Beilinson–Bloch conjecture and arithmetic inner product formula

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The BSD conjecture

- $E: y^2 = x^3 + Ax + B$ an elliptic curve over $\mathbb{Q}$.
- Algebraic rank: the rank of the finitely generated abelian group $E(\mathbb{Q})$

$$r_{\mathrm{alg}}(E) := \mathrm{rank}\, E(\mathbb{Q}).$$

- Analytic rank: the order of vanishing of $L(E, s)$ at the central point $s = 1$

$$r_{\mathrm{an}}(E) := \mathrm{ord}_{s=1} L(E, s).$$

Conjecture (Birch–Swinnerton-Dyer, 1960s)

(1) (Rank)
$$r_{\mathrm{an}}(E) \stackrel{?}{=} r_{\mathrm{alg}}(E),$$

(2) (Leading coefficient) For $r = r_{\mathrm{an}}(E)$,
$$\frac{L^{(r)}(E,1)}{r!} \stackrel{?}{=} \frac{\Omega(E)R(E)\prod_p c_p(E)\cdot|\text{Ш}(E)|}{|E(\mathbb{Q})_{\mathrm{tor}}|^2}$$
where $R(E) = \det(\langle P_i, P_j\rangle_{\mathrm{NT}})_{r\times r}$ is the regulator for the Néron–Tate height pairing
$$\langle\ ,\ \rangle_{\mathrm{NT}} : E(\mathbb{Q}) \times E(\mathbb{Q}) \to \mathbb{R}$$
and $\text{Ш}(E)$ is the Tate–Shafarevich group.

Remark (Tate, *The Arithmetic of Elliptic Curves*, 1974)

> This remarkable conjecture relates the behavior of a function L at a point where it is not at present known to be defined to the order of a group Ш which is not known to be finite!

What is known about BSD?

The BSD conjecture is still widely open in general, but much progress has been made in the rank 0 or 1 case.

Theorem (Gross–Zagier, Kolyvagin, 1980s)

$$r_{\mathrm{an}}(E) = 0 \Rightarrow r_{\mathrm{alg}}(E) = 0, \quad r_{\mathrm{an}}(E) = 1 \Rightarrow r_{\mathrm{alg}}(E) = 1,$$

Remark. When $r = r_{\mathrm{an}}(E) \in \{0, 1\}$, many cases of the formula for $L^{(r)}(E, 1)$ are known.

The proof combines two inequalities:

(1) (Gross–Zagier formula)

$$r_{\mathrm{an}}(E) = 1 \Rightarrow r_{\mathrm{alg}}(E) \geq 1.$$

(2) (Kolyvagin's Euler system)

$$r_{\mathrm{an}}(E) \in \{0, 1\} \Rightarrow r_{\mathrm{alg}}(E) \leq r_{\mathrm{an}}(E).$$

Both steps rely on Heegner points on modular curves.

The Beilinson–Bloch conjecture

- X: smooth projective variety over a number field K.
- $\mathrm{CH}^m(X)$: the Chow group of algebraic cycles of codimension m on X.
- $\mathrm{CH}^m(X)^0 \subseteq \mathrm{CH}^m(X)$: the subgroup of geometrically cohomologically trivial cycles.
- Beilinson–Bloch height pairing

$$\langle\ ,\ \rangle_{\mathrm{BB}} : \mathrm{CH}^m(X)^0 \times \mathrm{CH}^{\dim X+1-m}(X)^0 \to \mathbb{R}.$$

- $L(H^{2m-1}(X), s)$: the motivic L-function for $H^{2m-1}(X_{\bar{K}}, \mathbb{Q}_\ell)$.

Conjecture (Beilinson–Bloch, 1980s)

(1) (Rank) $\quad \mathrm{ord}_{s=m} L(H^{2m-1}(X), s) \stackrel{?}{=} \mathrm{rank}\, \mathrm{CH}^m(X)^0.$

(2) (Leading coefficient) $\quad L^{(r)}(H^{2m-1}(X), m) \stackrel{?}{\sim} \det(\langle Z_i, Z_j' \rangle_{\mathrm{BB}})_{r\times r}$

Example ($X/K = E/\mathbb{Q}$ and $m = 1$)

BB recovers the BSD conjecture as

$$\mathrm{CH}^1(E)^0 \simeq E(\mathbb{Q}), \quad L(H^1(E), s) = L(E, s), \quad \langle\ ,\ \rangle_{\mathrm{BB}} = -\langle\ ,\ \rangle_{\mathrm{NT}}.$$

Remark. In general both sides are only conditionally defined!

(1) $L(H^{2m-1}(X), s)$ is not known to be analytically continued to the central point $s = m$.

(2) $\mathrm{CH}^m(X)^0$ is not known to be finitely generated.

Testable BB conjecture: X = Shimura varieties

- Langlands–Kottwitz/Langlands–Rapoport: express the motivic L-functions of Shimura varieties $X = \mathrm{Sh}_G$, as a product of automorphic L-functions $L(s,\pi)$ on G,

$$L(H^{2m-1}(X), s+m) = \prod_{\pi} L(s+1/2, \pi).$$

- Assume from now (the most interesting case):
  (1) $2m-1 = \dim X$ (middle cohomology).
  (2) π is tempered cuspidal.
- Analytic properties of $L(s,\pi)$ can be established.
- $\mathrm{CH}^m(X)^0$ is not known to be finitely generated, but we can test if it is nontrivial.

Unconditional prediction of BB conjecture, in the same spirit of Gross–Zagier:

Conjecture (Beilinson–Bloch for Shimura varieties)

$$\mathrm{ord}_{s=1/2} L(s,\pi) = 1 \overset{?}{\Longrightarrow} \mathrm{rank}\, \mathrm{CH}^m(X)^0_\pi \geq 1.$$

What is known about BB?

Conjecture (Beilinson–Bloch for Shimura varieties)

$$\mathrm{ord}_{s=1/2} L(s,\pi) = 1 \stackrel{?}{\Longrightarrow} \mathrm{rank}\, \mathrm{CH}^m(X)^0_\pi \geq 1.$$

Remark. Conjecture was only known for:

(1) X = modular curves (Gross–Zagier)

(2) X = Shimura curves (S. Zhang, Kudla–Rapoport–Yang, Yuan–Zhang–Zhang, Liu).

(3) $X = \mathrm{U}(1,1) \times \mathrm{U}(2,1)$ Shimura threefolds and π = endoscopic (Xue).

Theorem A (L.-Liu, impressionist version)

Conjecture holds for $\mathrm{U}(2m-1,1)$-Shimura varieties and π satisfying local assumptions.

Remark (Kudla, *Central derivatives of Eisenstein series and height pairings*, 1997)

> Our results suggest that it might be possible to establish Gross-Zagier type formulas for Shimura varieties of type $O(2n-1,2)$ and $U(2n-1,1)$, and in certain other sporadic cases, along the same lines. One of these sporadic cases — that of the central derivative of the triple product L-function attached to a triple of modular forms of weight 2 — was considered jointly with Gross and Zagier some time ago [12], [13], and was the origin of this program. Many very serious problems currently block the way, but one might at least hope to obtain further evidence for our conjectures in the higher dimensional cases.

Unitary Shimura varieties X

- E/F: CM extension of a totally real number field.
- $\mathbb{V}$: totally definite incoherent $\mathbb{A}_E/\mathbb{A}_F$-hermitian space of rank n.
- Incoherent: $\mathbb{V}$ is not the base change of a global E/F-hermitian space, or equivalently $\prod_v \varepsilon(\mathbb{V}_v) = -1$, where $\mathbb{V}_v := \mathbb{V} \otimes_{\mathbb{A}_F} F_v$.
- Any place $w|\infty$ of F gives a nearby coherent E/F-hermitian space V such that
$$V_v \cong \mathbb{V}_v, v \neq w, \quad \text{but } V_w \text{ has signature } (n-1, 1).$$
- $G = \mathrm{U}(\mathbb{V})$.
- $K \subseteq G(\mathbb{A}_F^\infty) \cong \mathrm{U}(V)(\mathbb{A}_F^\infty)$: open compact subgroup.
- X: unitary Shimura variety of dimension $n-1$ over E such that for any place $w|\infty$ inducing $\iota_w : E \hookrightarrow \mathbb{C}$,
$$X(\mathbb{C}) = \mathrm{U}(V)(F)\backslash[\mathbb{D} \times \mathrm{U}(V)(\mathbb{A}_F^\infty)/K],$$
where
$$\mathbb{D} := \{z \in \mathbb{C}^{n-1} : |z| < 1\} \cong \frac{\mathrm{U}(n-1,1)}{\mathrm{U}(n-1) \times \mathrm{U}(1)}.$$
- X is a Shimura variety of abelian type.
- Its étale cohomology and L-function are computed in the forthcoming work of Kisin–Shin–Zhu, under the help of the endoscopic classification for unitary groups (Mok, Kaletha–Minguez–Shin–White).

Automorphic representations π

- $W = E^{2m}$: the standard E/F-skew-hermitian space with matrix $\begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix}$.
- $\mathrm{U}(W)$: quasi-split unitary group of rank $n = 2m$.
- π: cuspidal automorphic representation of $\mathrm{U}(W)(\mathbb{A}_F)$.

Assumptions.

(1) E/F is split at all 2-adic places and $F \neq \mathbb{Q}$. If $v \nmid \infty$ is ramified in E, then v is unramified over $\mathbb{Q}$. Assume that $E/\mathbb{Q}$ is Galois or contains an imaginary quadratic field (for simplicity).

(2) For $v|\infty$, π_v is the holomorphic discrete series with Harish-Chandra parameter $\{\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{-n+3}{2}, \frac{-n+1}{2}\}$.

(3) For $v \nmid \infty$, π_v is tempered.

(4) For $v \nmid \infty$ ramified in E, π_v is spherical with respect to the stabilizer of $O_{E_v}^{2m}$.

(5) For $v \nmid \infty$ inert in E, π_v is unramified or almost unramified. If π_v is almost unramified, then v is unramified over $\mathbb{Q}$.

Remark. π_v is almost unramified: π_v has a nonzero Iwahori-fixed vector and its Satake parameter contains $\{q_v, q_v^{-1}\}$ and $2m-2$ complex numbers of norm 1. Equivalently, the theta lift of π_v to the non-quasi-split unitary group of same rank is spherical with respect to the stabilizer of an almost self-dual lattice.

Main result A: BB conjecture

Let $S_\pi = \{v \text{ inert} : \pi_v \text{ almost unramified}\}$. Then under Assumptions, the global root number for the (complete) standard L-function $L(s,\pi)$ equals

$$\varepsilon(\pi) = (-1)^{|S_\pi|} \cdot (-1)^{m[F:\mathbb{Q}]}$$

by epsilon dichotomy (Harris–Kudla–Sweet, Gan–Ichino). When $\mathrm{ord}_{s=1/2} L(s,\pi) = 1$:

- $\varepsilon(\pi) = -1$,
- $\mathbb{V} = \mathbb{V}_\pi$: totally definite incoherent space of rank $n = 2m$ such that for $v \nmid \infty$,
  $$\varepsilon(\mathbb{V}_v) = -1 \text{ exactly for } v \in S_\pi.$$
- Associated unitary Shimura variety X of dimension $n - 1 = 2m - 1$ over E.
- $\mathrm{CH}^m(X)^0_\pi$ the localization of $\mathrm{CH}^m(X)^0_{\mathbb{C}}$ at the maximal ideal $\mathfrak{m}_\pi$ of the Hecke algebra associated to π.

Theorem A (L.–Liu, 2020, 2021)

Let π be a cuspidal automorphic representation of $\mathrm{U}(W)(\mathbb{A}_F)$ satisfying Assumptions. Then the implication

$$\mathrm{ord}_{s=1/2} L(s,\pi) = 1 \Longrightarrow \mathrm{rank}\, \mathrm{CH}^m(X)^0_\pi \geq 1$$

holds when the level $K \subseteq G(\mathbb{A}_F^\infty)$ is sufficiently small.

Example: Symmetric power L-function of elliptic curves

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Example

Let A/F be a modular elliptic curve without CM such that

(1) $\mathrm{Sym}^{2m-1} A$ is modular (Newton–Thorne, Clozel–Thorne, Kim–Shahidi, ...)

(2) A has bad reduction only at places v split in E
(e.g., $A =$ LMFDB.64.1-a7 over $F = \mathbb{Q}(\zeta_7 + \zeta_7^{-1}) \subseteq E = \mathbb{Q}(\zeta_7)$).

Assume that E/F satisfies Assumptions. Then there exists π satisfying Assumptions such that

$$L(s+1/2,\pi) = L(\mathrm{Sym}^{2m-1} A_E, s+m).$$

As $S_\pi = \varnothing$ and $\varepsilon(\pi) = (-1)^{m[F:\mathbb{Q}]}$, Theorem A applies to π when $m[F:\mathbb{Q}]$ is odd.

Nontrivial cycles constructed via the method of arithmetic theta lifting (Kudla, Liu).
Next: a baby example of Heegner points.

The Gross–Zagier formula

- Modular curve

$$X_0(N) = \Gamma_0(N)\backslash\mathbb{H} \cup \{\text{cusps}\} = \{(E_1 \xrightarrow[N\text{-isogeny}]{\text{cyclic}} E_2)\}$$

- For certain imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-d})$, get a Heegner divisor

$$Z(d) := \{(E_1 \to E_2) \text{ with endomorphisms by } O_K\} \in \mathrm{CH}^1(X_0(N)).$$

- The theory of complex multiplication: $Z(d)$ is defined over K.
- $E/\mathbb{Q}$ elliptic curve of conductor N has a modular parametrization

$$\varphi_E : X_0(N) \to E.$$

- Get a Heegner point

$$P_K \in \varphi_E(Z(d) - \deg Z(d) \cdot \infty) \in E(K).$$

Theorem (Gross–Zagier, 1980s)

Up to simpler nonzero factors,

$$L'(E_K, 1) \sim \langle P_K, P_K \rangle_{\mathrm{NT}}.$$

Remark

Choosing K suitably gives the implications $r_{\mathrm{an}}(E) = 1 \Rightarrow r_{\mathrm{alg}}(E) \geq 1$.

Generating series of Heegner points

Take $P_d = \mathrm{tr}_{K/\mathbb{Q}} P_K \in E(\mathbb{Q})$. It may depend on the choice of d, even when $E(\mathbb{Q}) \cong \mathbb{Z}$.

Example ($E = 37a1 = X_0^+(37) : y^2 + y = x^3 - x$)

- $E(\mathbb{Q}) \cong \mathbb{Z}$ with a generator $P = (0, 0)$.
- E corresponds to the modular form $f \in S_2(37)$,

$$f = \sum_{n \geq 1} a_n q^n = q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 - q^7 + 6q^9 + 4q^{10} - 5q^{11} - 6q^{12} - 2q^{13} + \cdots$$

- Table of Heegner points P_d:

d	3	4	7	11	12	16	27	$\cdots$	67	$\cdots$
P_d	$(0, -1)$	$(0, -1)$	$(0, 0)$	$(0, -1)$	$(0, 0)$	$(1, 0)$	$(-1, -1)$	$\cdots$	$(6, -15)$	$\cdots$
c_d	-1	-1	1	-1	1	2	3	$\cdots$	-6	$\cdots$

where $P_d = c_d \cdot P$.

Miracle. The coefficients c_d appear as the Fourier coefficients of $\phi \in S^+_{3/2}(4 \cdot 37)$,

$$\phi = \sum_{d \geq 1} c_d q^d = -q^3 - q^4 + q^7 - q^{11} + q^{12} + 2q^{16} + 3q^{27} + \cdots - 6q^{67} + \cdots,$$

which maps to f under the Shimura–Waldspurger–Kohnen correspondence

$$\theta : S^+_{3/2}(4N) \to S_2(N), \quad \theta(\phi) = f.$$

Arithmetic theta lifting

- The generating series of Heegner points

$$\sum_{d\geq 1} P_d \cdot q^d = \sum_{d\geq 1} c_d P \cdot q^d = \phi \cdot P \in S^+_{3/2}(4\cdot 37) \otimes E(\mathbb{Q})_{\mathbb{C}}$$

  is a modular form valued in $E(\mathbb{Q})_{\mathbb{C}}$.
- More generally, we may define a generating series of Heegner divisors on $X_0(N)$,

$$Z := \sum_{d\geq 0} Z(d) q^d \in M_{3/2}(4N) \otimes \mathrm{CH}^1(X_0(N))_{\mathbb{C}},$$

  which may be viewed as an arithmetic theta series.
- Use Z as the kernel to define arithmetic theta lifting

$$\Theta(\phi) := (Z, \phi)_{\mathrm{Pet}} \in \mathrm{CH}^1(X_0(N))^0_{f,\mathbb{C}} = E(\mathbb{Q})_{\mathbb{C}}.$$

- Now $\Theta(\phi)$ does not depend on any particular choice of d or K.

Theorem (Gross–Kohnen–Zagier, 1980s)

Up to simpler nonzero factors,

$$L'(E,1) \sim \langle \Theta(\phi), \Theta(\phi) \rangle_{\mathrm{NT}}.$$

Special cycles on X

- For any $y \in V$ with $(y, y) > 0$. Its orthogonal complement $V_y \subseteq V$ has rank $n-1$. The embedding $\mathrm{U}(V_y) \hookrightarrow \mathrm{U}(V)$ defines a Shimura subvariety of codimension 1
$$\mathrm{Sh}_{\mathrm{U}(V_y)} \to X = \mathrm{Sh}_{\mathrm{U}(V)}\,.$$
- For any $x \in V(\mathbb{A}_F^\infty)$ with $(x, x) \in F_{>0}$, there exists $y \in V$ and $g \in \mathrm{U}(V)(\mathbb{A}_F^\infty)$ such that $y = gx$. Define the special divisor
$$Z(x) \to X$$
to be the g-translate of $\mathrm{Sh}_{\mathrm{U}(V_y)}$.
- For any $\mathbf{x} = (x_1, \ldots, x_m) \in V(\mathbb{A}_F^\infty)^m$ with $T(\mathbf{x}) = ((x_i, x_j)) \in \mathrm{Herm}_m(F)_{>0}$, define the special cycle (of codimension m)
$$Z(\mathbf{x}) = Z(x_1) \cap \cdots \cap Z(x_m) \to X.$$
- More generally, for a Schwartz function $\varphi \in \mathcal{S}(V(\mathbb{A}_F^\infty)^m)^K$ and $T \in \mathrm{Herm}_m(F)_{>0}$, define the weighted special cycle
$$Z_\varphi(T) = \sum_{\substack{\mathbf{x} \in K \backslash V(\mathbb{A}_F^\infty)^m \\ T(\mathbf{x}) = T}} \varphi(\mathbf{x}) Z(\mathbf{x}) \in \mathrm{CH}^m(X)_{\mathbb{C}}.$$
- With extra care, we can also define $Z_\varphi(T) \in \mathrm{CH}^m(X)_{\mathbb{C}}$ for any $T \in \mathrm{Herm}_m(F)_{\geq 0}$.

Arithmetic theta lifting

Define Kudla's generating series of special cycles

$$Z_\varphi(\tau) = \sum_{T \in \mathrm{Herm}_m(E)_{\geq 0}} Z_\varphi(T) q^T.$$

Conjecture (Kudla's modularity, 1990s)

The formal generating series $Z_\varphi(\tau)$ converges absolutely and defines a modular form on $\mathrm{U}(W)$ valued in $\mathrm{CH}^m(X)_{\mathbb{C}}$.

Remark

(1) The analogous modularity in Betti cohomology is known (Kudla–Millson, 1980s).

(2) Conjecture is known for $m = 1$. For general m, the modularity follows from the absolute convergence (Liu, 2011).

(3) The analogous conjecture for orthogonal Shimura varieties over $\mathbb{Q}$ is known (Bruinier–Raum, 2014).

(4) Conjecture is known when $E = \mathbb{Q}(\sqrt{-d})$ for $d = 1, 2, 3, 7, 11$ (Xia, 2021).

Assuming Kudla's modularity conjecture, for holomorphic $\phi \in \pi$, define arithmetic theta lifting

$$\Theta_\varphi(\phi) = (Z_\varphi(\tau), \phi)_{\mathrm{Pet}} \in \mathrm{CH}^m(X)^0_\pi.$$

Main result B: Arithmetic inner product formula

Theorem B (L.–Liu, 2020, 2021)

Let π be a cuspidal automorphic representation of $\mathrm{U}(W)(\mathbb{A}_F)$ satisfying Assumptions. Assume $\varepsilon(\pi) = -1$. Assume Kudla's modularity. Then for any holomorphic $\phi \in \pi$ and $\varphi \in \mathcal{S}(\mathbb{V}(\mathbb{A}_F^\infty)^m)$, up to simpler factors depending on ϕ and φ,

$$L'(1/2, \pi) \sim \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi) \rangle_{\mathrm{BB}}.$$

Remark. The simpler factors can be further made explicit. For example, if

- π: unramified or almost unramified at all finite places,
- $\phi \in \pi$: holomorphic newform such that $(\phi, \overline{\phi})_\pi = 1$,
- φ: characteristic function of self-dual or almost self-dual lattices at all finite places.

Then
$$\frac{L'(1/2,\pi)}{\prod_{i=1}^{2m} L(i, \eta_{E/F}^i)} C_m^{[F:\mathbb{Q}]} \prod_{v \in S_\pi} \frac{q_v^{m-1}(q_v+1)}{(q_v^{2m-1}+1)(q_v^{2m}-1)} = (-1)^m \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi) \rangle_{\mathrm{BB}},$$

where $C_m = 2^{-2m} \pi^{m^2} \frac{\Gamma(1)\cdots\Gamma(m)}{\Gamma(m+1)\cdots\Gamma(2m)}$.

Remark

- Riemann hypothesis predicts $L'(1/2, \pi) \geq 0$.
- Beilinson's Hodge index conjecture predicts $(-1)^m \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi) \rangle_{\mathrm{BB}} \geq 0$.

Compatible with our formula!

Summary

BSD conjecture	BB conjecture
Modular curves $X_0(N)$	Unitary Shimura varieties X
Heegner points $Z(d) \in \mathrm{CH}^1(X_0(N))_{\mathbb{C}}$	Special cycles $Z_\varphi(T) \in \mathrm{CH}^m(X)_{\mathbb{C}}$
$Z = \sum_d Z(d) q^d$	$Z_\varphi = \sum_T Z_\varphi(T) q^T$
$\Theta(\phi) \in E(\mathbb{Q})_{\mathbb{C}}$	$\Theta_\varphi(\phi) \in \mathrm{CH}^m(X)^0_\pi$
Gross–Zagier formula $L'(E,1) \sim \langle \Theta(\phi), \Theta(\phi) \rangle_{\mathrm{NT}}$	Arithmetic inner product formula $L'(1/2,\pi) \sim \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi) \rangle_{\mathrm{BB}}$

Proof strategy: doubling method

- Doubling method (Piatetski-Shapiro–Rallis, Yamana)

$$L(s+1/2,\pi) \sim (\phi\otimes\overline{\phi}, \mathrm{Eis}(s,g))_{\mathrm{U}(W)^2},$$

  where $\mathrm{Eis}(s,g)$ is a Siegel Eisenstein series on $\mathrm{U}(W\oplus W)$.
- By definition $\Theta_\varphi(\phi) = (Z_\varphi, \phi)_{\mathrm{Pet}}$ gives

$$\langle \Theta_\varphi(\phi), \Theta_\varphi(\phi)\rangle_{\mathrm{BB}} = (\phi\otimes\overline{\phi}, \langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB}})_{\mathrm{U}(W)^2}.$$

- To prove $L'(1/2,\pi) \sim \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi)\rangle_{\mathrm{BB}}$, it suffices to compare

$$\mathrm{Eis}'(0,g) \overset{?}{=} \langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB}}.$$

  This can be viewed as an arithmetic Siegel–Weil formula.
- The Beilinson–Bloch height pairing is a sum of local indexes

$$\langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB}} = \sum_v \langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB},v}.$$

- The nonsingular Fourier coefficient decomposes as

$$\mathrm{Eis}'_T(0,g) = \sum_v \mathrm{Eis}'_{T,v}(0,g)$$

Proof strategy: comparison

- Nonsingular terms: it suffices to compare

$$\mathrm{Eis}'_{T,v}(0,g) \stackrel{?}{=} \langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB},T,v}.$$

- $m = 1$: both sides can be computed explicitly.
- Explicit computation infeasible for general m.
- $v \nmid \infty$ (source of Assumptions)
  1. relate $\langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB},T,v}$ to arithmetic intersection numbers (on regular integral models of PEL type) : sophisticated use of Hecke operators and proof of vanishing of certain $\mathfrak{m}_\pi$-localized ℓ-adic cohomology of integral models.
  2. relate arithmetic intersection numbers to $\mathrm{Eis}'_{T,v}(0,g)$: proof of Kudla–Rapoport conjecture (L.–Zhang) and our more recent extension to ramified places for exotic smooth integral models.
- $v \mid \infty$:
  1. archimedean arithmetic Siegel–Weil formula (Liu, Garcia–Sankaran).
  2. avoidance of holomorphic projections.

To finish:

- Kill singular terms on both sides: Prove the existence of special $\varphi \in \mathcal{S}(V(\mathbb{A}_F^\infty)^m)$ with regular support at two split places with nonvanishing local zeta integrals.
- Theorem B for special φ: comparison of nonsingular terms.
- Theorem B for arbitrary φ: multiplicity one of doubling method (tempered case).
- Theorem A: same computation without Kudla's modularity (proof by contradiction).

Remarks on Assumptions

- When $v \nmid \infty$, the local index $\langle \ , \ \rangle_{\mathrm{BB},v}$ is defined as a ℓ-adic linking number. It is defined on a certain subspace $\mathrm{CH}^m(X)^{\langle \ell \rangle} \subseteq \mathrm{CH}^m(X)^0$ (conjecturally equal) and its independence on ℓ is not known in general.
- Find a Hecke operator $t \notin \mathfrak{m}_\pi$ such that $t^* Z \in \mathrm{CH}^m(X)^{\langle \ell \rangle}$, so BB height is defined.
- Find another Hecke operator $s \notin \mathfrak{m}_\pi$, so BB height of $s^* t^* Z$ can be computed in terms of the arithmetic intersection number of a nice extension $\mathcal{Z}$ on $\mathcal{X}$. Here $\mathcal{X}$ is a regular integral model of a related unitary Shimura variety of PEL type. This step requires to prove certain vanishing of $\mathfrak{m}_\pi$-localized ℓ-adic cohomology of $\mathcal{X}$.
- Kudla–Rapoport conjecture: arithmetic intersection number equals $\mathrm{Eis}'_{T,v}(0, g)$.
- The ℓ-independence of $\langle Z_\varphi, Z_\varphi \rangle_{\mathrm{BB},T,v}$ then follows.
- Construction of Hecke operators and the proof of Kudla–Rapoport conjecture requires Assumptions.
- $F \neq \mathbb{Q}$ is needed to prove vanishing of $\mathfrak{m}_\pi$-localized cohomology of integral models with Drinfeld level structures at split places (with input from Mantovan, Caraiani–Scholze).