Lecture 3

Saturday, August 6, 2022 6:17 PM

§1. Gross-Zagier formula on Shimura curves

Fix $K/\mathbb{Q}$ im quad. Let $N = N^+ N^-$ where

$p \mid N^+ \Rightarrow p$ split in K.

$p \mid N^- \Rightarrow p$ inert in K

Assume N^- is square-free, then

$$\varepsilon(E_K) = -(-1)^{\#\{p \mid N^-\}}.$$

so $$\varepsilon(E_K) = \begin{cases} +1 & \#\{p \mid N^-\} \text{ odd.} \\ -1 & \#\{p \mid N^-\} \text{ even.} \end{cases}$$

(Heegner hypothesis $\Rightarrow N^- = 1 \Rightarrow \varepsilon(E_K) = -1$).

Def. K satisfies generalized Heegner hypothesis if N^- is a square-free product of even number of primes. ($\Rightarrow \varepsilon(E_K) = -1$).

Next: construct $y_K \in E(K)$ for K satisfies generalized Heegner hypothesis (Heeg^*) for short.

Since $\#\{p \mid N^-\}$ is even, we can make a quaternion alg $/\mathbb{Q}$:

Def. Let $B = B_{N^-}$ be the quaternion algebra $/\mathbb{Q}$. ramified at exactly $p \mid N^-$. So:

$$B_v \simeq \begin{cases} M_2(\mathbb{Q}_v) & v \nmid N^- \\ \ldots \end{cases}$$

$$B_p \simeq \begin{cases} M_2(\mathbb{Q}_p) & p \nmid N^- \\ \text{quaternion alg} & w \mid N^-. \end{cases}$$

Ex Consider $\mathbb{Q}(1, i, j, ij)$

$B(a,b) : i^2 = a,\ j^2 = b,\ ij = -ji.$ $\quad a, b \in \mathbb{Q}^\times$

If $a = 1$, then $B(1,b) \simeq M_2(\mathbb{Q})$.

$$i \longmapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$
$$j \longmapsto \begin{pmatrix} & 1 \\ b & \end{pmatrix}.$$
$$ij \longmapsto \begin{pmatrix} & 1 \\ -b & \end{pmatrix}.$$

If $a = p \equiv 1 \pmod 4$. $b = q$ s.t. $\left(\frac{q}{p}\right) \neq 1$. Then

$B(p,q)$ is ramified exactly at pq.

(e.g. $a = 5$, $b = 11$).

Def. An Eichler order of level N^+ is a subring $\mathcal{O} \underset{\text{finite index}}{\subseteq} \mathcal{O}_B$ (a max'l order of B) s.t.

$$\mathcal{O} \otimes \mathbb{Z}_p \simeq \begin{cases} M_0(N^+)_p & p \nmid N^- \\ \mathcal{O}_{B_p} & p \mid N^- \end{cases}$$

where $M_0(N^+) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : N^+ \mid c \right\} \subseteq M_2(\mathbb{Z})$

Def. $B_1^\times := \{ x \in B^\times : x\bar{x} = 1 \}$. (norm 1 elements)

Def. Define $\Gamma(N^+, N^-) := \mathcal{O}_1^\times \subseteq B_1^\times(\mathbb{Q})$.

Via $\Gamma(N^+, N^-) \hookrightarrow SL_2(\mathbb{R})$ it acts on $\mathcal{H}$.

Def We define the Shimura curve

$$X(N^+, N^-) := \Gamma(N^+, N^-) \backslash \mathcal{H}^*.$$

Rem. When $N^- = 1$. $\Gamma(N^+, N^-) = \Gamma_0(N)$

and $X(N^+, N^-) = X_0(N)$

Rem. Shimura curve is the simplest example of a Shimura variety. One can write it adelically as follows. Consider open compact subgps

$$K_p = (\mathcal{O} \otimes \mathbb{Z}_p)^\times$$

$$= \begin{cases} \Gamma_0(p^{\mathrm{ord}_p N^+}) \subseteq GL_2(\mathbb{Z}_p) & p \nmid N^- \\ \mathcal{O}_{B_p}^\times & p \mid N^- \end{cases}$$

Then $K = \prod_p K_p$ is an open compact subgp $\subseteq B^\times(\mathbb{A}_f)$.

and $K \cap B_1^\times(\mathbb{Q}) = \Gamma(N^+, N^-)$.

$$X(N^+, N^-) = \Gamma(N^+, N^-) \backslash \mathcal{H}^+$$

$$(\det K \twoheadrightarrow \hat{\mathbb{Z}}^\times) \qquad = B^\times(\mathbb{Q}) \backslash B^\times(\mathbb{A}_f) \times \mathcal{H}^\pm / K \cup \{\text{cusps}\}$$

As in the modular curve case, it has a moduli interpretation (in terms of abelian surfaces with $\mathcal{O}_B$-action + level structure)

$$X(1, N^-)(\mathbb{C}) = \{ A/\mathbb{C} \text{ abelian surface} + \mathcal{O}_B \hookrightarrow \mathrm{End}(A) \}.$$

$$\tau \in \mathcal{H} \longmapsto A_\tau = \mathbb{C}^2 / \mathcal{O}_B \binom{\tau}{1} \quad \text{where } \mathcal{O}_B \hookrightarrow M_2(\mathbb{C})$$

$$X(N^+, N^-)(\mathbb{C}) = \{ A \longrightarrow A' \text{ cyclic } \mathcal{O}_B\text{-isogeny of degree } (N^+)^2 \}.$$

Again one can define Heegner pts

$x_K = (A \to A') \in X(N^+, N^-)(H_K)$.

s.t. $A \sim E^2$, $A' \sim E'^2$, $\mathrm{End}(E) = \mathrm{End}(E') = \mathcal{O}_K$.

and using a modular parametrization (suitably normalized using Hodge divisor)

$$\varphi : X(N^+, N^-) \to E$$

to define $y_K \in E(K)$. (Heegner pt from Shimura curve)

Thm (Yuan-Zhang-Zhang) Assume K satisfies (Heeg^*).

$$L'(E_K, 1) = \frac{(f, f)}{|d_K|^{-\frac{1}{2}} \left|\frac{\mathcal{O}_K^\times}{\{\pm 1\}}\right|^2} \frac{\langle y_K, y_K\rangle}{\deg \varphi}.$$

$$\left(= \frac{\|w\|^2}{|d_K|^{-\frac{1}{2}} \left|\frac{\mathcal{O}_K^\times}{\{\pm 1\}}\right|^2} \frac{\deg \varphi_{X_0(N)}}{\deg \varphi} \langle y_K, y_K\rangle \right)$$

§2. Waldspurger formula

Now let us look at the case $\varepsilon(E_K) = +1$. This happens if

(Wald). N^- = square-free product of odd # of primes.

We would like to construct an analogue of $X(N^+, N^-)$ and y_K s.t.

$$y_K \overset{\wedge}{\longleftrightarrow} L(E_K, 1).$$

Since $\#\{p \mid N^-\}$ is odd, we can no longer make a quaternion alg ramified exactly at N^-. But we can consider $\{p \mid N^-\} \cup \{\infty\}$ of even cardinality!

<u>Def</u>. Let $B_{N^-\infty}$ be the quaternion ramified exactly

Def. Let $B_{N^-\infty}$ be the quaternion ramified exactly at N^- and ∞. So in particular:

$$B_{N^-\infty}(\mathbb{R}) \simeq \mathbb{H} \quad (\text{Hamilton's quaternion } i^2 = j^2 = -1).$$

Rem. $B/\mathbb{Q}$ is called indefinite if $B(\mathbb{R}) \simeq M_2(\mathbb{R})$

definite if $B(\mathbb{R}) \simeq \mathbb{H}$.

(as the quadratic form given by reduced norm

$(ui + vj + wij) \longmapsto -au^2 - bv^2 + ab\,w^2$

is indefinite / definite).

Since $B_1^\times(\mathbb{R}) = \mathbb{H}_1^\times \simeq SU_2 = \left\{ \begin{pmatrix} x & -y \\ \bar{y} & \bar{x} \end{pmatrix} \in M_2(\mathbb{C}) : |x|^2 + |y|^2 = 1 \right\}$

no longer acts on $\mathfrak{H}$, we can no longer construct a Shimura curve. Nevertheless, $B_1^\times(\mathbb{R})$ itself is compact, and the quotient

$$B^\times(\mathbb{Q}) \backslash B^\times(\mathbb{A}_f) / K.$$

is already finite. (A 0-dim'l Shimura variety).

Def. Let $\mathcal{O} \subseteq \mathcal{O}_{B_{N^-\infty}}$ be an Eichler order of level N^+. We define the Shimura set

$$X(N^+, N^-) = B^\times(\mathbb{Q}) \backslash B^\times(\mathbb{A}_f) \Big/ \prod_p (\mathcal{O} \otimes \mathbb{Z}_p)^\times.$$

Rem. Like $F^\times \backslash \mathbb{A}_{F,f}^\times / \prod_p \mathcal{O}_{F,p}^\times \simeq \mathrm{Cl}(\mathcal{O}_F)$

Rem. Like $F^\times \backslash \mathbb{A}_{F,f}^\times / \prod_p \mathcal{O}_{F,p}^\times \simeq Cl(\mathcal{O}_F)$ for a number field F.

$$X(N^+, N^-) \cong Cl(\mathcal{O})$$

$$= \{\text{right ideal classes of } \mathcal{O}\}.$$

Def. Fix an embedding $K \hookrightarrow B = B_{N^-\infty}$ s.t. $K \cap \mathcal{O} = \mathcal{O}_K$.

we have an map

$$Cl(\mathcal{O}_K) = K^\times \backslash \mathbb{A}_{K,f}^\times / \prod \mathcal{O}_{K,p}^\times \longrightarrow X(N^+, N^-).$$

$$I \subseteq \mathcal{O}_K \longmapsto I\mathcal{O}.$$

A point $x_K \in Cl(\mathcal{O}_K)$ is called a CM point (or Gross point)

(like the modular curve case, $Cl(\mathcal{O}_K)$ permutes X_K)

Def. Let $f = f_E \in S_2^{new}(N)$. Jacquet-Langlands gives a transfer f to a function on the finite set.

$$\varphi : X(N^+, N^-) \to \mathbb{C}.$$

s.t the Hecke eigenvalues of φ matches with f.

(this is analogue of the modular parametrization

$$\varphi : X(N^+, N^-) \to E \quad \text{under (Heeg}^*\text{)})$$

We define $$y_K = \sum_{x_K \in Cl(\mathcal{O}_K)} \varphi(x_K) \cdot |Aut(x_K)| \in \mathbb{C}$$

(known as Waldspurger's toric period)

$$\deg \varphi := \sum |\varphi(x)|^2 \cdot |Aut(x)|.$$

$$\deg \varphi := \sum_{x \in X(N^+, N^-)} |\varphi(x)|^2 \cdot |\mathrm{Aut}(x)|.$$

Here $\mathrm{Aut}(x) := \{ \gamma \in B^\times : \gamma x = x \} / \{\pm 1\}$.

Thm (Waldspurger, Gross) Assume K satisfies (Wald).

$$L(E_K, 1) = \frac{(f, f)}{|d_K|^{\frac{1}{2}} \left| \frac{\mathcal{O}_K^\times}{\{\pm 1\}} \right|^2} \cdot \frac{|y_K|^2}{\deg \varphi}.$$

Cor $r_{an}(E_K) = 0 \iff y_K \neq 0$.

Rem Waldspurger's formula gives an accessible way to study $L(E_K, 1)$: y_K is a finite sum of functions on CM points!

Rem Using Waldspurger formula and its p-adic variant:

Thm (Kriz - L.)

$$r_{an}(E_d : y^2 = x^3 + d) = \begin{cases} 0 & \text{for at least } \frac{1}{6} \text{ d's.} \\ 1 & \text{for at least } \frac{1}{6} \text{ d's.} \end{cases}$$

§3. Examples for Waldspurger formula

Ex. $E = X_0(11)$: $y^2 + y = x^3 - x^2 - 10x - 20$.

$$f = f_E = q \prod_{n \geq 1} (1 - q^n)^2 (1 - q^{11n})^2$$

$$= q - 2q^2 - q^3 + 2q^4 + q^5 \ldots$$

$K = \mathbb{Q}(\sqrt{-7})$ satisfies (Wald) (11 inert in K)

$N^+ = 1$, $N^- = 11$.

$B = B_{11, \infty}$ $\mathcal{O} = \mathcal{O}_B$

$B = B_{11,\infty}$. $\mathcal{O} = \mathcal{O}_B$.

If $N^- =$ prime, Eichler's mass formula says

$$\sum_{x \in \mathrm{Cl}(\mathcal{O}_B)} \frac{1}{|\mathrm{Aut}(x)|} = \frac{N-1}{12}.$$

so when $N = 11$. $\frac{N-1}{12} = \frac{5}{6} = \frac{1}{2} + \frac{1}{3}$.

$$\begin{array}{cc} \updownarrow & \updownarrow \\ x_1 & x_2 \end{array}$$

$X_{N^+, N^-} = \mathrm{Cl}(\mathcal{O}_B) = \mathbb{Z}/2$.

$\mathrm{Cl}(\mathcal{O}_K) = \mathbb{Z}/2 = \{x_1, x_2\}$

$\varphi : \begin{array}{l} x_1 \longrightarrow -1 \\ x_2 \longrightarrow 1 \end{array}$ has the correct Hecke eigenvalues.

Then $|y_K|^2 = 1$, $\deg \varphi = 5$.

$\Rightarrow L(E_K, 1) = \frac{(f, f)}{\sqrt{20}} \cdot \frac{1}{5} = \frac{(f, f)}{10\sqrt{5}}$

$r_{an}(E_K) = 0$.

Indeed $L(E_K, 1) = L(E, 1) \cdot L(E^{(-5)}, 1)$

$= (0.253841860\cdots) \cdot (0.65240264\ldots)$

$(f, f) = 3.70309\ldots$

$\frac{L(E_K, 1)}{(f, f)} = 0.044721\cdots = \frac{1}{10\sqrt{5}}$. !

Q. How does $|y_K|$ change when K varies?

A They fit into another modular form of wt $\frac{3}{2}$!

Def: Let $\mathrm{Cl}(\mathcal{O}) = \{x_1, \ldots x_n\}$. (right ideals of $\mathcal{O}$)

For each i, define $R_i = \{ \gamma \in B : b x_i \le x_i \}$ (left order of x_i).

define $S_i = \{ \gamma \in \mathbb{Z} + 2R_i : \mathrm{Tr}(\gamma) = 0 \}$

(rank 3 quadratic space $/\mathbb{Z}$).

Define a weight $\frac{3}{2}$ θ-series :

$$g_i = \frac{1}{2} \sum_{v \in S_i} q^{\mathrm{N} v}$$

($|d_K|$-th coefficient encode $X_K \longmapsto x_i \in X(N^+, N^-)$)

Let $g = \sum_{x_i \in \mathrm{Cl}(G)} \varphi(x_i)\, g_i$ weight $\frac{3}{2}$ modular form.

$$= \sum b_n q^n .$$

Then $b_{|d_K|} = y_K$. and thus

Thm $$L(E_K, 1) = \frac{(f, f)}{|d_K|^{\frac{1}{2}} \left| \frac{\mathcal{O}_K^\times}{\pm 1} \right|^2} \frac{|b_{|d_K|}|^2}{\deg \varphi}$$

Ex $E = X_0(11)$. $n = 2$

$$g_1 = \frac{1}{2} \sum_{x \equiv y \ (\mathrm{mod}\ 2)} q^{x^2 + 11y^2 + 11z^2}$$

$(x, y, z) = (3, 1, 0), (3, 0, 1)$

$$= \frac{1}{2} + q^4 + q^{11} + 2q^{12} + 2q^{15} + q^{16} + \boxed{2q^{20}} + 4q^{23} + 2q^{27} + 4q^{31}$$
$$+ 3q^{36} + 2q^{44} + 6q^{47} + 4q^{48} + 2q^{55} + 4q^{56} + 4q^{59} + 6q^{60} + 5q^{64}$$
$$+ 8q^{71} + 2q^{75} + 8q^{80} + 2q^{88} + 4q^{91} + 6q^{92} + q^{99} + 3q^{100} + \cdots$$

$$g_2 = \frac{1}{2} \sum_{\substack{x \equiv y \ (\mathrm{mod}\ 3) \\ y \equiv z \ (\mathrm{mod}\ 2)}} q^{\frac{x^2 + 11y^2 + 33z^2}{3}}$$

$$x \equiv y \pmod 3$$
$$y \equiv z \pmod 2$$

$(x, y, z) = (7, 1, 0), (2, 2, 0), (4, 1, 1).$

$$= \frac{1}{2} + q^3 + q^{12} + 3q^{15} + 3q^{16} + \boxed{3q^{20}} + 3q^{23} + q^{27} + 3q^{31} + 3q^{36} + 3q^{44} + 6q^{47} + 4q^{48} + 3q^{55} + 6q^{56} + 3q^{59} + 3q^{60} + 3q^{64} + 3q^{67} + 9q^{71} + 4q^{75} + 6q^{80} + 9q^{92} + 3q^{99} + 3q^{100} + \cdots$$

$$g = g_2 - g_1$$

$$q^3 - q^4 - q^{11} - q^{12} + q^{15} + 2q^{16} + \boxed{q^{20}} - q^{23} - q^{27} - q^{31} + q^{44} + q^{55} + 2q^{56} - q^{59} - 3q^{60} - 2q^{64} + \boxed{3q^{67}} + q^{71} + 2q^{75} - 2q^{80} - 2q^{88} - 4q^{91} + 3q^{92} + 2q^{99} + \cdots$$

BSD formula for $E_K \Longleftrightarrow$

$$|Ш(E/K)| \overset{?}{\doteq} b_{|d_K|}^2$$

Exercise Prove (as predicted by BSD)

$$Ш(E/\mathbb{Q}(\sqrt{-5})) = 0.$$

$$Ш(E/\mathbb{Q}(\sqrt{-17})) = (\mathbb{Z}/3)^2 \ !$$

Rem By studying the Fourier coefficients of

$$g = \sum b_n q^n$$

one can deduce strong nonvanishing results about $L(E^{(K)}, 1)$.

§4 Proof idea of Gross–Zagier formula

$$GZ: \quad L'(E_K, 1) \doteq (f_E, f_E) \cdot \langle y_K, y_K \rangle_{NT}.$$

$$L'(E_K, 1) = (f_E, Z_{an}) \qquad (f_E, f_E)\cdot\langle y_K, y_K\rangle_{NT} = (f_E, Z_{geo})$$

Idea: realize both sides as an integral against kernel functions

Analytic side: $\quad L(E_K, s) = L(f_E \times \Theta_K, s)$

Analytic side : $L(E_K, s) = L(f_E \times \underbrace{\theta_K}, s)$

weight 1 θ-series

$\overset{\text{Rankin}}{=} \left(f_E(\tau), \ \theta_K(\tau) \underbrace{E(\tau, s)} \right)$

weight 1 Eisenstein series

$\Rightarrow \ Z_{an} = \theta_K(\tau) \cdot \text{holomorphic projection}(E'(\tau, 1))$

Geometric side : define a generating series (assume $H_K = K$)

$$Z_{geo} = \sum_{n \geq 0} \langle x_K - \infty, \ T_n(x_K - \infty) \rangle_{NT} \, q^n$$

$$\Rightarrow \ (f_{E'}, Z_{geo}) \doteq (f_E, f_E) \langle y_K, y_K \rangle_{NT}.$$

Comparison prove $Z_{an} \doteq Z_{geo}$

This is done by computing the Fourier coefficients of both sides completely explicitly !

(advantage : both sides are independent of E)

Rem Explicit computation can be simplified / conceptualized nowadays using more sophiscated tools from analysis / arithmetic geometry / representation theory.