma_2$ has conductor 2 ($\mathrm{ord}_2(N) = 2$), we have $\sigma_2^{R_2^\times} \neq 0$. Since $\pi_2$ is supercuspidal, we know that $\sigma_2$ is the unique 2-dimensional irreducible representation of $B_2^\times/R_2^\times\mathbb{Q}_2^\times \cong S_3$.

Let $\hat{R}^\times = (R \otimes \hat{\mathbb{Z}})^\times$. The following proposition follows immediately from the previous local description of $\sigma$.

**Proposition 3.13.**
*(1) The space of invariants* $\sigma_f^{\hat{R}^\times}$ *is 2-dimensional.*
*(2) For* $p \mid D$*,* $W_p \cong \mathbb{Z}/2\mathbb{Z} = \langle w_p \rangle$ *acts on* $\sigma_f^{\hat{R}^\times}$ *via 2 copies of the character*

$$w_p \mapsto \left(\frac{-1}{p}\right)\varepsilon_p(E/\mathbb{Q}).$$

*(3) For* $p = 2$*,* $W_2 \cong S_3$ *acts on* $\sigma_f^{\hat{R}^\times}$ *via the unique 2-dimensional representation of* $S_3$*.*

3.14. **Shimura curves and Heegner points.** Let $\mathcal{H}^\pm = \mathbb{C} - \mathbb{R}$ be the union of the upper and lower half plane. Associated to the order $R$ we have a Shimura curve

$$X = R^\times \backslash \mathcal{H}^\pm,$$

where $R^\times$ acts on $\mathcal{H}^\pm$ via an embedding $R^\times \hookrightarrow (B \otimes \mathbb{R})^\times \cong \mathrm{GL}_2(\mathbb{R})$. Let $T/\mathbb{Q}$ be the maximal torus in $B^\times$ induced by the natural embedding $\mathbb{Z}[i] \hookrightarrow R$ (so $T$ is split by $\mathbb{Q}(i)$). Let $M$ be the $\mathrm{GL}_2(\mathbb{R})$-conjugates of the natural homomorphism $h_0 : T(\mathbb{R}) = \mathbb{C}^\times \hookrightarrow \mathrm{GL}_2(\mathbb{R})$, then $M \cong \mathcal{H}^\pm$ and $h_0$ is naturally identified with $i \in \mathcal{H}^+$. The Shimura curve $X$ has the adelic description

$$X \cong B^\times(\mathbb{Q}) \backslash M \times B^\times(\mathbb{A}_f)/\hat{R}^\times. \tag{3.2}$$

It is a well-known fact due to Shimura [Shi67] that the points of $X$ classify abelian surfaces together with endomorphisms by $R$ and this moduli interpretation provides the Shimura curve $X$ with a canonical smooth projective model over $\mathbb{Q}$.

*Remark* 3.15. Shimura curves associated to Eichler orders are well studied before. Let $Y$ be the Shimura curve associated to the Eichler order $S$ (Remark 3.12). Then the natural covering map $X \to Y$ has degree $[S_2^\times : R_2^\times] = [\mathbb{F}_4^\times : \mathbb{F}_2^\times] = 3$.

**Definition 3.16.** Let $K/\mathbb{Q}$ be an imaginary quadratic field with an embedding $\tau : \mathcal{O}_K \hookrightarrow R$. Then the induced homomorphism $h : (K \otimes \mathbb{R})^\times = \mathbb{C}^\times \hookrightarrow (B \otimes \mathbb{R})^\times \cong \mathrm{GL}_2(\mathbb{R})$ corresponds to a point $y_K$ on $X$, known as a *Heegner point*. Notice that $y_K$ depends on the choice of the embedding $\tau$. In terms of the moduli interpretation, $y_K$ corresponds to an abelian surface which is isomorphic to a product of two elliptic curves with complex multiplication by $\mathcal{O}_K$. By the theory of complex multiplication, $y_K$ is defined over the Hilbert class field of $K$.

We specialize to the case $K = \mathbb{Q}(i)$ and the natural embedding

$$\tau : \mathcal{O}_K = \mathbb{Z}[i] \hookrightarrow R = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}ij.$$

The associated Heegner point $y_K$ is represented by the point $[h_0, 1]$ under the double quotient (3.2) and it is $K$-rational since $K$ has class number one.

The finite group $W = \prod_{p \mid 4D} W_p$ acts on $X$ as automorphisms defined over $\mathbb{Q}$. The generator $w_p$ of $W_p$ for $p \mid D$ are known as the *Atkin–Lehner involution*.

**Proposition 3.17.** *Let* $w = \prod_{p \mid D} w_p \in W$*. Then* $w(y_K) = \overline{y_K}$*, the complex conjugate of* $y_K$*.*

*Proof.* In view of the moduli interpretation, the point $\overline{y_K}$ corresponds to the complex conjugate embedding $\bar{\tau} : \mathbb{Z}[i] \hookrightarrow R$ of $\tau$. Since $jij^{-1} = -i$, we know that conjugating $\tau$ by $j \in B^\times$ gives $\bar{\tau}$ and thus $jy_K = \overline{y_K}$. On the other hand, the reduced norm of $j^2$ is $-D$, so we see that

$$jR_p^\times \mathbb{Q}_p^\times / R_p^\times \mathbb{Q}_p^\times = \begin{cases} w_p, & p \mid D, \\ 1, & p \nmid D. \end{cases}$$

It follows that $jy_K = j[h_0, 1] = [h_0, w] = w(y_K)$. We conclude that $w(y_K) = \overline{y_K}$. □

**Proposition 3.18.** *Each point in the set* $W(y_K) = \{\sigma(y_K), \sigma \in W\}$ *has a stabilizer of order 2 (contained in* $W_2$*) under the action of* $W$*. In particular,* $W(y_K)$ *has size* $3 \cdot 2^{\#\{p|D\}}$ *and* $W_2(y_K)$ *has size* 3.

*Proof.* The stabilizer of $y_K = [h_0, 1]$ under the action of $B^\times$ is $T(\mathbb{Q}) = \mathbb{Q}(i)^\times \subseteq B^\times$. Since $\mathbb{Q}(i)$ is unramified at $p \neq 2$, it follows that for $p \neq 2$, $\mathbb{Q}_p(i)^\times R_p^\times / R_p^\times \mathbb{Q}_p^\times = \{1\}$ and thus $W_p$ acts on $W(y_K)$ freely. For $p = 2$, $\mathbb{Q}(i)$ is ramified at 2 and $\mathbb{Q}_2(i)^\times R_2^\times / R_2^\times \mathbb{Q}_2^\times \cong \mathbb{Z}/2\mathbb{Z}$ is generated by the class of a uniformizer of $\mathbb{Q}_2(i)$. So $W_2$ acts on $y_K$ with a stabilizer of order 2. □

3.19. **Uniformization by Shimura curves.** Let $J_X = \mathrm{Jac}(X)$ be the Jacobian of $X$ and $H = \mathrm{Hom}_{\mathbb{Q}}(J_X, E)$ be the group of homomorphisms (defined over $\mathbb{Q}$) from $J_X$ to $E$, then $W$ acts on $J_X$ and $H$. Notice that $H$ is a free abelian group and by [YZZ13, §3.2.3], we have

$$H \otimes \mathbb{C} \cong \sigma_f^{\hat{R}^\times},$$

as $W_2 \cong S_3$-representations. We know from Proposition 3.13 that $\sigma_f^{\hat{R}^\times}$ is the irreducible 2-dimensional representation of $S_3$, thus $H$ is a free abelian group of rank 2. In particular, the elliptic curve $E$ is uniformized by the Shimura curve $X$, in *two independent ways* which cannot be distinguished from each other.

3.20. **Heegner points on elliptic curves.** Let $D^0$ be the free abelian group of degree 0 divisors supported on the $K$-rational points $W(y_K)$. The following theorem shows that the image of these divisors under the projection maps $J_X \to E$ lies in the desired subgroup $E^-(\mathbb{Q})$ of $E(\mathbb{Q}(i))$.

**Theorem 3.21.** *Let* $d \in D^0$ *and* $\phi \in H$*. Then* $\phi(d) \in E^-(\mathbb{Q}) \subseteq E(\mathbb{Q}(i))$.

*Proof.* For any $d \in D^0$, it follows from Proposition 3.17 that $wd = \overline{d}$. Therefore

$$\overline{\phi(d)} = \phi(\overline{d}) = \phi(\sigma d) = \phi^\sigma(d) = w(\phi)(d).$$

By Proposition 3.13, this is equal to

$$\prod_{p|D} \left(\frac{-1}{p}\right) \varepsilon_p(E/\mathbb{Q}) \cdot \phi(d).$$

But $\varepsilon_2(E/\mathbb{Q}) = -1$ and $\prod_{p|D} \left(\frac{-1}{p}\right) = -1$ since $D \equiv 3 \pmod 4$, we obtain that

$$\overline{\phi(d)} = -\varepsilon(E/\mathbb{Q}) \cdot \phi(d).$$

Hence $\overline{\phi(d)} = \phi(d)$ if and only if $\varepsilon(E/\mathbb{Q}) = -1$. In other words, the image lies in $E^-(\mathbb{Q})$. □

The pairing between two free abelian groups

$$\langle \ , \ \rangle : H \times D^0 \to E(\mathbb{Q}(i)), \quad \langle \phi, d \rangle = \phi(d)$$

is bilinear and satisfies $\langle \phi^\sigma, d \rangle = \langle \phi, \sigma d \rangle$ for any $\sigma \in W$. Hence it induces a map

$$H \otimes_{\mathbb{Z}[W]} D^0 \to E(\mathbb{Q}(i)),$$

whose image lies in $E^-(\mathbb{Q})$ by Theorem 3.21.

3.22. **A canonical rational point.** We now use the extra automorphisms in $W_2$ to produce a canonical (up to sign) rational point $P$ in $E^-(\mathbb{Q})$. Let $D_2^0 \subseteq D^0$ be the subgroup of divisors supported on the set of three points $W_2(y_K)$.

**Theorem 3.23.** *$H \otimes_{\mathbb{Z}[W_2]} D_2^0$ is a free abelian group of rank one.*

*Proof.* There are two possibilities for the integral $S_3$-representation $H$: it is either the $A_2$-lattice

$$A_2 = \{a_1e_1 + a_2e_2 + a_3a_3 : \sum a_i = 0\} \subseteq \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$$

or its dual

$$A_2^\vee = \mathrm{Hom}(A_2, \mathbb{Z}) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3/\mathbb{Z}(e_1 + e_2 + e_3),$$

where $S_3$ permutes the basis vectors $\{e_1, e_2, e_3\}$ in the natural way. On the other hand, by Proposition 3.18, $W_2(y_K)$ consists of three points. Hence as a $W_2 \cong S_3$-representation, $D_2^0$ is isomorphic to the $A_2$-lattice.

If $H = A_2^\vee$, then the natural pairing between $A_2$ and $A_2^\vee$ induces an isomorphism

$$H \otimes_{\mathbb{Z}[W_2]} D_2^0 \cong A_2 \otimes_{\mathbb{Z}[S_3]} A_2^\vee \cong \mathbb{Z}.$$

It remains to check that the case $H = A_2$, i.e., it remains to show that

$$A_2 \otimes_{\mathbb{Z}[S_3]} A_2 \cong \mathbb{Z}.$$

Since $A_2$ is freely generated by the two vectors $u = e_1 - e_2$ and $v = e_2 - e_3$, it suffices to check that $u \otimes u$, $v \otimes v$, $u \otimes v$ and $v \otimes u$ in $A_2 \otimes_{\mathbb{Z}[S_3]} A_2$ generate a free abelian group of rank one. In fact, we have

$$u \otimes u = (e_1 - e_2) \otimes (e_1 - e_3 + e_3 - e_2) = \sigma_{12}(e_1 - e_2) \otimes \sigma_{12}(e_1 - e_3) - u \otimes v = -2u \otimes v,$$

$$v \otimes v = (e_2 - e_3) \otimes (e_2 - e_3) = \sigma_{13}(e_2 - e_3) \otimes \sigma_{13}(e_2 - e_3) = u \otimes u,$$

and

$$u \otimes v = (e_1 - e_2) \otimes (e_2 - e_3) = \sigma_{13}(e_1 - e_2) \otimes \sigma_{13}(e_2 - e_3) = v \otimes u,$$

where $\sigma_{ij} \in S_3$ denotes the transposition switching $e_i$ and $e_j$. It follows that

$$v \otimes v = u \otimes u = -2u \otimes v, \quad v \otimes u = u \otimes v,$$

and thus $A_2 \otimes_{\mathbb{Z}[S_3]} A_2$ is freely generated on one element $u \otimes v$. □

Finally, we define our desired canonical rational point $P \in E^-(\mathbb{Q})$ to be the image of the generator (up to sign) of $H \otimes_{\mathbb{Z}[W_2]} D_2^0$.

## 4. A conjecture on the canonical point $P$

In view of Conjecture 3.8, we propose the following conjecture.

**Conjecture 4.1.** *If the 2-rank of* $\mathrm{Cl}(L)$ *is at most 1, then* $P \in E^-(\mathbb{Q})$ *has infinite order.*

We now verify this conjecture for an explicit example.

**Example 4.2.** Consider the case $D = 11$. The Shimura curve $X$ associated to the quaternion order of discriminant 44 has genus 2. The Shimura curve $Y$ associated to the maximal order of discriminant 22 has genus 0. The degree 3 map $X \to Y$ (Remark 3.15) is ramified at the 4 elliptic points of order 3 on $Y$. Elkies in 2007 computed the elliptic points of $Y$ (see [Elk08] for his method). We can then deduce that $X$ has equation

$$-y^2 = x^6 - 7x^4 + 59x^2 + 11$$

and the three elliptic points in $W_2(y_K)$ are $\infty$, $(1,8i)$ and $(-1,8i)$ . The Jacobian $J_X$ is $(2,2)$-isogenous to $E\times\tilde{E}$, where

$$E=44a1: y^2=x^3+7x^2+59x-11,\quad \tilde{E}=44a2: y^2=x^3-59x^2-77x-121,$$

and the map $J_X\to E\times\tilde{E}$ is induced from the two maps

$$X\to E,\quad (x,y)\mapsto(-x^2,y),\qquad X\to\tilde{E},\quad (x,y)\mapsto\left(-\frac{11}{x^2},\frac{11y}{x^3}\right).$$

The two elliptic curves $E,\tilde{E}$ are 3-isogenous to each other and we see that $\mathrm{Hom}_{\mathbb{Q}}(J_X,E)$ is indeed of rank 2. The cubic field $L$ has discriminant $-44$ and class number $h_L=1$. The root number $\varepsilon(E/\mathbb{Q})=+1$. The quadratic twist of $E$ by $\mathbb{Q}(i)$ has equation

$$E^-=176c1: y^2=x^3-7x^2+59x+11.$$

As predicted by Conjecture 3.8, we have $r_{\mathrm{alg}}(E^-/\mathbb{Q})=1$. As predicted by Conjecture 4.1, the canonical point $P=(1,8)\in E^-(\mathbb{Q})$ we constructed is indeed a point of infinite order, generating a subgroup of index 2 in $E^-(\mathbb{Q})$.

## 5. A mod 2 congruence between elliptic curves and Artin representations

5.1. **A mod 2 congruence.** The mod 2 Galois representation $\bar{\rho}=\bar{\rho}_{E,2}: G_{\mathbb{Q}}\to\mathrm{GL}_2(\mathbb{F}_2)\cong S_3$ can also be viewed as a 2-dimensional irreducible Artin representation $\sigma$ via an embedding $S_3\hookrightarrow\mathrm{GL}_2(\mathbb{C})$. This Artin representation $\sigma$ has dihedral image and thus is induced from the imaginary quadratic field $K=\mathbb{Q}(\sqrt{-D})$. Then $\sigma$ is associated to a *weight one* newform with nebentypus the quadratic character $\varepsilon_K$, which is a Hecke theta series associated to $K$. On the other hand, there is a *weight two* newform $f$ with trivial nebentypus associated to the elliptic curve $E$ by the modularity theorem. By construction we have a congruence $f\equiv h \pmod 2$.

**Example 5.2** (cf. [Ser77, 7.3])**.** Consider the elliptic curve

$$E=92a1: y^2=F(x)=x^3+x^2+2x+1.$$

The polynomial $F(x)$ is irreducible and has square-free and negative discriminant $-D=-23$ and thus Assumption 2.1 holds. The elliptic curve $E$ has discriminant $\Delta=-2^4\cdot 23$, conductor $N=2^2\cdot 23$. The cubic field

$$L=\mathbb{Q}[x]/(x^3+x^2+2x+1)$$

has discriminant $d_L=-23$, hence the Artin representation $\sigma$ has conductor $N(\sigma)=23$. The class number of $L$ is 1 and we have $s_2(E/\mathbb{Q})=0$ as predicted by Theorem 2.18.

The the relevant Hecke character can be viewed as an order 3 character on the ideal class group $\mathrm{Cl}(K)$. We remark that $K$ is the cubic field of *smallest* (in the sense of the absolute value) discriminant with class number 3. The three ideal classes in $\mathrm{Cl}(K)$ are represented by the three integral binary quadratic forms of discriminant $d_K=-23$,

$$x^2+xy+6y^2,\quad 2x^2\pm xy+3y^2,$$

of order 1 and 3 respectively. We find that $h(z)$ is the following simple linear combination of theta series associated to these quadratic forms:

$$\begin{aligned} h(z) &= \frac{1}{2}\left(\sum_{m,n\in\mathbb{Z}} q^{m^2+mn+6n^2} - \sum_{m,n\in\mathbb{Z}} q^{2m^2+mn+3n^2}\right) \\ &= q - q^2 - q^3 + q^6 + q^8 - q^{13} - q^{16} + q^{23} - q^{24} + q^{25} \\ &\quad + q^{26} + q^{27} - q^{29} - q^{31} + q^{39} - q^{41} - q^{46} - q^{47} + q^{48} + q^{49} - q^{50} \cdots \end{aligned}$$

We remark that $\sigma$ is the irreducible 2-dimensional Artin representation of *smallest* conductor ([Ser77, 8.1]). Moreover, $h(z)$ can be written as the classical eta product:

$$h(z) = \eta(z)\eta(23z) = q\prod_{n\geq 1}(1-q^n)(1-q^{23n}),$$

where

$$\eta(z) = q^{1/24}\prod_{n\geq 1}(1-q^n) = \sum_{n\geq 1}\left(\frac{12}{n}\right)q^{\frac{n^2}{24}}$$

is Dedekind's eta function.

The first few Hecke eigenvalues of the newforms $f(z) \in S_2(92)$ and $h(z) \in S_1(23, \varepsilon_{-23})$ are listed in Table 1. We see that $a_p \equiv b_p \pmod 2$ for $p \neq 2$.

| $p$ | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 | 43 | 47 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a_p(f)$ | 0 | 1 | 0 | 2 | 0 | $-1$ | $-6$ | 2 | $-1$ | $-3$ | 5 | 8 | 3 | 8 | 9 |
| $b_p(h)$ | $-1$ | $-1$ | 0 | 0 | 0 | $-1$ | 0 | 0 | 1 | $-1$ | $-1$ | 0 | $-1$ | 0 | $-1$ |

TABLE 1. $E = 92a1$

**Example 5.3.** Notice the above mod 2 congruence does not require the conductor of $E$ to be of the form $N = 4D$. Consider the elliptic curve

$$E = X_0(11) = 11a1 : y^2 + y = x^3 - x^2 - 10x - 20,$$

which has the *smallest* conductor 11 among all elliptic curves over $\mathbb{Q}$. It has discriminant $\Delta = -11$. The cubic field

$$L \cong \mathbb{Q}[x]/(x^3 - x^2 + x + 1)$$

has discriminant $d_L = -2^2\cdot 11$. Hence the Artin representation $\sigma$ has conductor $N(\sigma) = 2^2\cdot 11$.

The ring class group of the quadratic order of discriminant $-44$ in $K = \mathbb{Q}(\sqrt{-11})$ has order 3, represented by the three binary quadratic form of discriminant $-44$,

$$2x^2 + 11y^2, \quad 3x^2 \pm 2xy + 4y^2$$

of order 1 and 3 respectively. We find that $h(z)$ is the following simple linear combination of theta series associated to these quadratic forms:

$$\begin{aligned} h(z) &= \frac{1}{2}\left(\sum_{m,n\in\mathbb{Z}} q^{2m^2+11n^2} - \sum_{m,n\in\mathbb{Z}} q^{3m^2+2mn+4n^2}\right) \\ &= q - q^3 - q^5 + q^{11} + q^{15} - q^{23} + q^{27} - q^{31} - q^{33} - q^{37} + 2q^{47} + q^{49} + \cdots \end{aligned}$$

Moreover, $h(z)$ is the classical eta product

$$h(z) = \eta(2z)\eta(22z) = q \prod_{n\geq 1}(1-q^{2n})(1-q^{22n}).$$

The newform $f(z) \in S_2(11)$ is also a classical eta product

$$\begin{aligned} f(z) = \eta^2(z)\eta^2(11z) &= q \prod_{n\geq 1}(1-q^n)^2(1-q^{11n})^2 \\ &= q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 - 2q^{10} + q^{11} - 2q^{12} \\ &\quad + 4q^{13} + 4q^{14} - q^{15} - 4q^{16} - 2q^{17} + 4q^{18} + 2q^{20} + 2q^{21} - 2q^{22} \\ &\quad - q^{23} - 4q^{25} - 8q^{26} + 5q^{27} - 4q^{28} + 2q^{30} + 7q^{31} + 8q^{32} - q^{33} \\ &\quad + 4q^{34} - 2q^{35} - 4q^{36} + 3q^{37} - 4q^{39} - 8q^{41} - 4q^{42} - 6q^{43} + 2q^{44} \\ &\quad - 2q^{45} + 2q^{46} + 8q^{47} + 4q^{48} - 3q^{49} + 8q^{50} \cdots \end{aligned}$$

The first few Hecke eigenvalues of the newforms $f(z) \in S_2(11)$ and $h(z) \in S_1(44, \varepsilon_{-44})$ are listed in Table 2. We see that $a_p \equiv b_p \pmod 2$ for all $p$ (which one can also see directly from the eta products above):

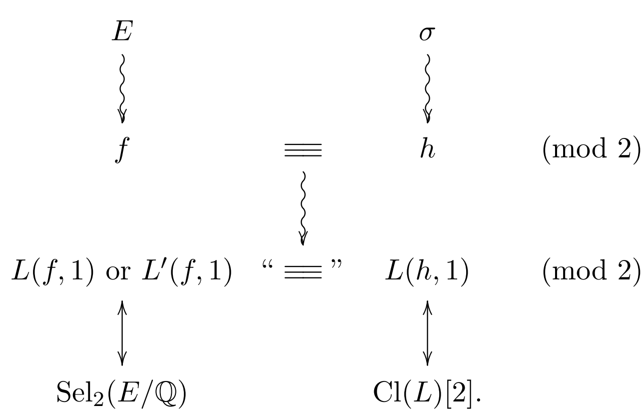
| $p$ | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 | 43 | 47 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a_p(f)$ | $-2$ | $-1$ | 1 | $-2$ | 1 | 4 | $-2$ | 0 | $-1$ | 0 | 7 | 3 | $-8$ | $-6$ | 8 |
| $b_p(h)$ | 0 | $-1$ | $-1$ | 0 | 1 | 0 | 0 | 0 | $-1$ | 0 | $-1$ | $-1$ | 0 | 0 | 2 |

TABLE 2. $E = 11a1$

5.4. **An interpretation of Theorem 1.4.** Under the BSD conjecture, Theorem 1.4 can be interpreted as a mod 2 congruence between (suitably defined algebraic parts of) the special values of $L$-functions of these two modular forms $f$ and $h$ (of different weights!):

- the 2-part of $L(f,1)$ or $L'(f,1)$ (depending on the sign) is related to $s_2(E/\mathbb{Q})$ by the BSD formula;
- the 2-part of $L(h,1)$ is related to the 2-part of the ideal class group $\mathrm{Cl}(L)$ by the class number formula, since $L(h,s) = \zeta_L(s)/\zeta(s)$ is the quotient of the Dekind zeta function of $L$ by the Riemann zeta function.

We depict this as follows.



We find this mod 2 congruence rather unusual: $L(E,s)$ is of symplectic type with the sign of the functional equation $\pm 1$ whereas the Artin $L$-function $L(\sigma,s)$ is of orthogonal type with the sign of the functional equation always $+1$. Congruences of this type is unique to $p=2$. As B. Mazur pointed out to us, this may suggest something much more general with intersections mod 2 of 2-adic eigenvarieties of different (symplectic versus orthogonal) reductive groups. Also, the point $s=1$ is the central critical point for $L(E,s)$ but there is *no* critical point for $L(\sigma,s)$ in the sense of Deligne, which makes even the formulation of the congruences more subtle.

We hope to formulate this type of mod 2 congruence between $L$-values more precisely in the future, which may shed light on Conjecture 1.9 by producing a direct mod 2 congruence between Heegner points on $E$ and the class group of $L$.

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