CHOW GROUPS AND L-DERIVATIVES OF AUTOMORPHIC MOTIVES FOR UNITARY GROUPS, II.

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Abstract. In this article, we improve our main results from [LL21] in two directions: First, we allow ramified places in the CM extension E/F at which we consider representations that are spherical with respect to a certain special maximal compact subgroup, by formulating and proving an analogue of the Kudla–Rapoport conjecture for exotic smooth Rapoport–Zink spaces. Second, we lift the restriction on the components at split places of the automorphic representation, by proving a more general vanishing result on certain cohomology of integral models of unitary Shimura varieties with Drinfeld level structures.

Contents

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1. Introduction

In 1986, Gross and Zagier [GZ86] proved a remarkable formula that relates the Néron–Tate heights of Heegner points on a rational elliptic curve to the central derivative of the corresponding Rankin–Selberg L-function. A decade later, Kudla [Kud97] revealed another striking relation between Gillet–Soulé heights of special cycles on Shimura curves and derivatives of Siegel Eisenstein series of genus two, suggesting an arithmetic version of theta lifting and the Siegel–Weil formula (see, for example, [Kud02, Kud03]). This was later further developed in his joint work with Rapoport and Yang [KRY06]. For the higher dimensional case, in a series of papers starting from the late 1990s, Kudla and Rapoport developed the theory of special cycles on integral models of Shimura varieties for GSpin groups in lower rank cases and for unitary groups of arbitrary ranks [KR11, KR14]. They also studied special cycles on the relevant Rapoport–Zink spaces over non-archimedean local fields. In particular, they formulated a conjecture relating the arithmetic intersection number of special cycles on the unitary Rapoport–Zink space to the first derivative of local Whittaker functions [KR11, Conjecture 1.3].

In his thesis work [Liu11a, Liu11b], one of us studied special cycles as elements in the Chow group of the unitary Shimura variety over its reflex field (rather than in the arithmetic Chow group of a certain integral model) and the Beilinson–Bloch height of the arithmetic theta lifting (rather than the Gillet–Soulé height). In particular, in the setting of unitary groups, he proposed an explicit conjectural formula for the Beilinson–Bloch height in terms of the central L-derivative and local doubling zeta integrals. Such formula is completely parallel to the Rallis inner product formula [Ral84], which computes the Petersson inner product of the global theta lifting, hence was named *arithmetic inner product formula* in [Liu11a], and can be regarded as a higher dimensional generalization of the Gross–Zagier formula.[1] In the case of $\mathrm{U}(1,1)$ over an arbitrary CM extension, such conjectural formula was completely confirmed in [Liu11b], while the case for $\mathrm{U}(r,r)$ with $r \geqslant 2$ is significantly harder. Recently, the Kudla–Rapoport conjecture has been proved by W. Zhang and one of us in [LZa];[2] and it has become possible to attack the cases for higher rank groups.

In [LL21], we proved that for certain cuspidal automorphic representations π of $\mathrm{U}(r,r)$, if the central derivative $L'(1/2,\pi)$ is nonvanishing, then the π-nearly isotypic localization of the Chow group of a certain unitary Shimura variety over its reflex field does not vanish. This proved part of the Beilinson–Bloch conjecture for Chow groups and L-functions (see [LL21, Section 1] for a precise formulation in our setting). Moreover, assuming the modularity of Kudla's generating functions of special cycles, we further proved the arithmetic inner product formula relating $L'(1/2,\pi)$ and the height of arithmetic theta liftings. In this article, we improve the main results from [LL21] in two directions: First, we allow ramified places in the CM extension E/F at which we consider representations that are spherical with respect to a certain special maximal compact subgroup, by formulating and proving an analogue of the Kudla–Rapoport conjecture for exotic smooth Rapoport–Zink spaces. Second, we lift the restriction on the components at split places of the automorphic representation, by proving

[1] By "generalization of the Gross–Zagier formula", we simply mean that they are both formulae relating Beilinson–Bloch heights of special cycles and central derivatives of L-functions. However, from a representation-theoretical point of view, the more accurate generalization of the Gross–Zagier formula should be the arithmetic Gan–Gross–Prasad conjecture.

[2] We remark that during the referee process of this article, the Kudla–Rapoport conjecture in the orthogonal case has also been formulated and proved by the same group of authors [LZb].

a more general vanishing result on certain cohomology of integral models of unitary Shimura varieties with Drinfeld level structures. However, for technical reasons, we will still assume $F \neq \mathbb{Q}$ (see Remark 4.33).

1.1. **Main results.** Let E/F be a CM extension of number fields with the complex conjugation $\mathtt{c}$. Denote by $\mathtt{V}_F^{(\infty)}$ and $\mathtt{V}_F^{\mathrm{fin}}$ the set of archimedean and non-archimedean places of F, respectively. Denote by $\mathtt{V}_F^{\mathrm{spl}}$, $\mathtt{V}_F^{\mathrm{int}}$, and $\mathtt{V}_F^{\mathrm{ram}}$ the subsets of $\mathtt{V}_F^{\mathrm{fin}}$ of those that are split, inert, and ramified in E, respectively.

Take an even positive integer $n = 2r$. We equip $W_r := E^n$ with the skew-hermitian form (with respect to the involution $\mathtt{c}$) given by the matrix $\begin{pmatrix} & 1_r \\ -1_r & \end{pmatrix}$. Put $G_r := \mathrm{U}(W_r)$, the unitary group of W_r, which is a quasi-split reductive group over F. For every $v \in \mathtt{V}_F^{\mathrm{fin}}$, we denote by $K_{r,v} \subseteq G_r(F_v)$ the stabilizer of the lattice $O_{E_v}^n$, which is a special maximal compact subgroup.

We start from an informal discussion on the arithmetic inner product formula. Let π be a tempered automorphic representation of $G_r(\mathbb{A}_F)$, which by theta dichotomy, gives rise to a unique up to isomorphism hermitian space V_π of rank n over $\mathbb{A}_E$. It is known that the hermitian space V_π is *coherent* (resp. *incoherent*), that is, V_π is (resp. is not) the base change of a hermitian space over E, if and only if the global root number $\varepsilon(\pi)$ equals 1 (resp. -1). When $\varepsilon(\pi) = 1$, we have the global theta lifting of π, which is a space of automorphic forms on $\mathrm{U}(V_\pi)(\mathbb{A}_F)$; and the famous Rallis' inner product formula [Ral84] computes the Petersson inner product of the global theta lifting in terms of the central L-value $L(\frac{1}{2}, \pi)$ of π. When $\varepsilon(\pi) = -1$, we have the arithmetic theta lifting of π, which is a space of algebraic cycles on the Shimura variety associated to V_π; and the conjectural *arithmetic inner product formula* [Liu11a] computes the height of the arithmetic theta lifting in terms of the central L-derivative $L'(\frac{1}{2}, \pi)$ of π. In our previous article [LL21], we verify the arithmetic inner product formula, under certain hypotheses, when E/F and π satisfy certain local conditions (see [LL21, Assumption 1.3]). In particular, we want $\mathtt{V}_F^{\mathrm{ram}} = \emptyset$, which forces $[F : \mathbb{Q}]$ to be even; and we want the representation π has to be either unramified or almost unramified at $v \in \mathtt{V}_F^{\mathrm{int}}$. Computing local root numbers, we have $\varepsilon(\pi_v) = (-1)^r$ if $v \in \mathtt{V}_F^{(\infty)}$, $\varepsilon(\pi_v) = 1$ if $v \in \mathtt{V}_F^{\mathrm{spl}}$ or π_v is unramified, $\varepsilon(\pi_v) = -1$ if ($v \in \mathtt{V}_F^{\mathrm{int}}$ and) π_v is almost unramified. It follows that $\varepsilon(\pi) = (-1)^{r[F:\mathbb{Q}]+|\mathtt{S}_\pi|}$, where $\mathtt{S}_\pi \subseteq \mathtt{V}_F^{\mathrm{int}}$ denotes the (finite) subset at which π is almost unramified, which equals $(-1)^{|\mathtt{S}_\pi|}$ as $[F : \mathbb{Q}]$ is even. In this article, we improve our results so that $\mathtt{V}_F^{\mathrm{ram}}$ can be nonempty, hence $[F : \mathbb{Q}]$ can be odd; and we will still have $\varepsilon(\pi) = (-1)^{r[F:\mathbb{Q}]+|\mathtt{S}_\pi|}$. To show the significance of such improvement, now we may have $\varepsilon(\pi) = -1$ but $\mathtt{S}_\pi = \emptyset$, so that we can accommodate π that comes from certain explicit motives like symmetric power of elliptic curves (see Example 1.10).

The readers may read the introduction of [LL21] for more background. Now we describe in more details our setup and main results in the current article.

Definition 1.1. We define the subset $\mathtt{V}_F^{\heartsuit}$ of $\mathtt{V}_F^{\mathrm{spl}} \cup \mathtt{V}_F^{\mathrm{int}}$ consisting of v satisfying that for every $v' \in \mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{ram}}$, where p is the underlying rational prime of v, the subfield of $\overline{F_v}$ generated by F_v and the Galois closure of $E_{v'}$ is unramified over F_v.

Remark 1.2. The purpose of this technical definition is that for certain places v in $\mathtt{V}_F^{\mathrm{spl}} \cup \mathtt{V}_F^{\mathrm{int}}$, we need to have a CM type of E such that its reflex field does not contain more ramification over p than F_v does – this is possible for $v \in \mathtt{V}_F^{\heartsuit}$. Note that

- the complement $(\mathtt{V}_F^{\mathrm{spl}} \cup \mathtt{V}_F^{\mathrm{int}}) \setminus \mathtt{V}_F^{\heartsuit}$ is finite;

- when E is Galois, or contains an imaginary quadratic field, or satisfies $\mathtt{V}_F^{\mathrm{ram}} = \emptyset$, we have $\mathtt{V}_F^{\heartsuit} = \mathtt{V}_F^{\mathrm{spl}} \cup \mathtt{V}_F^{\mathrm{int}}$.

Assumption 1.3. Suppose that $F \neq \mathbb{Q}$, that $\mathtt{V}_F^{\mathrm{spl}}$ contains all 2-adic places, and that every prime in $\mathtt{V}_F^{\mathrm{ram}}$ is unramified over $\mathbb{Q}$. We consider a cuspidal automorphic representation π of $G_r(\mathbb{A}_F)$ realized on a space $\mathcal{V}_\pi$ of cusp forms, satisfying:

(1) For every $v \in \mathtt{V}_F^{(\infty)}$, π_v is the holomorphic discrete series representation of Harish-Chandra parameter $\{\frac{1-n}{2}, \frac{3-n}{2}, \dots, \frac{n-3}{2}, \frac{n-1}{2}\}$ (see [LL21, Remark 1.4(1)]).
(2) For every $v \in \mathtt{V}_F^{\mathrm{ram}}$, π_v is spherical with respect to $K_{r,v}$, that is, $\pi_v^{K_{r,v}} \neq \{0\}$.
(3) For every $v \in \mathtt{V}_F^{\mathrm{int}}$, π_v is either unramified or almost unramified (see [LL21, Remark 1.4(3)]) with respect to $K_{r,v}$; moreover, if π_v is almost unramified, then v is unramified over $\mathbb{Q}$.
(4) For every $v \in \mathtt{V}_F^{\mathrm{fin}}$, π_v is tempered.
(5) We have $\mathtt{R}_\pi \cup \mathtt{S}_\pi \subseteq \mathtt{V}_F^{\heartsuit}$ (Definition 1.1), where
    - $\mathtt{R}_\pi \subseteq \mathtt{V}_F^{\mathrm{spl}}$ denotes the (finite) subset for which π_v is ramified,
    - $\mathtt{S}_\pi \subseteq \mathtt{V}_F^{\mathrm{int}}$ denotes the (finite) subset for which π_v is almost unramified.

Comparing Assumption 1.3 with [LL21, Assumption 1.3], we have lifted the restriction that $\mathtt{V}_F^{\mathrm{ram}} = \emptyset$ (by allowing π_v to be a certain type of representations for $v \in \mathtt{V}_F^{\mathrm{ram}}$), and also the restriction on π_v for $v \in \mathtt{V}_F^{\mathrm{spl}}$. Note that (5) is not really a new restriction since when $\mathtt{V}_F^{\mathrm{ram}} = \emptyset$, it is automatic by Remark 1.2.

Suppose that we are in Assumption 1.3. Denote by $L(s,\pi)$ the doubling L-function. Then we have $\varepsilon(\pi) = (-1)^{r[F:\mathbb{Q}]+|\mathtt{S}_\pi|}$ for the global (doubling) root number, so that the vanishing order of $L(s,\pi)$ at the center $s = \frac{1}{2}$ has the same parity as $r[F:\mathbb{Q}] + |\mathtt{S}_\pi|$. The cuspidal automorphic representation π determines a hermitian space V_π over $\mathbb{A}_E$ of rank n via local theta dichotomy (so that the local theta lifting of π_v to $\mathrm{U}(V_\pi)(F_v)$ is nontrivial for every place v of F), unique up to isomorphism, which is totally positive definite and satisfies that for every $v \in \mathtt{V}_F^{\mathrm{fin}}$, the local Hasse invariant $\epsilon(V_\pi \otimes_{\mathbb{A}_F} F_v) = 1$ if and only if $v \notin \mathtt{S}_\pi$.

Now suppose that $r[F:\mathbb{Q}]+|\mathtt{S}_\pi|$ is odd, hence $\varepsilon(\pi) = -1$, which is equivalent to that V_π is incoherent. In what follows, we take $V = V_\pi$ in the context of [LL21, Conjecture 1.1], hence $H = \mathrm{U}(V_\pi)$. Let $\mathtt{R}$ be a finite subset of $\mathtt{V}_F^{\mathrm{fin}}$. We fix a special maximal subgroup $L^{\mathtt{R}}$ of $H(\mathbb{A}_F^{\infty,\mathtt{R}})$ that is the stabilizer of a lattice $\Lambda^{\mathtt{R}}$ in $V \otimes_{\mathbb{A}_F} \mathbb{A}_F^{\infty,\mathtt{R}}$ (see Notation 4.2(H6) for more details). For a field $\mathbb{L}$, we denote by $\mathbb{T}_{\mathbb{L}}^{\mathtt{R}}$ the (abstract) Hecke algebra $\mathbb{L}[L^{\mathtt{R}} \backslash H(\mathbb{A}_F^{\infty,\mathtt{R}})/L^{\mathtt{R}}]$, which is a commutative $\mathbb{L}$-algebra. When $\mathtt{R}$ contains $\mathtt{R}_\pi$, the cuspidal automorphic representation π gives rise to a character

$$\chi_\pi^{\mathtt{R}} \colon \mathbb{T}_{\mathbb{Q}^{\mathrm{ac}}}^{\mathtt{R}} \to \mathbb{Q}^{\mathrm{ac}},$$

where $\mathbb{Q}^{\mathrm{ac}}$ denotes the subfield of $\mathbb{C}$ of algebraic numbers; and we put $\mathfrak{m}_\pi^{\mathtt{R}} \coloneqq \ker \chi_\pi^{\mathtt{R}}$, which is a maximal ideal of $\mathbb{T}_{\mathbb{Q}^{\mathrm{ac}}}^{\mathtt{R}}$.

In what follows, we will fix an arbitrary embedding $\boldsymbol{\iota} \colon E \hookrightarrow \mathbb{C}$ and denote by $\{X_L\}$ the system of unitary Shimura varieties of dimension $n-1$ over $\boldsymbol{\iota}(E)$ indexed by open compact subgroups $L \subseteq H(\mathbb{A}_F^\infty)$ (see Subsection 4.2 for more details). The following is the first main theorem of this article.

Theorem 1.4. *Let $(\pi, \mathcal{V}_\pi)$ be as in Assumption 1.3 with $r[F:\mathbb{Q}] + |\mathtt{S}_\pi|$ odd, for which we assume* [LL21, Hypothesis 6.6]. *If $L'(\frac{1}{2},\pi) \neq 0$, that is, $\mathrm{ord}_{s=\frac{1}{2}} L(s,\pi) = 1$, then as long as*

$\mathtt{R}$ *satisfies* $\mathtt{R}_\pi \subseteq \mathtt{R}$ *and* $|\mathtt{R}\cap \mathtt{V}_F^{\mathrm{spl}}\cap \mathtt{V}_F^{\heartsuit}| \geqslant 2$, *the nonvanishing*

$$\varinjlim_{L_{\mathtt{R}}} \left(\mathrm{CH}^r(X_{L_{\mathtt{R}}L^{\mathtt{R}}})^0_{\mathbb{Q}^{\mathrm{ac}}}\right)_{\mathfrak{m}_\pi^{\mathtt{R}}} \neq 0$$

holds, where the colimit is taken over all open compact subgroups $L_{\mathtt{R}}$ *of* $H(F_{\mathtt{R}})$.

Our remaining results rely on Hypothesis 4.11 on the modularity of Kudla's generating functions of special cycles, hence are conditional at this moment.

Theorem 1.5. *Let* $(\pi,\mathcal{V}_\pi)$ *be as in Assumption 1.3 with* $r[F:\mathbb{Q}]+|\mathtt{S}_\pi|$ *odd, for which we assume* [LL21, Hypothesis 6.6]. *Assume Hypothesis 4.11 on the modularity of generating functions of codimension* r.

(1) *For every test vectors*
- $\varphi_1=\otimes_v\varphi_{1v}\in\mathcal{V}_\pi$ *and* $\varphi_2=\otimes_v\varphi_{2v}\in\mathcal{V}_\pi$ *such that for every* $v\in\mathtt{V}_F^{(\infty)}$, φ_{1v} *and* φ_{2v} *have the lowest weight and satisfy* $\langle\varphi_{1v}^{\mathtt{c}},\varphi_{2v}\rangle_{\pi_v}=1$,
- $\phi_1^\infty=\otimes_v\phi_{1v}^\infty\in\mathscr{S}(V^r\otimes_{\mathbb{A}_F}\mathbb{A}_F^\infty)$ *and* $\phi_2^\infty=\otimes_v\phi_{2v}^\infty\in\mathscr{S}(V^r\otimes_{\mathbb{A}_F}\mathbb{A}_F^\infty)$,

the identity

$$\langle\Theta_{\phi_1^\infty}(\varphi_1),\Theta_{\phi_2^\infty}(\varphi_2)\rangle^\natural_{X,E}=\frac{L'(\frac12,\pi)}{b_{2r}(0)}\cdot C_r^{[F:\mathbb{Q}]}\cdot\prod_{v\in\mathtt{V}_F^{\mathrm{fin}}}\mathfrak{Z}^\natural_{\pi_v,V_v}(\varphi_{1v}^{\mathtt{c}},\varphi_{2v},\phi_{1v}^\infty\otimes(\phi_{2v}^\infty)^{\mathtt{c}})$$

holds. Here,
- $\Theta_{\phi_i^\infty}(\varphi_i)\in\varinjlim_L\mathrm{CH}^r(X_L)^0_{\mathbb{C}}$ *is the arithmetic theta lifting (Definition 4.12), which is only well-defined under Hypothesis 4.11;*
- $\langle\Theta_{\phi_1^\infty}(\varphi_1),\Theta_{\phi_2^\infty}(\varphi_2)\rangle^\natural_{X,E}$ *is the normalized height pairing (Definition 4.17),*[3] *which is constructed based on Beilinson's notion of height pairing;*
- $b_{2r}(0)$ *is defined in Notation 4.1(F4), which equals* $L(M_r^\vee(1))$ *where* M_r *is the motive associated to* G_r *by Gross* [Gro97], *and is in particular a positive real number;*
- $C_r=(-1)^r2^{-2r}\pi^{r^2}\frac{\Gamma(1)\cdots\Gamma(r)}{\Gamma(r+1)\cdots\Gamma(2r)}$, *which is the exact value of a certain archimedean doubling zeta integral; and*
- $\mathfrak{Z}^\natural_{\pi_v,V_v}(\varphi_{1v}^{\mathtt{c}},\varphi_{2v},\phi_{1v}^\infty\otimes(\phi_{2v}^\infty)^{\mathtt{c}})$ *is the normalized local doubling zeta integral* [LL21, Section 3], *which equals* 1 *for all but finitely many* v.

(2) *In the context of* [LL21, Conjecture 1.1], *take (*$V=V_\pi$ *and)* $\tilde\pi^\infty$ *to be the theta lifting of* π^∞ *to* $H(\mathbb{A}_F^\infty)$. *If* $L'(\frac12,\pi)\neq0$, *that is,* $\mathrm{ord}_{s=\frac12}L(s,\pi)=1$, *then*

$$\mathrm{Hom}_{H(\mathbb{A}_F^\infty)}\left(\tilde\pi^\infty,\varinjlim_L\mathrm{CH}^r(X_L)^0_{\mathbb{C}}\right)\neq0$$

holds.

Remark 1.6. We have the following remarks concerning Theorem 1.5.

(1) Part (1) verifies the so-called *arithmetic inner product formula*, a conjecture proposed by one of us [Liu11a, Conjecture 3.11].
(2) The arithmetic inner product formula in part (1) is perfectly parallel to the classical Rallis inner product formula. In fact, suppose that V is totally positive definite but *coherent*. We have the classical theta lifting $\theta_{\phi^\infty}(\varphi)$ where we use standard Gaussian

[3] Strictly speaking, $\langle\Theta_{\phi_1^\infty}(\varphi_1),\Theta_{\phi_2^\infty}(\varphi_2)\rangle^\natural_{X,E}$ relies on the choice of a rational prime ℓ and is a priori an element in $\mathbb{C}\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$. However, the above identity implicitly says that it belongs to $\mathbb{C}$ and is independent of the choice of ℓ.

functions at archimedean places. Then the Rallis inner product formula in this case reads as

$$\langle\theta_{\phi_1^\infty}(\varphi_1),\theta_{\phi_2^\infty}(\varphi_2)\rangle_H=\frac{L(\frac{1}{2},\pi)}{b_{2r}(0)}\cdot C_r^{[F:\mathbb{Q}]}\cdot\prod_{v\in\mathtt{V}_F^{\mathrm{fin}}}\mathfrak{Z}^\natural_{\pi_v,V_v}(\varphi_{1v}^{\mathtt{c}},\varphi_{2v},\phi_{1v}^\infty\otimes(\phi_{2v}^\infty)^{\mathtt{c}}),$$

in which $\langle\ ,\ \rangle_H$ denotes the Petersson inner product with respect to the *Tamagawa measure* on $H(\mathbb{A}_F)$.

In the case where $\mathtt{R}_\pi=\emptyset$, we have a very explicit height formula for test vectors that are new everywhere.

Corollary 1.7. *Let $(\pi,\mathcal{V}_\pi)$ be as in Assumption 1.3 with $r[F:\mathbb{Q}]+|\mathtt{S}_\pi|$ odd, for which we assume* [LL21, Hypothesis 6.6]*. Assume Hypothesis 4.11 on the modularity of generating functions of codimension r. In the situation of Theorem 1.5(1), suppose further that*

- $\mathtt{R}_\pi=\emptyset$;
- $\varphi_1=\varphi_2=\varphi\in\mathcal{V}_\pi^{[r]\emptyset}$ *(see Notation 4.3(G8) for the precise definition of the one-dimensional space $\mathcal{V}_\pi^{[r]\emptyset}$ of holomorphic new forms) such that for every $v\in\mathtt{V}_F$, $\langle\varphi_v^{\mathtt{c}},\varphi_v\rangle_{\pi_v}=1$; and*
- $\phi_1^\infty=\phi_2^\infty=\phi^\infty$ *such that for every $v\in\mathtt{V}_F^{\mathrm{fin}}$, $\phi_v^\infty=\mathbb{1}_{(\Lambda_v^\emptyset)^r}$.*

Then the identity

$$\langle\Theta_{\phi^\infty}(\varphi),\Theta_{\phi^\infty}(\varphi)\rangle^\natural_{X,E}=(-1)^r\cdot\frac{L'(\frac{1}{2},\pi)}{b_{2r}(0)}\cdot|C_r|^{[F:\mathbb{Q}]}\cdot\prod_{v\in\mathtt{S}_\pi}\frac{q_v^{r-1}(q_v+1)}{(q_v^{2r-1}+1)(q_v^{2r}-1)}$$

holds, where q_v is the residue cardinality of F_v.

Remark 1.8. Assuming the conjecture on the injectivity of the étale Abel–Jacobi map, one can show that the cycle $\Theta_{\phi^\infty}(\varphi)$ is a primitive cycle of codimension r. By [Beĭ87, Conjecture 5.5], we expect that $(-1)^r\langle\Theta_{\phi^\infty}(\varphi),\Theta_{\phi^\infty}(\varphi)\rangle^\natural_{X,E}\geqslant 0$ holds, which, in the situation of Corollary 1.7, is equivalent to $L'(\frac{1}{2},\pi)\geqslant 0$.

Remark 1.9. When $\mathtt{S}_\pi=\emptyset$, Theorem 1.4, Theorem 1.5, and Corollary 1.7 hold without [LL21, Hypothesis 6.6]. See Remark 4.32 for more details.

Example 1.10. Suppose that E/F satisfies the conditions in Assumption 1.3 and that $r\geqslant 2$. Consider an elliptic curve A over F without complex multiplication, satisfying that $\mathrm{Sym}^{2r-1}A$ and hence $\mathrm{Sym}^{2r-1}A_E$ are modular. Let Π be the cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ corresponding to $\mathrm{Sym}^{2r-1}A_E$, which satisfies $\Pi^\vee\simeq\Pi\circ\mathtt{c}$. Then there exists a cuspidal automorphic representation π of $G_r(\mathbb{A}_F)$ as in Assumption 1.3 with Π its base change if and only if A has good reduction at every $v\in\mathtt{V}_F^{\mathrm{fin}}\setminus\mathtt{V}_F^{\mathrm{spl}}$.[4] Moreover, if this is the case, then we have $\mathtt{S}_\pi=\emptyset$, hence $\varepsilon(\pi)=(-1)^{r[F:\mathbb{Q}]}$. In particular, above results apply when both r and $[F:\mathbb{Q}]$ are odd.

[4]Note that, when $r\geqslant 2$, the $(2r-1)$-th symmetric power of an irreducible admissible representation of $\mathrm{GL}_2(E_v)$ can never be the base change of an almost unramified representation of $G_r(F_v)$ for $v\in\mathtt{V}_F^{\mathrm{int}}$.

1.2. **Two new ingredients.** The proofs of our main theorems follow the same line in [LL21], with two new (main) ingredients, responsible for the two improvements we have mentioned at the beginning.

The first new ingredient is formulating and proving an analogue of the Kudla–Rapoport conjecture in the case where E/F is ramified and the level structure is the one that gives the exotic smooth model (see Subsection 2.1). Here, F is a p-adic field with p odd. Let $\boldsymbol{L}$ be an O_E-lattice of a nonsplit (nondegenerate) hermitian space $\boldsymbol{V}$ over E of (even) rank n. Then one can associate an intersection number $\mathrm{Int}(\boldsymbol{L})$ of special divisors on a formally smooth relative Rapoport–Zink space classifying quasi-isogenies of certain unitary O_F-divisible groups, and also the derivative of the representation density function $\partial\mathrm{Den}(\boldsymbol{L})$ given by $\boldsymbol{L}$. We show in Theorem 2.7 the formula

$$\mathrm{Int}(\boldsymbol{L}) = \partial\mathrm{Den}(\boldsymbol{L}).$$

This is parallel to the Kudla–Rapoport conjecture proved in [LZa], originally stated for the case where E/F is unramified. The proof follows from the same strategy as in [LZa], namely, we write $\boldsymbol{L} = L^\flat + \langle x\rangle$ for a sublattice $L^\flat$ of $\boldsymbol{L}$ such that $V_{L^\flat} := L^\flat \otimes_{O_F} F$ is nondegenerate, and regard x as a variable. Thus, it motivates us to define a function $\mathrm{Int}_{L^\flat}$ on $\boldsymbol{V} \setminus V_{L^\flat}$ by the formula $\mathrm{Int}_{L^\flat}(x) = \mathrm{Int}(L^\flat + \langle x\rangle)$ and similarly for $\partial\mathrm{Den}_{L^\flat}$. For $\mathrm{Int}_{L^\flat}$, there is a natural decomposition $\mathrm{Int}_{L^\flat} = \mathrm{Int}^{\mathrm{h}}_{L^\flat} + \mathrm{Int}^{\mathrm{v}}_{L^\flat}$ according to the horizontal and vertical parts of the special cycle defined by $L^\flat$. In a parallel manner, we have the decomposition $\partial\mathrm{Den}_{L^\flat} = \partial\mathrm{Den}^{\mathrm{h}}_{L^\flat} + \partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}$ by simply matching $\partial\mathrm{Den}^{\mathrm{h}}_{L^\flat}$ with $\mathrm{Int}^{\mathrm{h}}_{L^\flat}$. Thus, it suffices to show that $\mathrm{Int}^{\mathrm{v}}_{L^\flat} = \partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}$. By some sophisticated induction argument on $L^\flat$, it suffices to show the following remarkable property for both $\mathrm{Int}^{\mathrm{v}}_{L^\flat}$ and $\partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}$: they extend (uniquely) to compactly supported locally constant functions on $\boldsymbol{V}$, whose Fourier transforms are supported in the set $\{x \in \boldsymbol{V} \mid (x,x)_{\boldsymbol{V}} \in O_F\}$. However, there are some new difficulties in our case:

- The isomorphism class of an O_E-lattice is not determined by its fundamental invariants, and there is a parity constraint for the valuation of an O_E-lattice. This will make the induction argument on $L^\flat$ much more complicated than the one in [LZa] (see Subsection 2.7).
- The comparison of our relative Rapoport–Zink space to an (absolute) Rapoport–Zink space is not known. This is needed in the p-adic uniformization of Shimura varieties. We solve this problem when $F/\mathbb{Q}_p$ is unramified, which is the reason for us to assume that every prime in $\mathtt{V}_F^{\mathrm{ram}}$ is unramified over $\mathbb{Q}$ in Assumption 1.3. See Subsection 2.8.
- Due to the parity constraint, the computation of $\mathrm{Int}^{\mathrm{v}}_{L^\flat}$ can only be reduced to the case where $n = 4$ (rather than $n = 3$ in [LZa]). After that, we have to compute certain intersection multiplicity, for which we use a new argument based on the linear invariance of the K-theoretic intersection of special divisors. See Lemma 2.55.

Here come three more remarks:

- First, we need to extend the result of [CY20] on a counting formula for $\partial\mathrm{Den}(\boldsymbol{L})$ to hermitian spaces over a ramified extension E/F (Lemma 2.19).
- Second, we have found a simpler argument for the properties of $\partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}$ (Proposition 2.22), which does not use any functional equation or induction formula. This argument is applicable to [LZa] to give a new proof of the main result on the analytic side there.

Also note that we prove the vanishing property in Proposition 2.22 directly, while in [LZa] it is only deduced after proving $\mathrm{Int}^{\mathrm{v}}_{L^\flat} = \partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}$.[5]

- Finally, unlike the case in [LZa], the parity of the dimension of the hermitian space plays a crucial role in the exotic smooth case. In particular, we will not study the case where $\boldsymbol{V}$ has odd dimension.

The second new ingredient is a vanishing result on certain cohomology of integral models of unitary Shimura varieties with Drinfeld level structures. For $v \in \mathtt{V}_F^{\mathrm{spl}} \cap \mathtt{V}_F^{\heartsuit}$ with p the underlying rational prime, we have a tower of integral models $\{\mathcal{X}_m\}_{m\geqslant 0}$ defined by Drinfeld level structures (at v), with an action by $\mathbb{T}^{\mathtt{R}\cup\mathtt{V}_F^{(p)}}_{\mathbb{Q}^{\mathrm{ac}}}$ via Hecke correspondences. We show in Theorem 4.21 that

$$\mathrm{H}^{2r}(\mathcal{X}_m, \overline{\mathbb{Q}}_\ell(r))_{\mathfrak{m}} = 0$$

with $\ell \neq p$ and $\mathfrak{m} \coloneqq \mathfrak{m}^{\mathtt{R}}_\pi \cap \mathbb{S}^{\mathtt{R}\cup\mathtt{V}_F^{(p)}}_{\mathbb{Q}^{\mathrm{ac}}}$, where $\mathbb{S}^{\mathtt{R}\cup\mathtt{V}_F^{(p)}}_{\mathbb{Q}^{\mathrm{ac}}}$ is the subalgebra of $\mathbb{T}^{\mathtt{R}\cup\mathtt{V}_F^{(p)}}_{\mathbb{Q}^{\mathrm{ac}}}$ consisting of those supported at split places. We reduce this vanishing property to some other vanishing properties for cohomology of Newton strata of $\mathcal{X}_m$, by using a key result of Mantovan [Man08] saying that the closure of every refined Newton stratum is smooth. For the vanishing properties for Newton strata, we generalize an argument of [TY07, Proposition 4.4]. However, since in our case, the representation π_v has arbitrary level and our group has nontrivial endoscopy, we need a more sophisticated trace formula, which was provided in [CS17].

1.3. **Notation and conventions.**

- When we have a function f on a product set $A_1 \times \cdots \times A_m$, we will write $f(a_1, \dots, a_m)$ instead of $f((a_1, \dots, a_m))$ for its value at an element $(a_1, \dots, a_m) \in A_1 \times \cdots \times A_m$.
- For a set S, we denote by $\mathbb{1}_S$ the characteristic function of S.
- All rings are commutative and unital; and ring homomorphisms preserve units. However, we use the word *algebra* in the general sense, which is not necessarily commutative or unital.
- For a (formal) subscheme Z of a (formal) scheme X, we denote by $\mathscr{I}_Z$ the ideal sheaf of Z, which is a subsheaf of the structure sheaf $\mathscr{O}_X$ of X.
- For a ring R, we denote by $\mathrm{Sch}_{/R}$ the category of schemes over R, by $\mathrm{Sch}'_{/R}$ the subcategory of locally Noetherian schemes over R, and when R is discretely valued, by $\mathrm{Sch}^{\mathrm{v}}_{/R}$ the subcategory of schemes on which uniformizers of R are locally nilpotent.
- If a base ring is not specified in the tensor operation $\otimes$, then it is $\mathbb{Z}$.
- For an abelian group A and a ring R, we put $A_R \coloneqq A \otimes R$.
- For an integer $m \geqslant 0$, we denote by 0_m and 1_m the null and identity matrices of rank m, respectively. We also denote by $\mathtt{w}_m$ the matrix $\begin{pmatrix} & 1_m \\ -1_m & \end{pmatrix}$.
- We denote by $\mathtt{c}\colon \mathbb{C} \to \mathbb{C}$ the complex conjugation. For an element x in a complex space with a default underlying real structure, we denote by $x^{\mathtt{c}}$ its complex conjugation.
- For a field K, we denote by $\overline{K}$ the abstract algebraic closure of K. However, for aesthetic reason, we will write $\overline{\mathbb{Q}}_p$ instead of $\overline{\mathbb{Q}_p}$ and will denote by $\overline{\mathbb{F}}_p$ its residue field. On the other hand, we denote by $\mathbb{Q}^{\mathrm{ac}}$ the algebraic closure of $\mathbb{Q}$ *inside* $\mathbb{C}$.

[5] We have also tried to apply our argument to prove this vanishing property directly in the case considered in [LZa] as well, but the numerology seems much more complicated to make a success. Nevertheless, our argument does give a simpler proof of the weaker vanishing property in [LZa, Theorem 7.4.1].

- For a number field K, we denote by $\psi_K\colon K\backslash\mathbb{A}_K\to\mathbb{C}^\times$ the standard additive character, namely, $\psi_K:=\psi_\mathbb{Q}\circ\mathrm{Tr}_{K/\mathbb{Q}}$ in which $\psi_\mathbb{Q}\colon\mathbb{Q}\backslash\mathbb{A}\to\mathbb{C}^\times$ is the unique character such that $\psi_{\mathbb{Q},\infty}(x)=\mathrm{e}^{2\pi ix}$.
- Throughout the entire article, all parabolic inductions are unitarily normalized.

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2. Intersection of special cycles at ramified places

Throughout this section, we fix a *ramified* quadratic extension E/F of p-adic fields with p odd, with $\mathsf{c}\in\mathrm{Gal}(E/F)$ the Galois involution. We fix a uniformizer $u\in E$ satisfying $u^\mathsf{c}=-u$. Let k be the residue field of F and denote by q the cardinality of k. Let $n=2r$ be an even positive integer.

In Subsection 2.1, we introduce our relative Rapoport–Zink space and state the main theorem (Theorem 2.7) on the relation between intersection numbers and derivatives of representation densities. In Subsection 2.2, we study derivatives of representation densities. In Subsection 2.3, we recall the Bruhat–Tits stratification on the relative Rapoport–Zink space from [Wu] and deduce some consequences. In Subsection 2.4, we prove the linear invariance on the K-theoretic intersection of special divisors, following [How19]. In Subsection 2.5, we prove Theorem 2.7 when $r=1$, which is needed for the proof when $r>1$. In Subsection 2.6, we study intersection numbers. In Subsection 2.7, we prove Theorem 2.7 for general r. In Subsection 2.8, we compare our relative Rapoport–Zink space to certain (absolute) Rapoport–Zink space assuming $F/\mathbb{Q}_p$ is unramified.

Here are two preliminary definitions for this section:

- A *hermitian O_E-module* is a finitely generated free O_E-module L together with an O_F-bilinear pairing $(\ ,\ )_L\colon L\times L\to E$ such that the induced E-valued pairing on $L\otimes_{O_F}F$ is a nondegenerate hermitian pairing (with respect to c). When we say that a hermitian O_E-module L is contained in a hermitian O_E-module or a hermitian E-space M, we require that the restriction of the pairing $(\ ,\ )_M$ to L coincides with $(\ ,\ )_L$.
- Let X be an object of an additive category with a notion of dual.
  - We say that a morphism $\sigma_X\colon X\to X^\vee$ is a *symmetrization* if σ_X is an isomorphism and the composite morphism $X\to X^{\vee\vee}\xrightarrow{\sigma_X^\vee}X^\vee$ coincides with σ_X.
  - Given an action $\iota_X\colon O_E\to\mathrm{End}(X)$, we say that a morphism $\lambda_X\colon X\to X^\vee$ is *ι_X-compatible* if $\lambda_X\circ\iota_X(\alpha)=\iota_X(\alpha^\mathsf{c})^\vee\circ\lambda_X$ holds for every $\alpha\in O_E$.

2.1. A Kudla–Rapoport type formula.

We fix an embedding $\varphi_0\colon E\to\mathbb{C}_p$ and let $\breve{E}$ be the maximal complete unramified extension of $\varphi_0(E)$ in $\mathbb{C}_p$. We regard E as a subfield of $\breve{E}$ via φ_0, hence identify the residue field of $\breve{E}$ with an algebraic closure $\overline{k}$ of k.

Definition 2.1. Let S be an object of $\mathrm{Sch}_{/O_{\breve{E}}}$. We define a category $\mathrm{Exo}_{(n-1,1)}(S)$ whose objects are triples (X,ι_X,λ_X) in which

- X is an O_F-divisible group[6] over S of dimension $n=2r$ and (relative) height $2n$;

[6] An O_F-divisible group is also called a strict O_F-module.

- $\iota_X\colon O_E \to \mathrm{End}(X)$ is an action of O_E on X satisfying:
  - (Kottwitz condition): the characteristic polynomial of $\iota_X(u)$ on the locally free $\mathscr{O}_S$-module $\mathrm{Lie}(X)$ is $(T-u)^{n-1}(T+u) \in \mathscr{O}_S[T]$,
  - (Wedge condition): we have
$$\bigwedge^2 (\iota_X(u) - u \mid \mathrm{Lie}(X)) = 0,$$
  - (Spin condition): for every geometric point s of S, the action of $\iota_X(u)$ on $\mathrm{Lie}(X_s)$ is nonzero;
- $\lambda_X\colon X \to X^\vee$ is a ι_X-compatible polarization such that $\ker(\lambda_X) = X[\iota_X(u)]$.

A morphism (resp. quasi-morphism) from (X, ι_X, λ_X) to (Y, ι_Y, λ_Y) is an O_E-linear isomorphism (resp. quasi-isogeny) $\rho\colon X \to Y$ of height zero such that $\rho^*\lambda_Y = \lambda_X$.

When S belongs to $\mathrm{Sch}^{\mathrm{v}}_{/O_{\breve{E}}}$, we denote by $\mathrm{Exo}^{\mathrm{b}}_{(n-1,1)}(S)$ the subcategory of $\mathrm{Exo}_{(n-1,1)}(S)$ consisting of (X, ι_X, λ_X) in which X is supersingular.[7]

Remark 2.2. Giving a ι_X-compatible polarization λ_X of X satisfying $\ker(\lambda_X) = X[\iota_X(u)]$ is equivalent to giving a ι_X-compatible symmetrization σ_X of X. In fact, since $\ker(\lambda_X) = X[\iota_X(u)]$, there is a unique morphism $\sigma_X\colon X \to X^\vee$ satisfying $\lambda_X = \sigma_X \circ \iota_X(u)$, which is in fact an isomorphism, satisfying
$$\sigma_X^\vee = \iota_X(u^{-1})^\vee \circ \lambda_X^\vee = -\iota_X(u^{-1})^\vee \circ \lambda_X = -\lambda_X \circ \iota_X(u^{-1,\mathtt{c}}) = \lambda_X \circ \iota_X(u^{-1}) = \sigma_X,$$
and is clearly ι_X-compatible. Conversely, given a ι_X-compatible symmetrization σ_X of X, we may recover λ_X as $\sigma_X \circ \iota_X(u)$. In what follows, we call σ_X the symmetrization of λ_X.

To define our relative Rapoport–Zink space, we fix an object $(\boldsymbol{X}, \iota_{\boldsymbol{X}}, \lambda_{\boldsymbol{X}}) \in \mathrm{Exo}^{\mathrm{b}}_{(n-1,1)}(\overline{k})$.

Definition 2.3. We define a functor $\mathcal{N} \coloneqq \mathcal{N}_{(\boldsymbol{X}, \iota_{\boldsymbol{X}}, \lambda_{\boldsymbol{X}})}$ on $\mathrm{Sch}^{\mathrm{v}}_{/O_{\breve{E}}}$ such that for every object S of $\mathrm{Sch}^{\mathrm{v}}_{/O_{\breve{E}}}$, $\mathcal{N}(S)$ consists of quadruples $(X, \iota_X, \lambda_X; \rho_X)$ in which

- (X, ι_X, λ_X) is an object of $\mathrm{Exo}^{\mathrm{b}}_{(n-1,1)}(S)$;
- ρ_X is a quasi-morphism from $(X, \iota_X, \lambda_X) \times_S (S \otimes_{O_{\breve{E}}} \overline{k})$ to $(\boldsymbol{X}, \iota_{\boldsymbol{X}}, \lambda_{\boldsymbol{X}}) \otimes_{\overline{k}} (S \otimes_{O_{\breve{E}}} \overline{k})$ in the category $\mathrm{Exo}^{\mathrm{b}}_{(n-1,1)}(S \otimes_{O_{\breve{E}}} \overline{k})$.

Lemma 2.4. *The functor $\mathcal{N}$ is a separated formal scheme formally smooth over* $\mathrm{Spf}\, O_{\breve{E}}$ *of relative dimension $n-1$. Moreover, $\mathcal{N}$ has two connected components.*

Proof. It follows from [RZ96] that $\mathcal{N}$ is a separated formal scheme over $\mathrm{Spf}\, O_{\breve{E}}$. The formal smoothness of $\mathcal{N}$ follow from the smoothness of its local model, which is [RSZ17, Proposition 3.10]; and the dimension also follows. For the last assertion, our moduli functor $\mathcal{N}$ is the disjoint union of $\mathcal{N}_{(0,0)}$ and $\mathcal{N}_{(0,1)}$ from [Wu, Section 3.4], each of which is connected by [Wu, Theorem 5.18(2)].[8] ∎

To study special cycles on $\mathcal{N}$, we fix a triple $(X_0, \iota_{X_0}, \lambda_{X_0})$ where

- X_0 is a supersingular O_F-divisible group over $\mathrm{Spec}\, O_{\breve{E}}$ of dimension 1 and height 2;
- $\iota_{X_0}\colon O_E \to \mathrm{End}(X_0)$ is an O_E-action on X_0 such that the induced action on $\mathrm{Lie}(X_0)$ is given by φ_0;

[7] Here, the superscript "b" stands for *basic*, which is related to the basic locus in the Shimura variety that appears later.

[8] The article [Wu] only studied the case $F = \mathbb{Q}_p$. In fact, all arguments and results work for general F. This footnote applies to the proof of Proposition 2.28 as well.

- $\lambda_{X_0}\colon X_0 \to X_0^\vee$ is a ι_{X_0}-compatible principal polarization.

Note that ι_{X_0} induces an isomorphism $\iota_{X_0}\colon O_E \xrightarrow{\sim} \mathrm{End}_{O_E}(X_0)$. Put

$$\boldsymbol{V} \coloneqq \mathrm{Hom}_{O_E}(X_0 \otimes_{O_{\breve{E}}} \overline{k}, \boldsymbol{X}) \otimes \mathbb{Q},$$

which is a vector space over E of dimension n. We have a pairing

$$(\ ,\ )_{\boldsymbol{V}}\colon \boldsymbol{V} \times \boldsymbol{V} \to E \tag{2.1}$$

sending $(x, y) \in \boldsymbol{V}^2$ to the composition of quasi-homomorphisms

$$X_0 \xrightarrow{x} \boldsymbol{X} \xrightarrow{\lambda_{\boldsymbol{X}}} \boldsymbol{X}^\vee \xrightarrow{y^\vee} X_0^\vee \xrightarrow{u^{-2}\lambda_{X_0}^{-1}} X_0$$

as an element in $\mathrm{End}_{O_E}(X_0)\otimes\mathbb{Q}$, hence in E via $\iota_{X_0}^{-1}$. It is known that $(\ ,\ )_{\boldsymbol{V}}$ is a nondegenerate and nonsplit hermitian form on $\boldsymbol{V}$ [RSZ17, Lemma 3.5].[9]

Definition 2.5. For every nonzero element $x \in \boldsymbol{V}$, we define the *special divisor* $\mathcal{N}(x)$ of $\mathcal{N}$ to be the maximal closed formal subscheme over which the quasi-homomorphism

$$\rho_X^{-1} \circ x\colon (X_0 \otimes_{O_{\breve{E}}} \overline{k}) \otimes_k (S \otimes_{O_{\breve{E}}} \overline{k}) \to X \times_S (S \otimes_{O_{\breve{E}}} \overline{k})$$

lifts (uniquely) to a homomorphism $X_0 \otimes_{O_{\breve{E}}} S \to X$.

Definition 2.6. For an O_E-lattice $\boldsymbol{L}$ of $\boldsymbol{V}$, the Serre intersection multiplicity

$$\chi\left(\mathscr{O}_{\mathcal{N}(x_1)} \overset{\mathbb{L}}{\otimes}_{\mathscr{O}_{\mathcal{N}}} \cdots \overset{\mathbb{L}}{\otimes}_{\mathscr{O}_{\mathcal{N}}} \mathscr{O}_{\mathcal{N}(x_n)}\right)$$

does not depend on the choice of a basis $\{x_1, \dots, x_n\}$ of $\boldsymbol{L}$ by Corollary 2.35, which we define to be $\mathrm{Int}(\boldsymbol{L})$.

Theorem 2.7. *For every O_E-lattice $\boldsymbol{L}$ of $\boldsymbol{V}$, we have*

$$\mathrm{Int}(\boldsymbol{L}) = \partial\mathrm{Den}(\boldsymbol{L}),$$

where $\partial\mathrm{Den}(\boldsymbol{L})$ is defined in Definition 2.16.

The strategy of proving this theorem described in Subsection 1.2 motivates the following definition, which will be frequently used in the rest of Section 2.

Definition 2.8. We define $\flat(\boldsymbol{V})$ to be the set of hermitian O_E-modules contained in $\boldsymbol{V}$ of rank $n-1$. In what follows, for $L^\flat \in \flat(\boldsymbol{V})$, we put $V_{L^\flat} \coloneqq L^\flat \otimes_{O_F} F$ and write $V_{L^\flat}^\perp$ for the orthogonal complement of $V_{L^\flat}$ in $\boldsymbol{V}$.

Remark 2.9. Let S be an object of $\mathrm{Sch}_{/O_{\breve{E}}}$. We have another category $\mathrm{Exo}_{(n,0)}(S)$ whose objects are triples (X, ι_X, λ_X) in which

- X is an O_F-divisible group over S of dimension $n = 2r$ and (relative) height $2n$;
- $\iota_X\colon O_E \to \mathrm{End}(X)$ is an action of O_E on X such that $\iota_X(u) - u$ annihilates $\mathrm{Lie}(X)$;
- $\lambda_X\colon X \to X^\vee$ is a ι_X-compatible polarization such that $\ker(\lambda_X) = X[\iota_X(u)]$.

Morphisms are defined similarly as in Definition 2.1.

[9]The readers may notice that we have an extra factor u^{-2} in the definition of the hermitian form. This is because we want to ensure that $\mathcal{N}(x)$ is nonempty if and only if $(x, x)_{\boldsymbol{V}} \in O_F$.

For later use, we fix a nontrivial additive character $\psi_F\colon F\to\mathbb{C}^\times$ of conductor O_F. For a locally constant compactly supported function ϕ on a hermitian space V over E, its Fourier transform $\widehat{\phi}$ is defined by

$$\widehat{\phi}(x)=\int_V\phi(y)\psi_F(\mathrm{Tr}_{E/F}(x,y)_V)\,\mathrm{d}y$$

where $\mathrm{d}y$ is the self-dual Haar measure on V.

2.2. **Fourier transform of analytic side.** In this subsection, we study local densities of hermitian lattices. We first introduce some notion about O_E-lattices in hermitian spaces.

Definition 2.10. Let V be a hermitian space over E of dimension m, equipped with the hermitian form $(\ ,\ )_V$.

(1) For a subset X of V,
- we denote by X^{int} the subset $\{x\in X\mid (x,x)_V\in O_F\}$;
- we denote by $\langle X\rangle$ the O_E-submodule of V generated by X; when $X=\{x,\dots\}$ is explicitly presented, we simply write $\langle x,\dots\rangle$ instead of $\langle\{x,\dots\}\rangle$.

(2) For an O_E-lattice L of V, we put

$$\begin{aligned}L^\vee&:=\{x\in V\mid \mathrm{Tr}_{E/F}(x,y)_V\in O_F\text{ for every }y\in L\}\\&=\{x\in V\mid (x,y)_V\in u^{-1}O_E\text{ for every }y\in L\}.\end{aligned}$$

We say that L is
- *integral* if $L\subseteq L^\vee$;
- *vertex* if it is integral such that $L^\vee/L$ is annihilated by u; and
- *self-dual* if $L=L^\vee$.

(3) For an integral O_E-lattice L of V, we define
- the *fundamental invariants* of L unique integers $0\leqslant a_1\leqslant\cdots\leqslant a_m$ such that $L^\vee/L\simeq O_E/(u^{a_1})\oplus\cdots\oplus O_E/(u^{a_m})$ as O_E-modules;
- the *type* $t(L)$ of L to be the number of nonzero elements in its fundamental invariants; and
- the *valuation* of L to be $\mathrm{val}(L):=\sum_{i=1}^m a_i$; when L is generated by a single element x, we simply write $\mathrm{val}(x)$ instead of $\mathrm{val}(\langle x\rangle)$.

The above notation and definitions make sense without specifying V, namely, they apply to hermitian O_E-modules.

Definition 2.11. For a hermitian O_E-module L, we say that a basis $\{e_1,\dots,e_m\}$ of L is a *normal basis* if its moment matrix $T=((e_i,e_j)_L)_{i,j=1}^m$ is conjugate to

$$\left(\beta_1u^{2b_1}\right)\oplus\cdots\oplus\left(\beta_su^{2b_s}\right)\oplus\begin{pmatrix}0&u^{2c_1-1}\\-u^{2c_1-1}&0\end{pmatrix}\oplus\cdots\oplus\begin{pmatrix}0&u^{2c_t-1}\\-u^{2c_t-1}&0\end{pmatrix}$$

by a permutation matrix, for some $\beta_1,\dots,\beta_s\in O_F^\times$ and $b_1,\dots,b_s,c_1,\dots,c_t\in\mathbb{Z}$.

Lemma 2.12. *In the above definition, we have*

(1) normal basis exists;
(2) the invariants s,t and $b_1,\dots,b_s,c_1,\dots,c_t$ depend only on L;
(3) when L is integral, the fundamental invariants of L are the unique nondecreasing rearrangement of $(2b_1+1,\dots,2b_s+1,2c_1,2c_1,\dots,2c_t,2c_t)$.

Proof. Part (1) follows from [Jac62, Propositions 4.3 & 8.1]. Part (2) follows from the canonicity of the Jordan splitting on [Jac62, Page 449]. Part (3) follows from a direct calculation of $L^\vee$. □

Remark 2.13. The above lemma implies that for an integral hermitian O_E-module L of rank m with fundamental invariants $(a_1, \dots, a_m)$,

(1) L is vertex if and only if $a_m \leqslant 1$ and self-dual if and only if $a_m = 0$;
(2) $t(L)$ and $\mathrm{val}(L)$ must have the same parity with m.

Definition 2.14. Let M and L be two hermitian O_E-modules. We denote by $\mathrm{Herm}_{L,M}$ the scheme of hermitian O_E-module homomorphisms from L to M, which is a scheme of finite type over O_F. We define the *local density* to be

$$\mathrm{Den}(M,L) \coloneqq \lim_{N\to+\infty} \frac{\left|\mathrm{Herm}_{L,M}(O_F/(u^{2N}))\right|}{q^{N\cdot d_{L,M}}}$$

where $d_{L,M}$ is the dimension of $\mathrm{Herm}_{L,M} \otimes_{O_F} F$.

Denote by H the standard hyperbolic hermitian O_E-module (of rank 2) given by the matrix $\left(\begin{smallmatrix} 0 & u^{-1} \\ -u^{-1} & 0 \end{smallmatrix}\right)$. For an integer $s \geqslant 0$, put $H_s \coloneqq H^{\oplus s}$. Then H_s is a self-dual hermitian O_E-module of rank $2s$. The following lemma is a variant of a result of Cho–Yamauchi [CY20] when E/F is ramified.

Lemma 2.15. *Let L be a hermitian O_E-module of rank m. Then we have*

$$\mathrm{Den}(H_s, L) = \sum_{L\subseteq L'\subseteq L'^\vee} |L'/L|^{m-2s} \prod_{s-\frac{m+t(L')}{2}<i\leqslant s} (1-q^{-2i})$$

for every integer $s \geqslant m$, where the sum is taken over integral O_E-lattices of $L \otimes_{O_F} F$ containing L.

Proof. Put $V \coloneqq L \otimes_{O_F} F$. For an integral O_E-lattice L' of V, we equip the k-vector space $L'_k \coloneqq L' \otimes_{O_E} O_E/(u)$ with a k-valued pairing $\langle\ ,\ \rangle_{L'_k}$ by the formula

$$\langle x, y\rangle_{L'_k} \coloneqq u \cdot (x^\sharp, y^\sharp)_V \mod (u)$$

where $x^\sharp$ and $y^\sharp$ are arbitrary lifts of x and y, respectively. Then L'_k becomes a symplectic space over k of dimension m whose radical has dimension $t(L')$. Similarly, we have $H_{s,k}$, which is a nondegenerate symplectic space over k of dimension $2s$. We denote by $\mathrm{Isom}_{L'_k,H_{s,k}}$ the k-scheme of isometries from L'_k to $H_{s,k}$.

By the same argument in [CY20, Section 3.3], we have

$$\mathrm{Den}(H_s, L) = q^{-m(4s-m+1)/2} \cdot \sum_{L\subseteq L'\subseteq L'^\vee} |L'/L|^{m-2s} \left|\mathrm{Isom}_{L'_k,H_{s,k}}(k)\right|.$$

Thus, it remains to show that

$$\left|\mathrm{Isom}_{L'_k,H_{s,k}}(k)\right| = q^{m(4s-m+1)/2} \prod_{s-\frac{m+t(L')}{2}<i\leqslant s} (1-q^{-2i}). \tag{2.2}$$

We fix a decomposition $L'_k = L_0 \oplus L_1$ in which L_0 is nondegenerate and L_1 is the radical of L'_k. We have a morphism $\pi\colon \mathrm{Isom}_{L'_k,H_{s,k}} \to \mathrm{Isom}_{L_0,H_{s,k}}$ given by restriction, such that for every element $f \in \mathrm{Isom}_{L_0,H_{s,k}}(k)$, the fiber $\pi^{-1}f$ is isomorphic to $\mathrm{Isom}_{L_1,\mathrm{im}(f)^\perp}$. As $\mathrm{im}(f)^\perp$

is isomorphic to $H_{s-\frac{m-t(L')}{2},k}$, it suffices to show (2.2) in the two extremal cases: $t(L')=0$ and $t(L')=m$.

Suppose that $t(L')=0$, that is, L'_k is nondegenerate. Note that $\mathrm{Sp}(H_{s,k})$ acts on $\mathrm{Isom}_{L'_k,H_{s,k}}$ transitively, with the stabilizer isomorphic to $\mathrm{Sp}(H_{s-\frac{m}{2},k})$. Thus, we have

$$\begin{aligned}\left|\mathrm{Isom}_{L'_k,H_{s,k}}(k)\right| &= \frac{|\mathrm{Sp}(H_{s,k})(k)|}{\left|\mathrm{Sp}(H_{s-\frac{m}{2},k})(k)\right|}\\ &= \frac{q^{s^2}\prod_{i=1}^{s}(q^{2i}-1)}{q^{\left(s-\frac{m}{2}\right)^2}\prod_{i=1}^{s-\frac{m}{2}}(q^{2i}-1)}\\ &= q^{m(4s-m+1)/2}\prod_{s-\frac{m}{2}<i\leqslant s}(1-q^{-2i}),\end{aligned}$$

which confirms (2.2).

Suppose that $t(L')=m$, that is, L'_k is isotropic. Note that $\mathrm{Sp}(H_{s,k})$ acts on $\mathrm{Isom}_{L'_k,H_{s,k}}$ transitively, with the stabilizer Q fits into a short exact sequence

$$1\to U_m\to Q\to \mathrm{Sp}(H_{s-m,k})\to 1$$

in which U_m is a unipotent subgroup of $\mathrm{Sp}(H_{s,k})$ of Levi type $\mathrm{GL}_{m,k}\times\mathrm{Sp}(H_{s-m,k})$. Thus, we have

$$\begin{aligned}|\mathrm{Isom}_{L'_k,H_{s,k}}(k)| &= \frac{|\mathrm{Sp}(H_{s,k})(k)|}{|U_m(k)|\cdot|\mathrm{Sp}(H_{s-m,k})(k)|}\\ &= \frac{q^{s^2}\prod_{i=1}^{s}(q^{2i}-1)}{q^{m(2s-2m)+\frac{m(m+1)}{2}}\cdot q^{(s-m)^2}\prod_{i=1}^{s-m}(q^{2i}-1)}\\ &= q^{m(4s-m+1)/2}\prod_{s-m<i\leqslant s}(1-q^{-2i}),\end{aligned}$$

which confirms (2.2).

Thus, (2.2) is proved and the lemma follows. ☐

Now we fix a hermitian space $\boldsymbol{V}$ over E of dimension $n=2r$ that is *nonsplit*.

Definition 2.16. For an O_E-lattice $\boldsymbol{L}$ of $\boldsymbol{V}$, define the *(normalized) local Siegel series* of $\boldsymbol{L}$ to be the polynomial $\mathrm{Den}(X,\boldsymbol{L})\in\mathbb{Z}[X]$, which exists by Lemma 2.19 below, such that for every integer $s\geqslant 0$,

$$\mathrm{Den}(q^{-s},\boldsymbol{L})=\frac{\mathrm{Den}(H_{r+s},\boldsymbol{L})}{\prod_{i=s+1}^{r+s}(1-q^{-2i})},$$

where Den is defined in Definition 2.14. We then put

$$\partial\mathrm{Den}(\boldsymbol{L}):=-\left.\frac{\mathrm{d}}{\mathrm{d}X}\right|_{X=1}\mathrm{Den}(X,\boldsymbol{L}).$$

Remark 2.17. Since $\boldsymbol{V}$ is nonsplit, we have $\mathrm{Den}(1,\boldsymbol{L})=\mathrm{Den}(H_r,\boldsymbol{L})=0$.

Remark 2.18. Let $\boldsymbol{L}$ be an O_E-lattice of $\boldsymbol{V}$. Let $T\in\mathrm{GL}_n(E)$ be a representing matrix of $\boldsymbol{L}$, and consider the T-th Whittaker function $W_T(s,1_{4r},\mathbb{1}_{H_r^{2r}})$ of the Schwartz function $\mathbb{1}_{H_r^{2r}}$ at the identity element 1_{4r}. By [KR14, Proposition 10.1],[10] we have

$$W_T(s,1_{4r},\mathbb{1}_{H_r^{2r}})=\mathrm{Den}(H_{r+s},\boldsymbol{L})$$

[10] In [KR14, Proposition 10.1] and its proof, the lattice $L_{r,r}$ should be replaced by H_r.

for every integer $s \geqslant 0$. Thus, we obtain

$$\log q \cdot \partial\mathrm{Den}(\boldsymbol{L}) = \frac{W_T'(0, 1_{4r}, \mathbb{1}_{H_r^{2r}})}{\prod_{i=1}^{r}(1-q^{-2i})}$$

by Definition 2.16.

Lemma 2.19. *For every O_E-lattice $\boldsymbol{L}$ of $\boldsymbol{V}$, we have*

$$\mathrm{Den}(X, \boldsymbol{L}) = \sum_{\boldsymbol{L}\subseteq L\subseteq L^\vee} X^{2\mathrm{length}_{O_E}(L/\boldsymbol{L})} \prod_{i=0}^{\frac{t(L)}{2}-1}(1-q^{2i}X^2), \tag{2.3}$$

and

$$\partial\mathrm{Den}(\boldsymbol{L}) = 2\sum_{\boldsymbol{L}\subseteq L\subseteq L^\vee} \prod_{i=1}^{\frac{t(L)}{2}-1}(1-q^{2i}), \tag{2.4}$$

where both sums are taken over integral O_E-lattices of $\boldsymbol{V}$ containing $\boldsymbol{L}$.[11]

Proof. The identity (2.3) is a direct consequence of Lemma 2.15 and Definition 2.16. The identity (2.4) is a consequence of (2.3). □

Definition 2.20. Let $L^\flat$ be an element of $\flat(\boldsymbol{V})$ (Definition 2.8). For $x \in \boldsymbol{V} \setminus V_{L^\flat}$, we put

$$\partial\mathrm{Den}_{L^\flat}(x) := \partial\mathrm{Den}(L^\flat + \langle x\rangle),$$
$$\partial\mathrm{Den}^{\mathrm{h}}_{L^\flat}(x) := 2\sum_{\substack{L^\flat\subseteq L\subseteq L^\vee\\ t(L\cap V_{L^\flat})=1}} \mathbb{1}_L(x),$$
$$\partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}(x) := \partial\mathrm{Den}_{L^\flat}(x) - \partial\mathrm{Den}^{\mathrm{h}}_{L^\flat}(x).$$

Here in the second formula, L in the summation is an O_E-lattice of $\boldsymbol{V}$.

Remark 2.21. We have

1. (1) The summation in $\partial\mathrm{Den}^{\mathrm{h}}_{L^\flat}(x)$ equals twice the number of integral O_E-lattices L of $\boldsymbol{V}$ that contains $L^\flat + \langle x\rangle$ and such that $t(L \cap V_{L^\flat}) = 1$.
2. (2) There exists a compact subset $C_{L^\flat}$ of $\boldsymbol{V}$ such that $\partial\mathrm{Den}_{L^\flat}$, $\partial\mathrm{Den}^{\mathrm{h}}_{L^\flat}$, and $\partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}$ vanish outside $C_{L^\flat}$ and are locally constant functions on $C_{L^\flat} \setminus V_{L^\flat}$.
3. (3) For an integral O_E-lattice L of $\boldsymbol{V}$, if $t(L \cap V_{L^\flat}) = 1$, then $t(L) = 2$ by Lemma 2.23(1) below and the fact that $\boldsymbol{V}$ is nonsplit.
4. (4) By (3) and Lemma 2.19, we have

$$\partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}(x) = 2\sum_{\substack{L^\flat\subseteq L\subseteq L^\vee\\ t(L\cap V_{L^\flat})>1}} \left(\prod_{i=1}^{\frac{t(L)}{2}-1}(1-q^{2i})\right)\mathbb{1}_L(x)$$

for $x \in \boldsymbol{V} \setminus V_{L^\flat}$.

The following is our main result of this subsection.

Proposition 2.22. *Let $L^\flat$ be an element of $\flat(\boldsymbol{V})$. Then $\partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}$ extends (uniquely) to a (compactly supported) locally constant function on $\boldsymbol{V}$, which we still denote by $\partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}$. Moreover, the support of $\widehat{\partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}}$ is contained in $\boldsymbol{V}^{\mathrm{int}}$ (Definition 2.10).*

[11]In (2.4), when $t(L) = 2$, we regard the empty product $\prod_{i=1}^{\frac{t(L)}{2}-1}(1-q^{2i})$ as 1.

We need some lemmas for preparation.

Lemma 2.23. *Let L be an integral hermitian O_E-module of with fundamental invariants $(a_1,\dots,a_m)$.*

(1) If $T=((e_i,e_j)_L)_{i,j=1}^m$ is the moment matrix of an arbitrary basis $\{e_1,\dots,e_m\}$ of L, then for every $1\leqslant i\leqslant m$, $a_1+\cdots+a_i-i$ equals the minimal E-valuation of the determinant of all i-by-i minors of T.

(2) If $L=L'+\langle x\rangle$ for some (integral) hermitian O_E-module L' contained in L of rank $m-1$, then we have

$$t(L)=\begin{cases} t(L')+1, & \text{if } x'\in uL'^\vee+L',\\ t(L')-1, & \text{otherwise,}\end{cases}$$

where x' is the unique element in $L'^\vee$ such that $(x',y)_L=(x,y)_L$ for every $y\in L'$.

Proof. Part (1) is simply the well-known method of computing the Smith normal form of uT (over O_E) using ideals generated by determinants of minors. For (2), take a normal basis $\{x_1,\dots,x_{m-1}\}$ of L (Definition 2.11) such that $\langle x_1,\dots,x_{m-1-t(L')}\rangle$ is self-dual. Applying (1) to the basis $\{x_1,\dots,x_{m-1},x\}$ of L, we know that $t(L)=t(L')+1$ if $(x_i,x)_L\in O_E$ for every $m-t(L')\leqslant i\leqslant m-1$; otherwise, we have $t(L)=t(L')-1$. In particular, (2) follows. ☐

In the rest of this subsection, in order to shorten formulae, we put

$$\mu(t)\coloneqq\prod_{i=1}^{\frac{t}{2}-1}(1-q^{2i})$$

for every positive even integer t.

Lemma 2.24. *Take $L^\flat\in\flat(\boldsymbol{V})$ that is integral. For every compact subset X of $\boldsymbol{V}$ not contained in $V_{L^\flat}$, we denote by δ_X the maximal integer such that the image of X under the projection map $\boldsymbol{V}\to V_{L^\flat}^\perp$ induced by the orthogonal decomposition $\boldsymbol{V}=V_{L^\flat}\oplus V_{L^\flat}^\perp$ is contained in $u^{\delta_X}(V_{L^\flat}^\perp)^{\mathrm{int}}$. We denote by $\mathfrak{L}$ the set of O_E-lattices of $\boldsymbol{V}$ containing $L^\flat$, and by $\mathfrak{E}$ the set of triples $(L^{\flat\prime},\delta,\varepsilon)$ in which $L^{\flat\prime}$ is an O_E-lattice of $V_{L^\flat}$ containing $L^\flat$, $\delta\in\mathbb{Z}$, and $\varepsilon\colon u^\delta(V_{L^\flat}^\perp)^{\mathrm{int}}\to L^{\flat\prime}\otimes_{O_F}F/O_F$ is an O_E-linear map.*

(1) The map $\mathfrak{L}\to\mathfrak{E}$ sending L to the triple $(L\cap V_{L^\flat},\delta_L,\varepsilon_L)$ is a bijection, where ε_L is the extension map $u^{\delta_L}(V_{L^\flat}^\perp)^{\mathrm{int}}\to(L\cap V_{L^\flat})\otimes_{O_F}F/O_F$ induced by the short exact sequence

$$0\to L\cap V_{L^\flat}\to L\to u^{\delta_L}(V_{L^\flat}^\perp)^{\mathrm{int}}\to 0.$$

Moreover, L is integral if and only if the following hold:
- *$L\cap V_{L^\flat}$ is integral;*
- *the image of ε is contained in $(L\cap V_{L^\flat})^\vee/(L\cap V_{L^\flat})$;*
- *$\varepsilon_L(x)+x\subseteq\boldsymbol{V}^{\mathrm{int}}$ for every $x\in u^{\delta_L}(V_{L^\flat}^\perp)^{\mathrm{int}}$.*[12]

(2) For $L\in\mathfrak{L}$ that is integral and corresponds to $(L^{\flat\prime},\delta,\varepsilon)\in\mathfrak{E}$, we have

$$t(L)=\begin{cases} t(L^{\flat\prime})+1, & \text{if the image of } \varepsilon \text{ is contained in } (u(L^{\flat\prime})^\vee+L^{\flat\prime})/L^{\flat\prime},\\ t(L^{\flat\prime})-1, & \text{otherwise.}\end{cases}$$

[12]For $(L^{\flat\prime},\delta,\varepsilon)\in\mathfrak{E}$, we regard $\varepsilon(x)+x$ as an $L^{\flat\prime}$-coset in $\boldsymbol{V}$ as long as we write $\varepsilon(x)+x\subseteq\Omega$ for a subset Ω of $\boldsymbol{V}$.

(3) *For every fixed integral O_E-lattice $L^{\flat\prime}$ of $V_{L^\flat}$ containing $L^\flat$, the sum*

$$\sum_{\substack{L\subseteq L^\vee \\ L\cap V_{L^\flat}=L^{\flat\prime}}} q^{-\delta_L}|\mu(t(L))|$$

is convergent; and if $t(L^{\flat\prime})>1$, then we have

$$\sum_{\substack{L\subseteq L^\vee \\ L\cap V_{L^\flat}=L^{\flat\prime} \\ z\in L^\vee}} q^{-\delta_L}\mu(t(L))=0$$

for every $z\in \boldsymbol{V}\setminus \boldsymbol{V}^{\mathrm{int}}$.

(4) *For every fixed integral O_E-lattice $L^{\flat\prime}$ of $V_{L^\flat}$ containing $L^\flat$ with $t(L^{\flat\prime})>1$, we have*

$$\sum_{\substack{L\subseteq L^\vee \\ L\cap V_{L^\flat}=L^{\flat\prime} \\ \delta_L=0}} \mu(t(L))=0.$$

Proof. For (1), the inverse map $\mathfrak{E}\to\mathfrak{L}$ is the one that sends $(L^{\flat\prime},\delta,\varepsilon)$ to the O_E-lattice L generated by $L^{\flat\prime}$ and $\varepsilon_L(x)+x$ for every $x\in u^{\delta_L}(V_{L^\flat}^\perp)^{\mathrm{int}}$. The rest of (1) is straightforward.

Part (2) is simply Lemma 2.23(2).

Part (4) follows by applying (3) to generators z of O_E-modules $u^{-1}(V_{L^\flat}^\perp)^{\mathrm{int}}$ and $u^{-2}(V_{L^\flat}^\perp)^{\mathrm{int}}$ and then taking the difference.

Now we prove (3), which is the most difficult one. For every $x\in\boldsymbol{V}$, we denote by $x'\in V_{L^\flat}$ the first component of x with respect to the orthogonal decomposition $\boldsymbol{V}=V_{L^\flat}\oplus V_{L^\flat}^\perp$. Put

$$\Omega\coloneqq\{x\in\boldsymbol{V}^{\mathrm{int}}\mid x'\in(L^{\flat\prime})^\vee\},\qquad \Omega^\circ\coloneqq\{x\in\boldsymbol{V}^{\mathrm{int}}\mid x'\in u(L^{\flat\prime})^\vee+L^{\flat\prime}\}.$$

Note that both Ω and Ω° are open compact subsets of $\boldsymbol{V}$ stable under the translation by $L'^\flat$. For an element $L\in\mathfrak{L}$ corresponding to $(L^{\flat\prime},\delta,\varepsilon)\in\mathfrak{E}$ from (1), L is integral if and only $\varepsilon(x)+x\subseteq\Omega$ for every $x\in u^\delta(V_{L^\flat}^\perp)^{\mathrm{int}}$. By (2), for such L,

$$t(L)=\begin{cases} t(L^{\flat\prime})+1, & \text{if } \varepsilon(x)+x\subseteq\Omega^\circ \text{ for every } x\in u^\delta(V_{L^\flat}^\perp)^{\mathrm{int}}\setminus u^{\delta+1}(V_{L^\flat}^\perp)^{\mathrm{int}},\\ t(L^{\flat\prime})-1, & \text{if } \varepsilon(x)+x\subseteq\Omega\setminus\Omega^\circ \text{ for every } x\in u^\delta(V_{L^\flat}^\perp)^{\mathrm{int}}\setminus u^{\delta+1}(V_{L^\flat}^\perp)^{\mathrm{int}}.\end{cases}$$

Thus, we may replace the term corresponding to L in the summation in (3) by an integration over the region $\bigcup_{x\in u^\delta(V_{L^\flat}^\perp)^{\mathrm{int}}\setminus u^{\delta+1}(V_{L^\flat}^\perp)^{\mathrm{int}}}(\varepsilon(x)+x)$ of Ω. It follows that

$$\sum_{\substack{L\subseteq L^\vee \\ L\cap V_{L^\flat}=L^{\flat\prime}}} q^{-\delta_L}|\mu(t(L))|=\frac{1}{C}\left(\int_{\Omega^\circ\setminus V_{L^\flat}}|\mu(t(L^{\flat\prime})+1)|\,\mathrm{d}x+\int_{\Omega\setminus(\Omega^\circ\cup V_{L^\flat})}|\mu(t(L^{\flat\prime})-1)|\,\mathrm{d}x\right),$$

which is convergent, where

$$C=\mathrm{vol}(L^{\flat\prime})\cdot\mathrm{vol}((V_{L^\flat}^\perp)^{\mathrm{int}}\setminus u(V_{L^\flat}^\perp)^{\mathrm{int}}).$$

Now we take an element $z\in\boldsymbol{V}\setminus\boldsymbol{V}^{\mathrm{int}}$. We may assume $z'\in(L^{\flat\prime})^\vee$ since otherwise the summation in (3) is empty. Put

$$\Omega_z\coloneqq\{x\in\Omega\mid(x,z)_{\boldsymbol{V}}\in u^{-1}O_E\},\qquad \Omega_z^\circ\coloneqq\{x\in\Omega^\circ\mid(x,z)_{\boldsymbol{V}}\in u^{-1}O_E\},$$

both stable under the translation by $L'^{\flat}$ since $z' \in (L^{\flat\prime})^\vee$. Similarly, we have

$$\sum_{\substack{L\subseteq L^\vee \\ L\cap V_{L^\flat}=L^{\flat\prime} \\ z\in L^\vee}} q^{-\delta_L}\mu(t(L)) = \frac{1}{C}\left(\int_{\Omega_z^\circ\setminus V_{L^\flat}} \mu(t(L^{\flat\prime})+1)\,\mathrm{d}x + \int_{\Omega_z\setminus(\Omega_z^\circ\cup V_{L^\flat})} \mu(t(L^{\flat\prime})-1)\,\mathrm{d}x\right)$$

$$= \frac{\mu(t(L^{\flat\prime})-1)}{C}\left(\mathrm{vol}(\Omega_z\setminus\Omega_z^\circ) + \left(1-q^{t(L^{\flat\prime})-1}\right)\mathrm{vol}(\Omega_z^\circ)\right)$$

$$= \frac{\mu(t(L^{\flat\prime})-1)}{C}\left(\mathrm{vol}(\Omega_z) - q^{t(L^{\flat\prime})-1}\,\mathrm{vol}(\Omega_z^\circ)\right),$$

where we have used $t(L^{\flat\prime}) > 1$ in the second equality. Thus, it remains to show that

$$\mathrm{vol}(\Omega_z) = q^{t(L^{\flat\prime})-1}\,\mathrm{vol}(\Omega_z^\circ). \tag{2.5}$$

We fix an orthogonal decomposition $L^{\flat\prime} = L_0 \oplus L_1$ in which L_0 is self-dual and L_1 is of both rank and type $t(L^{\flat\prime})$. Since both Ω_z and Ω_z° depend only on the coset $z + L^{\flat\prime}$, we may assume $z' \in L_1^\vee$ and anisotropic. Let $V_2 \subseteq \boldsymbol{V}$ be the orthogonal complement of $L_0 + \langle z\rangle$. We claim

($*$) There exists an integral O_E-lattice L_2 of V_2 of of type $t(L^{\flat\prime})$ such that

$$(u^i L_2^\vee)^{\mathrm{int}} = \{x \in V_2^{\mathrm{int}} \mid x' \in u^i L_1^\vee\} \tag{2.6}$$

for $i = 0, 1$.

Assuming ($*$), by construction, we have

$$\{x \in \boldsymbol{V} \mid (x,z)_{\boldsymbol{V}} \in u^{-1}O_E\} = L_0 \otimes_{O_F} F \oplus \langle z\rangle^\vee \oplus V_2.$$

Now we use the condition $z \notin \boldsymbol{V}^{\mathrm{int}}$, which implies that $\langle z\rangle^\vee \subseteq u\langle z\rangle \cap \boldsymbol{V}^{\mathrm{int}}$. Combining with (2.6), we obtain

$$\Omega_z = L_0 \times \langle z\rangle^\vee \times (L_2^\vee)^{\mathrm{int}}, \qquad \Omega_z^\circ = L_0 \times \langle z\rangle^\vee \times (uL_2^\vee)^{\mathrm{int}}.$$

Thus, (2.5) follows from Lemma 2.25 below. Part (3) is proved.

Now we show ($*$). There are two cases.

First, we assume $z \neq z'$, that is, $z \notin V_{L^\flat}$. Let L_2 be the unique O_E-lattice of V_2 satisfying

$$L_2^\vee = \{x \in V_2 \mid x' \in L_1^\vee\}. \tag{2.7}$$

Then (2.6) clearly holds. Thus, it remains to show that L_2 is integral of type $t(L^{\flat\prime})$. Put $w \coloneqq z - z' \in V_{L^\flat}^\perp$, which is nonzero, hence anisotropic. Then

$$\bar{z} \coloneqq z' - \frac{(z',z')_{\boldsymbol{V}}}{(w,w)_{\boldsymbol{V}}} w$$

is the unique element in V_2 such that $\bar{z}' = z'$. To compute L_2, we write

$$L_1^\vee = M + \langle y + \alpha z'\rangle$$

for some $y \in V_{L^\flat} \cap V_2$ and $\alpha \in E \setminus uO_E$, where $M \coloneqq L_1^\vee \cap V_2$. Then

$$M^\dagger \coloneqq L_1 \cap V_2 = \{x \in M^\vee \mid (x,y)_{\boldsymbol{V}} \in u^{-1}O_E\}.$$

Since $M^\vee/M^\dagger$ is isomorphic to an O_E-submodule of $E/u^{-1}O_E$, we may take an element $y^\dagger \in M^\vee$ that generates $M^\vee/M^\dagger$. Then we have

$$L_1 = M^\dagger + \langle y^\dagger + \alpha^\dagger z'\rangle$$

for some $\alpha^\dagger\in E^\times$ such that $(y^\dagger,y)_{\boldsymbol{V}}+\alpha^\dagger\alpha^\mathtt{c}(z',z')_{\boldsymbol{V}}\in u^{-1}O_E$. Now by (2.7), we have

$$L_2^\vee=M+\langle y+\alpha\bar z\rangle.$$

By the same argument, we have

$$L_2=M^\dagger+\langle y^\dagger+\alpha^\dagger\rho\bar z\rangle,$$

where

$$\rho\coloneqq\frac{(z',z')_{\boldsymbol{V}}}{(\bar z,\bar z)_{\boldsymbol{V}}}.$$

By Lemma 2.23(2), we have $t(L_2)=t(L_1)=t(L^{\flat\prime})$ as long as L_2 is integral. Thus, it suffices to show that $y^\dagger+\alpha^\dagger\rho\bar z\in\boldsymbol{V}^{\mathrm{int}}$. We compute

$$\begin{aligned}
&(y^\dagger+\alpha^\dagger\rho\bar z,y^\dagger+\alpha^\dagger\rho\bar z)_{\boldsymbol{V}}-(y^\dagger+\alpha^\dagger z',y^\dagger+\alpha^\dagger z')_{\boldsymbol{V}}\\
&=(\alpha^\dagger\rho\bar z,\alpha^\dagger\rho\bar z)_{\boldsymbol{V}}-(\alpha^\dagger z',\alpha^\dagger z')_{\boldsymbol{V}}\\
&=\mathrm{Nm}_{E/F}(\alpha^\dagger)\left(\frac{(z',z')^2_{\boldsymbol{V}}}{(\bar z,\bar z)_{\boldsymbol{V}}}-(z',z')_{\boldsymbol{V}}\right)\\
&=\mathrm{Nm}_{E/F}(\alpha^\dagger)(z',z')_{\boldsymbol{V}}\left(\frac{(z',z')_{\boldsymbol{V}}}{(z',z')_{\boldsymbol{V}}+\frac{(z',z')^2_{\boldsymbol{V}}}{(w,w)_{\boldsymbol{V}}}}-1\right)\\
&=\mathrm{Nm}_{E/F}(\alpha^\dagger)(z',z')_{\boldsymbol{V}}\left(\frac{(w,w)_{\boldsymbol{V}}}{(z',z')_{\boldsymbol{V}}+(w,w)_{\boldsymbol{V}}}-1\right)\\
&=\frac{-(\alpha^\dagger)^\mathtt{c}}{\alpha^\dagger}\frac{(\alpha^\dagger z',z')^2_{\boldsymbol{V}}}{(z,z)_{\boldsymbol{V}}}.
\end{aligned}$$

As $z'\in L_1^\vee$, we have $(\alpha^\dagger z',z')_{\boldsymbol{V}}\in u^{-1}O_E$. As $z\notin\boldsymbol{V}^{\mathrm{int}}$, we have $(z,z)_{\boldsymbol{V}}\notin u^{-1}O_E$. Together, we have $\frac{(\alpha^\dagger z',z')^2_{\boldsymbol{V}}}{(z,z)_{\boldsymbol{V}}}\in O_F$. Thus, $y^\dagger+\alpha^\dagger\rho\bar z\in\boldsymbol{V}^{\mathrm{int}}$ as $y^\dagger+\alpha^\dagger z'\in\boldsymbol{V}^{\mathrm{int}}$, hence L_2 meets the requirement in $(*)$.

Second, we assume $z=z'$, that is, $z\in V_{L^\flat}$. Take $L_2=(L_1^\vee\cap V_2)^\vee\oplus u^\delta(V_{L^\flat}^\perp)^{\mathrm{int}}$ for some integer $\delta\geqslant 0$ determined later. We show that $(L_1^\vee\cap V_2)^\vee$ is an integral hermitian O_E-module of type $t(L^{\flat\prime})-1$. As in the previous case, we write

$$L_1^\vee=M+\langle y+\alpha z'\rangle$$

for some $y\in V_{L^\flat}\cap V_2$ and $\alpha\in E\setminus uO_E$, where $M\coloneqq L_1^\vee\cap V_2$. Then

$$L_1=M^\dagger+\langle y^\dagger+\alpha^\dagger z'\rangle,$$

so that $M^\vee$ is generated by $M^\dagger$ and $y^\dagger$. As L_1 is of type $t(L^{\flat\prime})$, which is its rank, we have $L_1\subseteq uL_1^\vee$, that is,

$$M^\dagger+\langle y^\dagger+\alpha^\dagger z'\rangle\subseteq uM+u\langle y+\alpha z'\rangle,$$

hence $M^\dagger\subseteq uM$. As $z'\in L_1^\vee$, we have $(\alpha z',z')_{\boldsymbol{V}}\in u^{-1}O_E$. As $z'=z\notin\boldsymbol{V}^{\mathrm{int}}$, we have $(z',z')_{\boldsymbol{V}}\notin u^{-1}O_E$, hence $\alpha^\dagger\in uO_E$. Again as $z'\in L_1^\vee$, we have $\alpha^\dagger z'\in uL_1^\vee$, hence $y^\dagger\in uL_1^\vee\cap V_2=uM$. Together, we obtain $M^\vee\subseteq uM$, that is, $(L_1^\vee\cap V_2)^\vee$ is an integral hermitian O_E-module of type $t(L^{\flat\prime})-1$.

Consequently, L_2 is an integral O_E-lattice of V_2 of type $t(L^{\flat\prime})$. Since $L_2^\vee=(L_1^\vee\cap V_2)\oplus u^{-\delta-1}(V_{L^\flat}^\perp)^{\mathrm{int}}$, it is clear that for δ sufficiently large, (2.6) holds for $i=0,1$. Thus, $(*)$ is proved.

The lemma is all proved. □

Lemma 2.25. *Let L be an integral hermitian O_E-module of rank $2m+1$ for some integer $m\geqslant 0$ with $t(L)=2m+1$. Then we have*

$$\left|(L^\vee)^{\mathrm{int}}/L\right| = q^{2m}\cdot\left|(uL^\vee)^{\mathrm{int}}/L\right|. \tag{2.8}$$

Note that both $(L^\vee)^{\mathrm{int}}$ and $(uL^\vee)^{\mathrm{int}}$ are stable under the translation by L as $t(L)=2m+1$.

Proof. Put $V\coloneqq L\otimes_{O_F}F$. We prove by induction on $\mathrm{val}(L)$ for integral O_E-lattices L of V with $t(L)=2m+1$ that (2.8) holds.

The initial case is such that $\mathrm{val}(L)=2m+1$, that is, $L^\vee=u^{-1}L$. The pairing $u^2(\ ,\ )_V$ induces a nondegenerate quadratic form on $L^\vee/L$. It is clear that $(L^\vee)^{\mathrm{int}}/L$ is exactly the set of isotropic vectors in $L^\vee/L$ under the previous form. In particular, we have

$$\left|(L^\vee)^{\mathrm{int}}/L\right| = q^{2m} = q^{2m}\cdot\left|(uL^\vee)^{\mathrm{int}}/L\right|.$$

Now we consider L with $\mathrm{val}(L)>2m+1$, and suppose that (2.8) holds for such L' with $\mathrm{val}(L')<\mathrm{val}(L)$. Choose an orthogonal decomposition $L=L_0\oplus L_1$ in which L_0 is an integral hermitian O_E-module with fundamental invariants $(1,\dots,1)$ and such that all fundamental invariants of L_1 are at least 2. In particular, L_1 has positive rank. It is easy to see that we may choose a hermitian O_E-module L_1' contained in $u^{-1}L_1$ satisfying $L_1\subsetneq L_1'$ and $t(L_1')=t(L_1)$. Put $L'\coloneqq L_0\oplus L_1'$. By the induction hypothesis, we have

$$\left|(L'^\vee)^{\mathrm{int}}/L'\right| = q^{2m}\cdot\left|(uL'^\vee)^{\mathrm{int}}/L'\right|.$$

It remains to show that

$$\left|((L^\vee)^{\mathrm{int}}\setminus(L'^\vee)^{\mathrm{int}})/L\right| = q^{2m}\cdot\left|((uL^\vee)^{\mathrm{int}}\setminus(uL'^\vee)^{\mathrm{int}})/L\right|. \tag{2.9}$$

We claim that the map

$$((L^\vee)^{\mathrm{int}}\setminus(L'^\vee)^{\mathrm{int}})/L\to((uL^\vee)^{\mathrm{int}}\setminus(uL'^\vee)^{\mathrm{int}})/L$$

given by the multiplication by u is q^{2m}-to-1. Take an element $x\in(uL^\vee)^{\mathrm{int}}\setminus(uL'^\vee)^{\mathrm{int}}$. Its preimage is bijective to the set of elements $(y_0,y_1)\in L_0/uL_0\oplus L_1/uL_1$ such that $u^{-1}(x+(y_0,y_1))\in V^{\mathrm{int}}$, which amounts to the equation

$$(x,x)_V+\mathrm{Tr}_{E/F}(x,y_0)_V+\mathrm{Tr}_{E/F}(x,y_1)_V+(y_0,y_0)_V\in u^2O_F.$$

Since $x\in(uL_0^\vee)\times((uL_1^\vee)^{\mathrm{int}}\setminus(u^2L_1^\vee)^{\mathrm{int}})$, there exists $y_1\in L_1$ such that $(x,y_1)_V\in O_E^\times$. In other words, for each y_0, the above relation defines a nontrivial linear equation on L_1/uL_1. Thus, the preimage of x has cardinality q^{2m}. We obtain (2.9), hence complete the induction process. ∎

Proof of Proposition 2.22. We fix an element $L^\flat\in\flat(\boldsymbol{V})$. If $L^\flat$ is not integral, then $\partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}\equiv 0$, hence the proposition is trivial. Thus, we now assume $L^\flat$ integral and will freely adopt notation from Lemma 2.24.

To show that $\partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}$ extends to a compactly supported locally constant function on $\boldsymbol{V}$, it suffices to show that for every $y\in V_{L^\flat}/L^\flat$, there exists an integer $\delta(y)>0$ such that $\partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}(y+x)$ is constant for $x\in u^{\delta(y)}(V_{L^\flat}^\perp)^{\mathrm{int}}\setminus\{0\}$. If $L^\flat+\langle y\rangle$ is not integral, then there exists $\delta(y)>0$ such that $L^\flat+\langle y+x\rangle$ is not integral for $x\in u^{\delta(y)}(V_{L^\flat}^\perp)^{\mathrm{int}}\setminus\{0\}$, which implies $\partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}(y+x)=0$.

Now we fix an element $y\in V_{L^\flat}/L^\flat$ such that $L^\flat+\langle y\rangle$ is integral. We claim that we may take $\delta(y)=a_{n-1}$, which is the maximal element in the fundamental invariants of $L^\flat$. It amounts to showing that for every fixed pair (f_1, f_2) of generators of the O_E-module $(V_{L^\flat}^\perp)^{\mathrm{int}}$, we have

$$\partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}(y+u^\delta f_1) - \partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}(y+u^{\delta-1} f_2) = 0 \tag{2.10}$$

for $\delta > a_{n-1}$. For every $\delta' \in \mathbb{Z}$, we define two sets

$$\mathfrak{L}_1^{\delta'} \coloneqq \{L \in \mathfrak{L} \mid L \subseteq L^\vee, \delta_L = \delta', y + u^\delta f_1 \in L\},$$
$$\mathfrak{L}_2^{\delta'} \coloneqq \{L \in \mathfrak{L} \mid L \subseteq L^\vee, \delta_L = \delta', y + u^{\delta-1} f_2 \in L\}.$$

By Remark 2.21(4), we have

$$\partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}(y+u^\delta f_1) = 2\sum_{\delta'\leqslant\delta}\sum_{\substack{L\in\mathfrak{L}_1^{\delta'}\\ t(L\cap V_{L^\flat})>1}}\mu(t(L)) = 2\sum_{\substack{L^\flat\subseteq L^{\flat\prime}\subseteq(L^{\flat\prime})^\vee\\ t(L^{\flat\prime})>1}}\sum_{\delta'\leqslant\delta}\sum_{\substack{L\in\mathfrak{L}_1^{\delta'}\\ L\cap V_{L^\flat}=L^{\flat\prime}}}\mu(t(L)),$$

$$\partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}(y+u^{\delta-1} f_2) = 2\sum_{\delta'\leqslant\delta-1}\sum_{\substack{L\in\mathfrak{L}_2^{\delta'}\\ t(L\cap V_{L^\flat})>1}}\mu(t(L)) = 2\sum_{\substack{L^\flat\subseteq L^{\flat\prime}\subseteq(L^{\flat\prime})^\vee\\ t(L^{\flat\prime})>1}}\sum_{\delta'\leqslant\delta-1}\sum_{\substack{L\in\mathfrak{L}_2^{\delta'}\\ L\cap V_{L^\flat}=L^{\flat\prime}}}\mu(t(L)).$$

Now we claim that

$$\sum_{\delta'\leqslant\delta}\sum_{\substack{L\in\mathfrak{L}_1^{\delta'}\\ L\cap V_{L^\flat}=L^{\flat\prime}}}\mu(t(L)) - \sum_{\delta'\leqslant\delta-1}\sum_{\substack{L\in\mathfrak{L}_2^{\delta'}\\ L\cap V_{L^\flat}=L^{\flat\prime}}}\mu(t(L)) = 0 \tag{2.11}$$

for every $L^{\flat\prime}$ in the summation. Since $\delta > a_{n-1}$, it follows that for $\delta' < 0$, we have

$$\mathfrak{L}_1^{\delta'} = \mathfrak{L}_2^{\delta'} = \{L \in \mathfrak{L} \mid L \subseteq L^\vee, \delta_L = \delta', y \in L\}.$$

Thus, the left-hand side of (2.11) equals

$$\sum_{\delta'=0}^{\delta}\sum_{\substack{L\in\mathfrak{L}_1^{\delta'}\\ L\cap V_{L^\flat}=L^{\flat\prime}}}\mu(t(L)) - \sum_{\delta'=0}^{\delta-1}\sum_{\substack{L\in\mathfrak{L}_2^{\delta'}\\ L\cap V_{L^\flat}=L^{\flat\prime}}}\mu(t(L)). \tag{2.12}$$

However, we also have $\mathfrak{L}_1^{0} = \{L \in \mathfrak{L} \mid L \subseteq L^\vee, \delta_L = \delta', y \in L\}$, which implies

$$\sum_{\substack{L\in\mathfrak{L}_1^{0}\\ L\cap V_{L^\flat}=L^{\flat\prime}}}\mu(t(L)) = \mathbb{1}_{L^{\flat\prime}}(y)\sum_{\substack{L\subseteq L^\vee\\ L\cap V_{L^\flat}=L^{\flat\prime}\\ \delta_L=0}}\mu(t(L)),$$

which vanishes by Lemma 2.24(4). Thus, we obtain

$$(2.12) = \sum_{\delta'=1}^{\delta}\sum_{\substack{L\in\mathfrak{L}_1^{\delta'}\\ L\cap V_{L^\flat}=L^{\flat\prime}}}\mu(t(L)) - \sum_{\delta'=0}^{\delta-1}\sum_{\substack{L\in\mathfrak{L}_2^{\delta'}\\ L\cap V_{L^\flat}=L^{\flat\prime}}}\mu(t(L)). \tag{2.13}$$

Finally, the automorphism of $\mathfrak{E}$ sending $(L^{\flat\prime}, \delta', \varepsilon)$ to $(L^{\flat\prime}, \delta'-1, \varepsilon\circ(u\alpha\cdot))$, where $\alpha \in O_E^\times$ is the element satisfying $f_1 = \alpha f_2$, induces a bijection from $\mathfrak{L}_1^{\delta'}$ to $\mathfrak{L}_2^{\delta'-1}$ preserving both $L\cap V_{L^\flat}$ and $t(L)$. Thus, (2.13) vanishes, hence (2.11) and (2.10) hold.

Now we show that the support of $\widehat{\partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}}$ is contained in $\boldsymbol{V}^{\mathrm{int}}$. Take an element $z \in \boldsymbol{V} \setminus \boldsymbol{V}^{\mathrm{int}}$. Using Remark 2.21(4), we have

$$\begin{aligned}\widehat{\partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}}(z) &= \int_{\boldsymbol{V}} \widehat{\partial\mathrm{Den}^{\mathrm{v}}_{L^\flat}}(x)\psi(\mathrm{Tr}_{E/F}(x,z)_{\boldsymbol{V}})\,\mathrm{d}z \\ &= 2\sum_{\substack{L^\flat\subseteq L\subseteq L^\vee \\ t(L\cap V_{L^\flat})>1}} \mu(t(L))\,\mathrm{vol}(L)\mathbb{1}_{L^\vee}(z) \\ &= 2\sum_{\substack{L^\flat\subseteq L^{\flat\prime}\subseteq (L^{\flat\prime})^\vee \\ t(L^{\flat\prime})>1}} \sum_{\substack{L\subseteq L^\vee \\ L\cap V_{L^\flat}=L^{\flat\prime} \\ z\in L^\vee}} \mu(t(L))\,\mathrm{vol}(L) \\ &= 2\sum_{\substack{L^\flat\subseteq L^{\flat\prime}\subseteq (L^{\flat\prime})^\vee \\ t(L^{\flat\prime})>1}} \mathrm{vol}(L^{\flat\prime})\,\mathrm{vol}((V_{L^\flat}^\perp)^{\mathrm{int}}) \sum_{\substack{L\subseteq L^\vee \\ L\cap V_{L^\flat}=L^{\flat\prime} \\ z\in L^\vee}} q^{-\delta_L}\mu(t(L)),\end{aligned}$$

which is valid and vanishes by Lemma 2.24(3).

Proposition 2.22 is proved. □

2.3. Bruhat–Tits stratification.

Let the setup be as in Subsection 2.1. We first generalize Definition 2.5 to a more general context. For every subset X of $\boldsymbol{V}$ such that $\langle X\rangle$ is finitely generated, we put

$$\mathcal{N}(X) \coloneqq \bigcap_{x\in X}\mathcal{N}(x),$$

which is always a finite intersection, and depends only on $\langle X\rangle$. Clearly, we have $\mathcal{N}(X')\subseteq\mathcal{N}(X)$ if $\langle X\rangle\subseteq\langle X'\rangle$. When $X=\{x,\dots\}$ is explicitly presented, we simply write $\mathcal{N}(x,\dots)$ instead of $\mathcal{N}(\{x,\dots\})$.

Remark 2.26. When $\langle X\rangle$ is an O_E-lattice of $\boldsymbol{V}$, the formal subscheme $\mathcal{N}(X)$ is a proper closed subscheme of $\mathcal{N}$. This can be proved by the same argument for [LZa, Lemma 2.10.1].

Definition 2.27. Let Λ be a vertex O_E-lattice of $\boldsymbol{V}$ (Definition 2.10).

(1) We equip the k-vector space $\Lambda^\vee/\Lambda$ with a k-valued pairing $(\ ,\ )_{\Lambda^\vee/\Lambda}$ by the formula

$$(x,y)_{\Lambda^\vee/\Lambda} \coloneqq u^2\,\mathrm{Tr}_{E/F}(x^\sharp,y^\sharp)_{\boldsymbol{V}} \mod (u^2)$$

where $x^\sharp$ and $y^\sharp$ are arbitrary lifts of x and y, respectively. Then $\Lambda^\vee/\Lambda$ becomes a nonsplit (nondegenerate) quadratic space over k of (even positive) dimension $t(\Lambda)$.

(2) Let $\mathcal{V}_\Lambda$ be the reduced subscheme of $\mathcal{N}(\Lambda)$, and put

$$\mathcal{V}^\circ_\Lambda \coloneqq \mathcal{V}_\Lambda - \bigcup_{\Lambda\subsetneq\Lambda'}\mathcal{V}_{\Lambda'}.$$

Proposition 2.28 (Bruhat–Tits stratification, [Wu])**.** *The reduced subscheme $\mathcal{N}_{\mathrm{red}}$ is a disjoint union of $\mathcal{V}^\circ_\Lambda$ for all vertex O_E-lattices Λ of $\boldsymbol{V}$ in the sense of stratification, such that $\mathcal{V}_\Lambda\cap\mathcal{V}_{\Lambda'}$ coincides with $\mathcal{V}_{\Lambda+\Lambda'}$ (resp. is empty) if $\Lambda+\Lambda'$ is (resp. is not) a vertex O_E-lattice.*

Moreover, for every vertex O_E-lattice Λ,

(1) $\mathcal{V}_\Lambda$ is canonically isomorphic to the generalized Deligne–Lusztig variety of $\mathrm{O}(\Lambda^\vee/\Lambda)$ over $\overline{k}$ classifying maximal isotropic subspaces U of $(\Lambda^\vee/\Lambda)\otimes_k\overline{k}$ satisfying

$$\dim(U\cap\delta(U)) = \tfrac{t(\Lambda)}{2}-1,$$

where $\delta\in\mathrm{Gal}(\overline{k}/k)$ denotes the Frobenius element;

(2) the intersection of $\mathcal{V}_\Lambda$ with each connected component of $\mathcal{N}_{\mathrm{red}}$ is connected, nonempty, and smooth projective over $\overline{k}$ of dimension $\frac{t(\Lambda)}{2}-1$.

Proof. This follows from [Wu, Proposition 5.13 & Theorem 5.18]. Note that we use lattices in $\boldsymbol{V}$, which is different from the hermitian space C used in [Wu], to parameterize strata. By the obvious analogue of [KR11, Lemma 3.9], we may naturally identify $\boldsymbol{V}$ with C, after which the stratum $\mathcal{S}_\Lambda$ in [Wu] coincides with our stratum $\mathcal{V}_{u\Lambda^\vee}$. □

Remark 2.29. In the above proposition, when $t(\Lambda)=4$, $\mathcal{V}_\Lambda$ is isomorphic to two copies of $\mathbb{P}^1_{\overline{k}}$, though we do not need this explicit description in the following.

Corollary 2.30. *For every nonzero element $x\in\boldsymbol{V}$, we have*

$$\mathcal{N}(x)_{\mathrm{red}}=\bigcup_{x\in\Lambda}\mathcal{V}_\Lambda^\circ$$

where the union is taken over all vertex O_E-lattices of $\boldsymbol{V}$ containing x.

Proof. Since $\mathcal{N}(x)_{\mathrm{red}}$ is a reduced closed subscheme of $\mathcal{N}_{\mathrm{red}}$, it suffices to check that

$$\mathcal{N}(x)(\overline{k})=\bigcup_{x\in\Lambda}\mathcal{V}_\Lambda^\circ(\overline{k}).$$

By Definition 2.27(2), we have

$$\mathcal{N}(x)(\overline{k})\supseteq\bigcup_{x\in\Lambda}\mathcal{V}_\Lambda^\circ(\overline{k}).$$

For the other direction, by Proposition 2.28, we have to show that if Λ does not contain x, then $\mathcal{N}(x)(\overline{k})\cap\mathcal{V}_\Lambda^\circ(\overline{k})=\emptyset$. Suppose that we have $s\in\mathcal{N}(x)(\overline{k})\cap\mathcal{V}_\Lambda^\circ(\overline{k})$, then s should belong to $\mathcal{V}_{\Lambda'}(\overline{k})$ where Λ' is the O_E-lattice generated by Λ and x. In particular, Λ' is vertex and strictly contains Λ. But this contradicts with the definition of $\mathcal{V}_\Lambda^\circ$. The corollary follows. □

Corollary 2.31. *Suppose that $r\geqslant 2$. For every nonzero element $x\in\boldsymbol{V}$, the intersection of $\mathcal{N}(x)$ with each connected component of $\mathcal{N}_{\mathrm{red}}$ is strictly a closed subscheme of the latter.*

Proof. By Corollary 2.30 and Proposition 2.28(2), it suffices to show that the intersection of all vertex O_E-lattices of $\boldsymbol{V}$ is $\{0\}$.

Take a nonsplit hermitian subspace V_2 of $\boldsymbol{V}$ of dimension 2 and an O_E-lattice L_2 of V_2 of fundamental invariants $(1,1)$. Then the orthogonal complement $V_2^\perp$ of V_2 in $\boldsymbol{V}$ admits a self-dual O_E-lattice L_1. Choose a normal basis (Definition 2.11) $\{e_1,\dots,e_{2r-2}\}$ of L_1 under which the moment matrix is given by $\begin{pmatrix} 0 & u^{-1}\\ -u^{-1} & 0\end{pmatrix}^{\oplus r-1}$. For every tuple $a=(a_1,\dots,a_{2r-2})\in\mathbb{Z}^{2r-2}$ satisfying $a_{2i-1}+a_{2i}=0$ for $1\leqslant i\leqslant r-1$, the O_E-lattice

$$\Lambda_a\coloneqq L_2\oplus\langle u^{a_1}e_1,\dots,u^{a_{2r-2}}e_{2r-2}\rangle$$

is integral with fundamental invariants $(0,\dots,0,1,1)$, hence vertex. It is clear that the intersection of all such Λ_a is L_2. Since $r\geqslant 2$, the intersection of all 2-dimensional nonsplit hermitian subspaces of $\boldsymbol{V}$ is $\{0\}$. Thus, the intersection of all vertex O_E-lattices of $\boldsymbol{V}$ is $\{0\}$. □

Lemma 2.32. *Let Λ be a vertex O_E-lattice of $\boldsymbol{V}$. For each connected component $\mathcal{V}_\Lambda^+$ of $\mathcal{V}_\Lambda$ and integer $d\geqslant 0$, the group of d-cycles of $\mathcal{V}_\Lambda^+$, up to ℓ-adic homological equivalence for every rational prime $\ell\neq p$, is generated by $\mathcal{V}_{\Lambda'}\cap\mathcal{V}_\Lambda^+$ for all vertex O_E-lattices Λ' containing Λ with $t(\Lambda')=2d+2$.*

Proof. Let k' be the quadratic extension of k in $\overline{k}$. Note that $\mathcal{V}_\Lambda^+$ has a canonical structure over k', so that $\mathcal{V}_\Lambda^{\circ+} := \mathcal{V}_\Lambda^\circ \cap \mathcal{V}_\Lambda^+$ (over k') is the classical Deligne–Lusztig variety of $\mathrm{SO}(\Lambda^\vee/\Lambda)$ of Coxeter type.

Recall that δ is the Frobenius element of $\mathrm{Gal}(\overline{k}/k)$. Fix a rational prime ℓ different from p. For every finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space V with an action by δ^2, we denote by $V^\dagger$ the subspace consisting of elements on which δ^2 acts by roots of unity. Then for the lemma, it suffices to show that for every $d \geqslant 0$, $\mathrm{H}_{2d}(\mathcal{V}_\Lambda^+, \overline{\mathbb{Q}}_\ell(-d))^\dagger$ is generated by (the cycle class of) $\mathcal{V}_{\Lambda'} \cap \mathcal{V}_\Lambda^+$ for all vertex O_E-lattices Λ' containing Λ with $t(\Lambda') = 2d+2$. By the same argument for [LZa, Theorem 5.3.2], it reduces to the following claim:

$(*)$ The action of δ^2 on $V := \bigoplus_{j\geqslant 0} \mathrm{H}^{2j}(\mathcal{V}_\Lambda^{\circ+}, \overline{\mathbb{Q}}_\ell(j))$ is semisimple, and $V^\dagger = \mathrm{H}^0(\mathcal{V}_\Lambda^{\circ+}, \overline{\mathbb{Q}}_\ell)$.

There are three cases.

When $t(\Lambda) = 2$, $\mathcal{V}_\Lambda^{\circ+}$ is isomorphic to $\mathrm{Spec}\,\overline{k}$, hence $(*)$ is trivial.

When $t(\Lambda) = 4$, $\mathcal{V}_\Lambda^{\circ+}$ is an affine curve, hence $(*)$ is again trivial.

When $t(\Lambda) \geqslant 6$, by Case ${}^2D_n$ (with $n = \frac{t(\Lambda)}{2} \geqslant 3$) in [Lus76, Section 7.3], the action of δ^2 on $\bigoplus_{j\geqslant 0} \mathrm{H}_c^j(\mathcal{V}_\Lambda^{\circ+}, \overline{\mathbb{Q}}_\ell)$ has eigenvalues $\{1, q^2, q^4, \dots, q^{t(\Lambda)-2}\}$ and that the eigenvalue q^{2j} appears in $\mathrm{H}_c^{j+\frac{t(\Lambda)}{2}-1}(\mathcal{V}_\Lambda^{\circ+}, \overline{\mathbb{Q}}_\ell)$. Moreover by [Lus76, Theorem 6.1], the action of δ^2 is semisimple. Thus, $(*)$ follows from the Poincaré duality.

The lemma is proved. □

2.4. **Linear invariance of intersection numbers.** Let the setup be as in Subsection 2.1. For every nonzero element $x \in \boldsymbol{V}$, we define a chain complex of locally free $\mathscr{O}_\mathcal{N}$-modules

$$C(x) := \left(\cdots \to 0 \to \mathscr{I}_{\mathcal{N}(x)} \to \mathscr{O}_\mathcal{N} \to 0\right)$$

supported in degrees 1 and 0 with the map $\mathscr{I}_{\mathcal{N}(x)} \to \mathscr{O}_\mathcal{N}$ being the natural inclusion. We extend the definition to $x = 0$ by setting

$$C(0) := \left(\cdots \to 0 \to \omega \xrightarrow{0} \mathscr{O}_\mathcal{N} \to 0\right) \tag{2.14}$$

supported in degrees 1 and 0, where ω is the line bundle from Definition 2.38.

The following is our main result of this subsection.

Proposition 2.33. *Let $0 \leqslant m \leqslant n$ be an integer. Suppose that $x_1, \dots, x_m \in \boldsymbol{V}$ and $y_1, \dots, y_m \in \boldsymbol{V}$ generate the same O_E-submodule. Then we have an isomorphism*

$$\mathrm{H}_i(C(x_1) \otimes_{\mathscr{O}_\mathcal{N}} \cdots \otimes_{\mathscr{O}_\mathcal{N}} C(x_m)) \simeq \mathrm{H}_i(C(y_1) \otimes_{\mathscr{O}_\mathcal{N}} \cdots \otimes_{\mathscr{O}_\mathcal{N}} C(y_m))$$

of $\mathscr{O}_\mathcal{N}$-modules for every i.

Proposition 2.33 has the following two immediate corollaries.

Corollary 2.34. *Let $0 \leqslant m \leqslant n$ be an integer. Suppose that $x_1, \dots, x_m \in \boldsymbol{V}$ and $y_1, \dots, y_m \in \boldsymbol{V}$ generate the same O_E-submodule. Then we have*

$$[C(x_1) \otimes_{\mathscr{O}_\mathcal{N}} \cdots \otimes_{\mathscr{O}_\mathcal{N}} C(x_m)] = [C(y_1) \otimes_{\mathscr{O}_\mathcal{N}} \cdots \otimes_{\mathscr{O}_\mathcal{N}} C(y_m)]$$

in $\mathrm{K}_0(\mathcal{N})$, where $\mathrm{K}_0(\mathcal{N})$ denotes the K-group of $\mathcal{N}$ [LL21, Section B].

Corollary 2.35. *Suppose that $x_1, \dots, x_n \in \boldsymbol{V}$ generate an O_E-lattice of $\boldsymbol{V}$. The Serre intersection multiplicity*

$$\chi\left(\mathscr{O}_{\mathcal{N}(x_1)} \overset{\mathbb{L}}{\otimes}_{\mathscr{O}_{\mathcal{N}}} \cdots \overset{\mathbb{L}}{\otimes}_{\mathscr{O}_{\mathcal{N}}} \mathscr{O}_{\mathcal{N}(x_n)}\right)$$
$$\coloneqq \sum_{i,j \geqslant 0} (-1)^{i+j} \mathrm{length}_{O_{\breve{E}}} \mathrm{H}^j\left(\mathcal{N}, \mathrm{H}_i\left(\mathscr{O}_{\mathcal{N}(x_1)} \overset{\mathbb{L}}{\otimes}_{\mathscr{O}_{\mathcal{N}}} \cdots \overset{\mathbb{L}}{\otimes}_{\mathscr{O}_{\mathcal{N}}} \mathscr{O}_{\mathcal{N}(x_n)}\right)\right)$$

depends only on the O_E-lattice of $\boldsymbol{V}$ generated by $x_1, \dots, x_n$. Note that by construction $[C(x_1) \otimes_{\mathscr{O}_{\mathcal{N}}} \cdots \otimes_{\mathscr{O}_{\mathcal{N}}} C(x_m)]$ belongs to the image of the map $\mathrm{K}_0^{\mathcal{N}(x_1,\dots,x_m)}(\mathcal{N}) \to \mathrm{K}_0(\mathcal{N})$, hence the above number is finite by Remark 2.26.

Now we start to prove Proposition 2.33, following [How19]. Let (X, ι_X, λ_X) be the universal object over $\mathcal{N}$. We have a short exact sequence

$$0 \to \mathrm{Fil}(X) \to \mathrm{D}(X) \to \mathrm{Lie}(X) \to 0$$

of locally free $\mathscr{O}_{\mathcal{N}}$-modules, where $\mathrm{D}(X)$ denotes the covariant crystal of X restricted to the Zariski site of $\mathcal{N}$. Then ι_X induces actions of O_E on all terms such that the short exact sequence is O_E-linear.

We define an $\mathscr{O}_{\mathcal{N}}$-submodule F_X of $\mathrm{Lie}(X)$ as the kernel of $\iota_X(u) - u$ on $\mathrm{Lie}(X)$, which is stable under the O_E-action.

Lemma 2.36. *The $\mathscr{O}_{\mathcal{N}}$-submodule F_X is locally free of rank $n-1$ and is locally a direct summand of* $\mathrm{Lie}(X)$.

Proof. Let $s \in \mathcal{N}(\bar{k})$ be a closed point. By the Wedge condition and the Spin condition in Definition 2.1, we know that the map

$$\iota_X(u) - u \colon \mathrm{Lie}(X) \otimes_{\mathscr{O}_{\mathcal{N}}} \mathscr{O}_{\mathcal{N},s} \to \mathrm{Lie}(X) \otimes_{\mathscr{O}_{\mathcal{N}}} \mathscr{O}_{\mathcal{N},s}$$

has rank 1 on both generic and special fibers. Thus, $F_X \otimes_{\mathscr{O}_{\mathcal{N}}} \mathscr{O}_{\mathcal{N},s}$ is a direct summand of $\mathrm{Lie}(X) \otimes_{\mathscr{O}_{\mathcal{N}}} \mathscr{O}_{\mathcal{N},s}$ of rank $n-1$. The lemma follows. ☐

The symmetrization σ_X of the polarization λ_X (Remark 2.2) induces a perfect symmetric $\mathscr{O}_{\mathcal{N}}$-bilinear pairing

$$(\ ,\ ) \colon \mathrm{D}(X) \times \mathrm{D}(X) \to \mathscr{O}_{\mathcal{N}}$$

satisfying $(\iota_X(\alpha)x, y) = (x, \iota_X(\alpha^{\mathtt{c}})y)$ for every $\alpha \in O_E$ and $x, y \in \mathrm{D}(X)$. As $\mathrm{Fil}(X)$ is a maximal isotropic $\mathscr{O}_{\mathcal{N}}$-submodule of $\mathrm{D}(X)$ with respect to $(\ ,\ )$, we have an induced perfect $\mathscr{O}_{\mathcal{N}}$-bilinear pairing

$$(\ ,\ ) \colon \mathrm{Fil}(X) \times \mathrm{Lie}(X) \to \mathscr{O}_{\mathcal{N}},$$

under which we denote by $F_X^{\perp} \subseteq \mathrm{Fil}(X)$ the annihilator of F_X. Then the $\mathscr{O}_{\mathcal{N}}$-submodule $F_X^{\perp}$ is locally free of rank 1 and is locally a direct summand of $\mathrm{Fil}(X)$.

Following [How19, Section 3], we put

$$\epsilon \coloneqq u \otimes 1 + 1 \otimes u \in O_E \otimes_{O_F} \mathscr{O}_{\mathcal{N}},$$
$$\epsilon^{\mathtt{c}} \coloneqq -u \otimes 1 + 1 \otimes u \in O_E \otimes_{O_F} \mathscr{O}_{\mathcal{N}}.$$

Lemma 2.37. *There are inclusions of $\mathscr{O}_{\mathcal{N}}$-modules $F_X^{\perp} \subseteq \epsilon\mathrm{D}(X) \subseteq \mathrm{D}(X)$, which are locally direct summands. The map $\epsilon \colon \mathrm{D}(X) \to \epsilon\mathrm{D}(X)$ descends to a surjective map*

$$\mathrm{Lie}(X) \xrightarrow{\epsilon} \epsilon\mathrm{D}(X)/F_X^{\perp},$$

whose kernel L_X is locally a direct summand $\mathscr{O}_{\mathcal{N}}$-submodule of $\mathrm{Lie}(X)$ *of rank* 1*. Moreover, the O_E-action stabilizes L_X, and acts on* $\mathrm{Lie}(X)/L_X$ *and L_X via φ_0 and $\varphi_0^{\mathtt{c}}$, respectively.*

Proof. This follows from the same proof for [How19, Proposition 3.3]. □

Definition 2.38. We define the *line bundle of modular forms* ω to be L_X^{-1}, where L_X is the line bundle on $\mathcal{N}$ from Lemma 2.37.

For every closed formal subscheme Z of $\mathcal{N}$, we denote by $\widetilde{Z}$ the closed formal subscheme defined by the sheaf $\mathscr{I}_Z^2$. Take a nonzero element $x \in \boldsymbol{V}$. By the definition of $\mathcal{N}(x)$, we have a distinguished morphism

$$X_0|_{\mathcal{N}(x)} \xrightarrow{x} X|_{\mathcal{N}(x)}$$

of O_F-divisible groups, which induces an O_E-linear map

$$\mathrm{D}(X_0)|_{\mathcal{N}(x)} \xrightarrow{x} \mathrm{D}(X)|_{\mathcal{N}(x)}$$

of vector bundles. By the Grothendieck–Messing theory, the above map admits a canonical extension

$$\mathrm{D}(X_0)|_{\widetilde{\mathcal{N}(x)}} \xrightarrow{\tilde{x}} \mathrm{D}(X)|_{\widetilde{\mathcal{N}(x)}},$$

which further restricts to a map

$$\mathrm{Fil}(X_0)|_{\widetilde{\mathcal{N}(x)}} \xrightarrow{\tilde{x}} \mathrm{Lie}(X)\ |_{\widetilde{\mathcal{N}(x)}}\ . \tag{2.15}$$

From now on, we fix a generator γ of the rank 1 free $O_{\breve{E}}$-module $\mathrm{Fil}(X_0)$.

Lemma 2.39. *The image $\tilde{x}(\gamma)$ is a section of L_X over $\widetilde{\mathcal{N}(x)}$, whose vanishing locus coincides with $\mathcal{N}(x)$, where $\tilde{x}$ is the map* (2.15)*.*

Proof. This follows from the same proof for [How19, Proposition 4.1]. □

The following lemma is parallel to [KR11, Proposition 3.5].

Lemma 2.40. *For every nonzero element $x \in \boldsymbol{V}$, the closed formal subscheme $\mathcal{N}(x)$ of $\mathcal{N}$ is either empty or a relative Cartier divisor.*

Proof. The case $r = 1$ has been proved in [RSZ17, Proposition 6.6]. Thus, we now assume $r \geqslant 2$.

We may assume that $\mathcal{N}(x)$ is nonempty. By the same argument in the proof of [How19, Proposition 4.3], $\mathcal{N}(x)$ is locally defined by one equation. It remains to show that such equation is not divisible by u. Since $r \geqslant 2$, this follows from [KR11, Lemma 3.6], Lemma 2.4, and Corollary 2.31. □

Proof of Proposition 2.33. The proof of [How19, Theorem 5.1] can be applied in the same way to Proposition 2.33, using Lemma 2.39 and Lemma 2.40. □

To end this subsection, we prove some results that will be used later.

Lemma 2.41. *The $\mathscr{O}_{\mathcal{N}}$-submodule L_X from Lemma 2.37 coincides with the image of the map $\iota_X(u) - u\colon \mathrm{Lie}(X) \to \mathrm{Lie}(X)$.*

Proof. Denote by L'_X the image of the map $\iota_X(u) - u\colon \mathrm{Lie}(X) \to \mathrm{Lie}(X)$. As we have $L'_X \simeq \mathrm{Lie}(X)/F_X$, L'_X is a locally free $\mathscr{O}_{\mathcal{N}}$-submodule of $\mathrm{Lie}(X)$ of rank 1 by Lemma 2.36. By the Spin condition in Definition 2.1, for every closed point $s \in \mathcal{N}(\overline{k})$, the induced map

$L'_X \otimes_{\mathscr{O}_\mathcal{N}} \overline{k} \to \mathrm{Lie}(X) \otimes_{\mathscr{O}_\mathcal{N}} \overline{k}$ over the residue field at s is injective. Thus, the quotient $\mathscr{O}_\mathcal{N}$-module $\mathrm{Lie}(X)/L'_X$ is locally free. It remains to show that $L'_X \subseteq L_X$.

By definition, every section of L'_X can be locally written as the image of $(\iota_X(u) - u)x$ for some section x of $\mathrm{D}(X)$. We need to show that

(1) $\epsilon(\iota_X(u) - u)x$ is a section of $\mathrm{Fil}(X)$;
(2) $(\epsilon(\iota_X(u) - u)x, y) = 0$ for every section y of F_X.

For (1), we have $\epsilon(\iota_X(u)-u)x = (\iota_X(u)+u)(\iota_X(u)-u)x = (\iota_X(u^2)-u^2)x$. Since $\iota_X(u^2)-u^2$ acts by zero on $\mathrm{Lie}(X)$, (1) follows. For (2), we have

$$(\epsilon(\iota_X(u) - u)x, y) = ((\iota_X(u) - u)x, (-\iota_X(u) + u)y) = 0$$

as y is a section of $\ker(\iota_X(u) - u)$. Thus, (2) follows.

The lemma is proved. □

Lemma 2.42. *Let Λ be a vertex O_E-lattice of $\boldsymbol{V}$ with $t(\Lambda) = 4$. Then ω has degree $q - 1$ on each connected component of (the smooth projective curve) $\mathcal{V}_\Lambda$ (Definition 2.27).*

Proof. Let δ be the Frobenius element of $\mathrm{Gal}(\overline{k}/k)$.

Let $s \in \mathcal{N}(\overline{k})$ be a closed point represented by the quadruple $(X, \iota_X, \lambda_X; \rho_X)$. Let M be the covariant O_F-Dieudonné module of X equipped with the O_E-action ι_X, which becomes a free $O_{\breve{E}}$-module. We have $\mathrm{Lie}(X) = \mathsf{M}/\mathsf{VM}$. By Definition 2.38 and Lemma 2.41, the fiber $\omega^{-1}|_s$ is canonically identified with $((u \otimes 1)\mathsf{M} + \mathsf{VM})/\mathsf{VM}$. By the identification between $\mathcal{V}_\Lambda$ and the generalized Deligne–Lusztig variety of $\mathrm{O}(\Lambda^\vee/\Lambda)$ in Proposition 2.28 given in [Wu, Proposition 4.29 & Proposition 5.13], we know that $\omega^{-1}|_{\mathcal{V}_\Lambda}$ coincides with $(\delta(U)+U)/U$ where U is the tautological subbundle of $(\Lambda^\vee/\Lambda) \otimes_k \mathscr{O}_{\mathcal{V}_\Lambda}$.

To compute the degree of $(\delta(U)+U)/U$, let $\mathcal{V}_\Lambda^+$ and $\mathcal{V}_\Lambda^-$ be the two connected components of $\mathcal{V}_\Lambda$. Let $\mathcal{L}_\Lambda$ be the scheme over $\overline{k}$ classifying lines in $\Lambda^\vee/\Lambda$ with the tautological bundle L. We may identify $\mathcal{V}_\Lambda^+$ and $\mathcal{V}_\Lambda^-$ as two closed subschemes of $\mathcal{L}_\Lambda$ via the assignment $U \mapsto \delta(U) \cap U$ (see [HP14, Section 3.2] for more details). Then, $\mathcal{V}_\Lambda^+$ and $\mathcal{V}_\Lambda^-$ are the two irreducible components of the locus where L and $\delta(L)$ generate an isotropic subspace, and the assignment $L \mapsto \delta(L)$ switches $\mathcal{V}_\Lambda^+$ and $\mathcal{V}_\Lambda^-$. Let $\mathcal{I}_\Lambda$ be the locus where L is isotropic and $L = \delta(L)$. Then $\mathcal{I}_\Lambda$ is a disjoint union of q^2+1 copies of $\mathrm{Spec}\,\overline{k}$ since there are exactly q^2+1 isotropic lines in $\Lambda^\vee/\Lambda$, and is contained in $\mathcal{V}_\Lambda^+ \cap \mathcal{V}_\Lambda^-$. Note that the map $\delta(U)/(\delta(U) \cap U) \to (\delta(U) + U)/U$ is an isomorphism, and there is a short exact sequence

$$0 \to \delta(\delta(U) \cap U) \to \delta(U)/(\delta(U) \cap U) \to \mathscr{O}_{\mathcal{I}_\Lambda} \to 0$$

of $\mathscr{O}_{\mathcal{V}_\Lambda^\pm}$-modules. Since $\delta(U) \cap U$ is the restriction of the tautological bundle L on $\mathcal{L}_\Lambda$, we have

$$\begin{aligned}\deg\left(\omega^{-1}|_{\mathcal{V}_\Lambda^\pm}\right) &= \deg\left((\delta(U) + U)/U|_{\mathcal{V}_\Lambda^\pm}\right) = \deg\left(\delta(\delta(U) \cap U)|_{\mathcal{V}_\Lambda^\pm}\right) + (q^2+1)\\ &= \deg\left(L^{\otimes q}|_{\mathcal{V}_\Lambda^\pm}\right) + (q^2+1) = -q\deg(\mathcal{V}_\Lambda^\pm) + (q^2+1),\end{aligned}$$

where $\deg(\mathcal{V}_\Lambda^\pm)$ denotes the degree of the curve $\mathcal{V}_\Lambda^\pm$ in the projective space $\mathcal{L}_\Lambda$. Thus, it remains to show that $\deg(\mathcal{V}_\Lambda^\pm) = q + 1$.

To compute the degree, take a 3-dimensional quadratic subspace H of $\Lambda^\vee/\Lambda$. Let $\mathcal{L}_\Lambda^H$ be the hyperplane of $\mathcal{L}_\Lambda$ that consists of lines contained in H. Then $\mathcal{L}_\Lambda^H \cap \mathcal{V}_\Lambda$ is the subscheme of lines $L \subseteq H$ that is isotropic and fixed by δ, which is a disjoint union of $q+1$ copies of $\mathrm{Spec}\,\overline{k}$ since there are exactly $q+1$ isotropic lines in H. As $\mathcal{L}_\Lambda^H \cap \mathcal{V}_\Lambda$ is contained in $\mathcal{I}_\Lambda$, it is contained in $\mathcal{V}_\Lambda^+ \cap \mathcal{V}_\Lambda^-$. Therefore, we have $\deg(\mathcal{V}_\Lambda^\pm) = q + 1$.

The lemma is proved. □

2.5. **Proof of Theorem 2.7 when $r = 1$.** Let the setup be as in Subsection 2.1. In this subsection, we assume $r = 1$. Note that since $\boldsymbol{V}$ is nonsplit, the fundamental invariants of an integral O_E-lattice of $\boldsymbol{V}$ must consist of two positive odd integers.

Lemma 2.43. *Let $\boldsymbol{L}$ be an integral O_E-lattice of $\boldsymbol{V}$ with fundamental invariants $(2b_1+1, 2b_2+1)$. Then*

$$\partial\mathrm{Den}(\boldsymbol{L}) = 2\sum_{j=0}^{b_1}\left(1+q+\cdots+q^j+(b_2-j)q^j\right).$$

Proof. We denote by $\mathfrak{L}$ the set of integral O_E-lattices of $\boldsymbol{V}$ containing $\boldsymbol{L}$. We now count $\mathfrak{L}$.

Fix an orthogonal basis $\{e_1, e_2\}$ of $\boldsymbol{V}$ with $(e_1,e_1)_{\boldsymbol{V}} \in O_F^\times$ and $(e_2,e_2)_{\boldsymbol{V}} \in O_F^\times$ and such that $\boldsymbol{L} = \langle u^{b_1}e_1\rangle + \langle u^{b_2}e_2\rangle$. For every $L \in \mathfrak{L}$, we let $j(L)$ be the unique integer such that $L \cap \langle e_1\rangle \otimes_{O_F} F = \langle u^{j(L)}e_1\rangle$ and let $k(L)$ be the unique integer such that image of L under the natural projection map $\boldsymbol{V} \to \langle e_2\rangle \otimes_{O_F} F$ is $\langle u^{k(L)}e_2\rangle$. Then by Lemma 2.23(1), L is uniquely determined by $j(L)$, $k(L)$, and the extension map $\varepsilon_L\colon \langle u^{k(L)}e_2\rangle \to \langle u^{j(L)}e_1\rangle \otimes_{O_F} F/O_F$. The condition that L contains $\boldsymbol{L}$ is equivalent to that $j(L) \leqslant b_1$, $k(L) \leqslant b_2$, and that ε_L vanishes on $\langle u^{b_2}e_2\rangle$. Since $\boldsymbol{L}$ is nonsplit, the condition that L is integral is equivalent to that $j(L) \geqslant 0$, $k(L) \geqslant 0$, and that the image of ε_L is contained $\langle e_1\rangle/\langle u^{j(L)}e_1\rangle$. Thus, the number of $L \in \mathfrak{L}$ with $j(L) = j$ for some fixed $0 \leqslant j \leqslant b_1$ equals $1+q+\cdots+q^j+(b_2-j)q^j$. Summing over all $0 \leqslant j \leqslant b_1$, we obtain

$$|\mathfrak{L}| = \sum_{j=0}^{b_1}\left(1+q+\cdots+q^j+(b_2-j)q^j\right).$$

The lemma then follows from (2.4) as $t(\boldsymbol{L}) = 2$. □

Proposition 2.44. *Theorem 2.7 holds when $r = 1$. More explicitly, for an integral O_E-lattice $\boldsymbol{L}$ of $\boldsymbol{V}$ with fundamental invariants $(2b_1+1, 2b_2+1)$, we have*

$$\mathrm{Int}(\boldsymbol{L}) = \partial\mathrm{Den}(\boldsymbol{L}) = 2\sum_{j=0}^{b_1}\left(1+q+\cdots+q^j+(b_2-j)q^j\right).$$

Proof. If $\boldsymbol{L}$ is not integral, then $\mathrm{Int}(\boldsymbol{L}) = \partial\mathrm{Den}(\boldsymbol{L}) = 0$. If $\boldsymbol{L}$ is integral with fundamental invariants $(2b_1+1, 2b_2+1)$. We may take an orthogonal basis $\{x_1, x_2\}$ of $\boldsymbol{L}$ such that $\mathrm{val}(x_1) = 2b_1+1$ and $\mathrm{val}(x_2) = 2b_2+1$.

Put $\boldsymbol{D} \coloneqq \mathrm{End}_{O_F}(X_0) \otimes \mathbb{Q}$, which is a division quaternion algebra over F with the F-linear embedding $\iota_{X_0}\colon E \to \boldsymbol{D}$. By the Serre construction, we may naturally identify $\boldsymbol{D}$ with $\boldsymbol{V}$, and we have an identity

$$\mathcal{N}(x_1) = \sum_{j=0}^{b_1}\mathcal{W}_{\overline{x_1}E,j} \tag{2.16}$$

of divisors, decomposing the special divisor as a sum of quasi-canonical lifting divisors (see [RSZ17, Section 6 & Proposition 7.1]).

We claim that for every $0 \leqslant j \leqslant b_1$, the identity

$$\mathrm{length}_{O_{\breve{E}}}\mathcal{W}_{\overline{x_1}E,j} \cap \mathcal{N}(x_2) = 2\left(1+q+\cdots+q^j+(b_2-l)q^j\right) \tag{2.17}$$

holds. In fact, this can be proved in the same way as for [KR11, Proposition 8.4] using Keating's formula [Vol07, Theorem 2.1]. Notice that in [KR11, Proposition 8.4] we replace

e_s by $2q^j$ since E/F is ramified, and that the factor 2 comes from the fact that $\mathcal{Z}_l$ has two connected components. By (2.16) and (2.17), we have

$$\mathrm{Int}(\boldsymbol{L}) = \mathrm{length}_{O_{\breve{E}}} \mathcal{N}(x_1) \cap \mathcal{N}(x_2) = \sum_{j=0}^{b_1} 2\left(1 + q + \cdots + q^j + (b_2 - l)q^j\right).$$

The proposition follows by Lemma 2.43. □

Definition 2.45. For $L^\flat \in \flat(\boldsymbol{V})$, we put

$$\mathcal{N}(L^\flat)^\circ \coloneqq \mathcal{N}(L^\flat) - \mathcal{N}(u^{-1}L^\flat)$$

as an effective divisor by (the $r = 1$ case of) Lemma 2.40.

Corollary 2.46. *Take an element $L^\flat \in \flat(\boldsymbol{V})$. For every $x \in \boldsymbol{V} \setminus V_{L^\flat}$, we have*

$$\mathrm{length}_{O_{\breve{E}}} \mathcal{N}(L^\flat)^\circ \cap \mathcal{N}(x) = 2 \sum_{\substack{L \subseteq L^\vee \\ L \cap V_{L^\flat} = L^\flat}} \mathbb{1}_L(x).$$

Proof. By Proposition 2.44, we have

$$\mathrm{length}_{O_{\breve{E}}} \mathcal{N}(L^\flat) \cap \mathcal{N}(x) = \mathrm{Int}(L^\flat + \langle x \rangle) = \partial\mathrm{Den}(L^\flat + \langle x \rangle) = 2 \sum_{\substack{L \subseteq L^\vee \\ L^\flat \subseteq L \cap V_{L^\flat}}} \mathbb{1}_L(x),$$

in which the last identity is due to (2.4). Similarly, we have

$$\mathrm{length}_{O_{\breve{E}}} \mathcal{N}(u^{-1}L^\flat) \cap \mathcal{N}(x) = 2 \sum_{\substack{L \subseteq L^\vee \\ u^{-1}L^\flat \subseteq L \cap V_{L^\flat}}} \mathbb{1}_L(x).$$

Taking the difference, we obtain the corollary. □

2.6. **Fourier transform of geometric side.** Let the setup be as in Subsection 2.1. We will freely use notation concerning K-groups of formal schemes from [LL21, Section B] and [Zha21, Appendix B], based on the work [GS87].

Definition 2.47. Let $\mathcal{X}$ be a formal scheme over $\mathrm{Spf}\, O_{\breve{E}}$.

1. We denote by $\mathcal{X}^{\mathrm{h}}$ the closed formal subscheme of $\mathcal{X}$ defined by the ideal sheaf $\mathscr{O}_{\mathcal{X}}[p^\infty]$.
2. For every closed formal subscheme $\mathcal{Z}$ of $\mathcal{X}$, we denote by $\mathrm{K}_0(\mathcal{X}, \mathcal{Z})$ the image of the map $\mathrm{K}_0^{\mathcal{Z}}(\mathcal{X}) \to \mathrm{K}_0(\mathcal{X})$, and similarly by $\mathrm{F}^m\mathrm{K}_0(\mathcal{X}, \mathcal{Z})$ the image of the map $\mathrm{F}^m\mathrm{K}_0^{\mathcal{Z}}(\mathcal{X}) \to \mathrm{K}_0(\mathcal{X})$ for $m \geqslant 0$.

Definition 2.48. Let X be a subset of $\boldsymbol{V}$ such that $\langle X \rangle$ is finitely generated of rank m.

1. We denote by ${}^{\mathrm{K}}\mathcal{N}(X) \in \mathrm{K}_0(\mathcal{N})$ the element $[C(x_1) \otimes_{\mathscr{O}_{\mathcal{N}}} \cdots \otimes_{\mathscr{O}_{\mathcal{N}}} C(x_m)]$ from Subsection 2.4 for a basis $\{x_1, \dots, x_m\}$ of the O_E-module generated by X, which is independent of the choice of the basis by Corollary 2.34.
2. We denote by ${}^{\mathrm{K}}\mathcal{N}(X)^{\mathrm{h}} \in \mathrm{K}_0(\mathcal{N})$ the class of $\mathcal{N}(X)^{\mathrm{h}}$.
3. We put ${}^{\mathrm{K}}\mathcal{N}(X)^{\mathrm{v}} \coloneqq {}^{\mathrm{K}}\mathcal{N}(X) - {}^{\mathrm{K}}\mathcal{N}(X)^{\mathrm{h}} \in \mathrm{K}_0(\mathcal{N})$.

Lemma 2.49. *Let $L^\flat$ be an element of $\flat(\boldsymbol{V})$ (Definition 2.8). We have*

(1) $\mathcal{N}(L^\flat)^{\mathrm{h}}$ is either empty or finite flat over $\mathrm{Spf}\, O_{\breve{E}}$;
(2) all of ${}^{\mathrm{K}}\mathcal{N}(L^\flat)$, ${}^{\mathrm{K}}\mathcal{N}(L^\flat)^{\mathrm{h}}$, and ${}^{\mathrm{K}}\mathcal{N}(L^\flat)^{\mathrm{v}}$ belong to $\mathrm{F}^{n-1}\mathrm{K}_0(\mathcal{N}, \mathcal{N}(L^\flat))$;
(3) there exist finitely many vertex O_E-lattices $\Lambda_1, \dots, \Lambda_m$ of $\boldsymbol{V}$ of type n such that ${}^{\mathrm{K}}\mathcal{N}(L^\flat)^{\mathrm{v}}$ belongs to $\sum_{i=1}^m \mathrm{F}^{n-1}\mathrm{K}_0(\mathcal{N}, \mathcal{V}_{\Lambda_i})$.

Proof. Part (1) follows from Lemma 2.54 and Lemma 2.53.

Take a basis $\{x_1,\dots,x_{n-1}\}$ of the O_E-module $L^\flat$.

For (2), it suffices to show ${}^{\mathbb{K}}\mathcal{N}(L^\flat)\in \mathrm{F}^{n-1}\mathrm{K}_0(\mathcal{N},\mathcal{N}(L^\flat))$ by (1). By definition, ${}^{\mathbb{K}}\mathcal{N}(L^\flat)$ is the cup product of the classes in $\mathrm{K}_0(\mathcal{N})$ of $\mathcal{N}(x_1),\dots,\mathcal{N}(x_{n-1})$, each being a divisor by Lemma 2.40. Thus, ${}^{\mathbb{K}}\mathcal{N}(L^\flat)$ belongs to $\mathrm{F}^{n-1}\mathrm{K}_0(\mathcal{N},\mathcal{N}(L^\flat))$ by (the analogue for formal schemes of) [GS87, Proposition 5.5].

For (3), by the same argument for [LZa, Lemma 5.1.1], we know that there exists a proper closed subscheme Z of $\mathcal{N}$ containing the reduced fiber of $\mathcal{N}(L^\flat)^{\mathrm{h}}$, such that $\mathcal{N}(L^\flat)$ is contained in $\mathcal{N}(L^\flat)^{\mathrm{h}}\bigcup Z$. By (1) and (2), there exists a closed reduced one-dimensional subscheme C of Z containing the reduced fiber of $\mathcal{N}(L^\flat)^{\mathrm{h}}$, such that ${}^{\mathbb{K}}\mathcal{N}(L^\flat)$ belongs to $\mathrm{K}_0(\mathcal{N},C\cup\mathcal{N}(L^\flat)^{\mathrm{h}})$. By [GS87, Lemma 1.9] (and its notation), ${}^{\mathbb{K}}\mathcal{N}(L^\flat)$ belongs to the image of the natural map $\mathrm{K}'_0(C\cup\mathcal{N}(L^\flat)^{\mathrm{h}})\to\mathrm{K}_0(\mathcal{X})$ that sends a coherent $\mathscr{O}_{C\cup\mathcal{N}(L^\flat)^{\mathrm{h}}}$-module M to any finite projective resolution of M on $\mathcal{X}$. It follows, by the definition of ${}^{\mathbb{K}}\mathcal{N}(L^\flat)^{\mathrm{v}}$, that ${}^{\mathbb{K}}\mathcal{N}(L^\flat)^{\mathrm{v}}$ can be represented by a finite complex of coherent sheaves on $C\cup\mathcal{N}(L^\flat)^{\mathrm{h}}$ that are Artinian on $\mathcal{N}(L^\flat)^{\mathrm{h}}$, which implies that ${}^{\mathbb{K}}\mathcal{N}(L^\flat)^{\mathrm{v}}$ belongs to the image of the map $\mathrm{K}'_0(C)\to\mathrm{K}_0(\mathcal{N})$. Let $C_1,\dots,C_m$ be the irreducible components of C. It is clear that the map $\bigoplus_{i=1}^m\mathrm{K}'_0(C_i)\to\mathrm{K}'_0(C)$ is surjective, which implies that ${}^{\mathbb{K}}\mathcal{N}(L^\flat)^{\mathrm{v}}$ belongs to $\sum_{i=1}^m\mathrm{K}_0(\mathcal{N},C_i)$. Finally, for each $1\leqslant i\leqslant m$, we may choose a vertex O_E-lattice Λ_i of $\boldsymbol{V}$ of type n such that $C_i\subseteq\mathcal{V}_{\Lambda_i}$ by Proposition 2.28. Then (3) follows. ∎

Definition 2.50. Let $L^\flat$ be an element of $\flat(\boldsymbol{V})$ (Definition 2.8). For $x\in\boldsymbol{V}\setminus V_{L^\flat}$, we put

$$\mathrm{Int}_{L^\flat}(x)\coloneqq{}^{\mathbb{K}}\mathcal{N}(L^\flat).{}^{\mathbb{K}}\mathcal{N}(x),$$
$$\mathrm{Int}^{\mathrm{h}}_{L^\flat}(x)\coloneqq{}^{\mathbb{K}}\mathcal{N}(L^\flat)^{\mathrm{h}}.{}^{\mathbb{K}}\mathcal{N}(x),$$
$$\mathrm{Int}^{\mathrm{v}}_{L^\flat}(x)\coloneqq{}^{\mathbb{K}}\mathcal{N}(L^\flat)^{\mathrm{v}}.{}^{\mathbb{K}}\mathcal{N}(x).$$

Here, the intersection numbers are well-defined since $\mathcal{N}(L^\flat)\cap\mathcal{N}(x)$ is a proper closed subscheme of $\mathcal{N}$ by Remark 2.26. Note that $\mathrm{Int}_{L^\flat}(x)=\mathrm{Int}(L^\flat+\langle x\rangle)$ (Definition 2.6).

The following is our main result of this subsection.

Proposition 2.51. *Let $L^\flat$ be an element of $\flat(\boldsymbol{V})$ (Definition 2.8).*

(1) We have $\mathrm{Int}^{\mathrm{h}}_{L^\flat}(x)=\partial\mathrm{Den}^{\mathrm{h}}_{L^\flat}(x)$ for $x\in\boldsymbol{V}\setminus V_{L^\flat}$, where $\partial\mathrm{Den}^{\mathrm{h}}_{L^\flat}$ is from Definition 2.20.

(2) The function $\mathrm{Int}^{\mathrm{v}}_{L^\flat}$ extends (uniquely) to a (compactly supported) locally constant function on $\boldsymbol{V}$, which we still denote by $\mathrm{Int}^{\mathrm{v}}_{L^\flat}$. Moreover, we have

$$\widehat{\mathrm{Int}^{\mathrm{v}}_{L^\flat}}=-\mathrm{Int}^{\mathrm{v}}_{L^\flat}.$$

In particular, the support of $\widehat{\mathrm{Int}^{\mathrm{v}}_{L^\flat}}$ is contained in $\boldsymbol{V}^{\mathrm{int}}$ (Definition 2.10).

The rest of this subsection is devoted to the proof of this proposition.

Remark 2.52 (Cancellation law for special cycles). Let $\boldsymbol{V}'$ be a hermitian subspace of $\boldsymbol{V}$ that is nonsplit and of positive even dimension n'. Let L be an integral hermitian O_E-module contained in $\boldsymbol{V}$ such that $L\cap\boldsymbol{V}'^{\perp}$ is a self-dual O_E-lattice of $\boldsymbol{V}'^{\perp}$. We may choose

- an object $(\boldsymbol{X}',\iota_{\boldsymbol{X}'},\lambda_{\boldsymbol{X}'})\in\mathrm{Exo}^{\mathrm{b}}_{(n'-1,1)}(\overline{k})$ (Definition 2.1),
- an object $(Y,\iota_Y,\lambda_Y)\in\mathrm{Exo}_{(n-n',0)}(O_{\breve{E}})$ (Remark 2.9),[13]

[13] When $n'=n$, we simply ignore (Y,ι_Y,λ_Y).

- a quasi-morphism ϱ from $(Y, \iota_Y, \lambda_Y) \otimes_{O_{\breve{E}}} \overline{k} \oplus (\boldsymbol{X}', \iota_{\boldsymbol{X}'}, \lambda_{\boldsymbol{X}'})$ to $(\boldsymbol{X}, \iota_{\boldsymbol{X}}, \lambda_{\boldsymbol{X}})$ in the category $\mathrm{Exo}^{\mathrm{b}}_{(n-1,1)}(S \otimes_{O_{\breve{E}}} \overline{k})$ satisfying
  - ϱ identifies $\mathrm{Hom}_{O_E}(X_0 \otimes_{O_{\breve{E}}} \overline{k}, \boldsymbol{X}') \otimes \mathbb{Q}$ with $\boldsymbol{V}'$ as hermitian spaces;
  - ϱ identifies $\mathrm{Hom}_{O_E}(X_0 \otimes_{O_{\breve{E}}} \overline{k}, Y \otimes_{O_{\breve{E}}} \overline{k})$ with $L \cap \boldsymbol{V}'^{\perp}$ as hermitian O_E-modules.

Let $\mathcal{N}' := \mathcal{N}_{(\boldsymbol{X}', \iota_{\boldsymbol{X}'}, \lambda_{\boldsymbol{X}'})}$ be the relative Rapoport–Zink space for the triple $(\boldsymbol{X}', \iota_{\boldsymbol{X}'}, \lambda_{\boldsymbol{X}'})$ (Definition 2.3). We have a morphism $\mathcal{N}' \to \mathcal{N}$ such that for every object S of $\mathrm{Sch}^{\mathrm{v}}_{/O_{\breve{E}}}$, $\mathcal{N}(S)$ it sends an object $(X', \iota_{X'}, \lambda_{X'}; \rho_{X'}) \in \mathcal{N}'(S)$ to the object

$$(Y \otimes_{O_{\breve{E}}} S \oplus X', \iota_Y \otimes_{O_{\breve{E}}} S \oplus \iota_{X'}, \lambda_Y \otimes_{O_{\breve{E}}} S \oplus \lambda_{X'}; \varrho \circ (\mathrm{id}_Y \otimes_{O_{\breve{E}}} S \oplus \rho_{X'})) \in \mathcal{N}(S).$$

We have

(1) The morphism $\mathcal{N}' \to \mathcal{N}$ above identifies $\mathcal{N}'$ with the closed formal subscheme $\mathcal{N}(L \cap \boldsymbol{V}'^{\perp})$ of $\mathcal{N}$.
(2) Suppose that $L \cap \boldsymbol{V}' \neq \{0\}$, then $\mathcal{N}(L)$ coincides with the image of $\mathcal{N}'(L \cap \boldsymbol{V}')$ under the morphism $\mathcal{N}' \to \mathcal{N}$ above.
(3) For a nonzero element $x \in \boldsymbol{V}$ written as $x = y + x'$ with respect to the orthogonal decomposition $\boldsymbol{V} = \boldsymbol{V}'^{\perp} \oplus \boldsymbol{V}'$, we have
$$\mathcal{N}' \times_{\mathcal{N}} \mathcal{N}(x) = \begin{cases} \emptyset, & \text{if } y \notin L \cap \boldsymbol{V}'^{\perp}, \\ \mathcal{N}', & \text{if } y \in L \cap \boldsymbol{V}'^{\perp} \text{ and } x' = 0, \\ \mathcal{N}'(x'), & \text{if } y \in L \cap \boldsymbol{V}'^{\perp} \text{ and } x' \neq 0. \end{cases}$$
(4) If L is an O_E-lattice of $\boldsymbol{V}$, then we have $\mathrm{Int}(L) = \mathrm{Int}(L \cap \boldsymbol{V}')$.

These follow from the similar argument for the cancellation law in [LZa, Section 2.11]. Indeed, we may choose compatible framing objects for $\mathcal{N}'$ and $\mathcal{N}$ as in [RSZ17, Page 2207]. Note that the hermitian form on $\boldsymbol{V}$ in [RSZ17] is the scaled form $u^2(\ ,\ )_{\boldsymbol{V}}$, and thus u-modular lattices in [RSZ17] correspond to our self-dual lattices.

Lemma 2.53. *Let $L^{\flat\prime} \in \flat(\boldsymbol{V})$ be an element that is integral and satisfies $t(L^{\flat\prime}) = 1$.*

(1) The formal subscheme $\mathcal{N}(L^{\flat\prime})$ is finite flat over $\mathrm{Spf}\, O_{\breve{E}}$.
(2) If we put $\mathcal{N}(L^{\flat\prime})^{\circ} := \mathcal{N}(L^{\flat\prime}) - \mathcal{N}(L^{\flat\prime}_{-})$ as an element in $\mathrm{F}^{n-1}\mathrm{K}_0(\mathcal{N})$, then for every $x \in \boldsymbol{V} \setminus V_{L^{\flat}}$,
$$\mathcal{N}(L^{\flat\prime})^{\circ} .^{\mathrm{K}} \mathcal{N}(x) = 2 \sum_{\substack{L \subseteq L^{\vee} \\ L \cap V_{L^{\flat}} = L^{\flat\prime}}} \mathbb{1}_L(x).$$
Here, $L^{\flat\prime}_{-}$ is the unique element in $\flat(\boldsymbol{V})$ satisfying $L^{\flat\prime} \subseteq L^{\flat\prime}_{-} \subseteq (L^{\flat\prime})^{\vee}$ with $|L^{\flat\prime}_{-}/L^{\flat\prime}| = q$ (so that $L^{\flat\prime}_{-}$ is either not integral, or is integral with $t(L^{\flat\prime}_{-}) = 1$).

Proof. Since $t(L^{\flat\prime}) = 1$, we may choose a 2-dimensional (nonsplit) hermitian subspace $\boldsymbol{V}'$ of $\boldsymbol{V}$ such that $L^{\flat\prime} \cap \boldsymbol{V}'^{\perp}$ is a self-dual O_E-lattice of $\boldsymbol{V}'^{\perp}$. We adopt the construction in Remark 2.52.

For (1), we have $\mathcal{N}(L^{\flat\prime}) = \mathcal{N}'(L^{\flat\prime} \cap \boldsymbol{V}')$, which is finite flat over $\mathrm{Spf}\, O_{\breve{E}}$ by (the $r = 1$ case of) Lemma 2.40.

For (2), we write $x = y + x'$ with respect to the orthogonal decomposition $\boldsymbol{V} = \boldsymbol{V}'^{\perp} \oplus \boldsymbol{V}'$. Since $x \notin V_{L^{\flat}}$, we have $x' \neq 0$. By Remark 2.52(2), $\mathcal{N}(L^{\flat\prime})^{\circ}$ coincides with (the class of) $\mathcal{N}'(L^{\flat\prime} \cap \boldsymbol{V}')^{\circ}$ in $\mathrm{F}^1\mathrm{K}_0(\mathcal{N}')$ under the map $\mathrm{F}^1\mathrm{K}_0(\mathcal{N}') \to \mathrm{F}^{n-1}\mathrm{K}_0(\mathcal{N})$. There are two cases.

If $y\notin L^{\flat\prime}\cap\boldsymbol{V}'^{\perp}$, then $\mathcal{N}(L^{\flat\prime})^{\circ}.{}^{\mathrm{K}}\mathcal{N}(x)=0$ by Remark 2.52(3), and there is no integral O_E-lattice of $\boldsymbol{V}$ containing $L^{\flat\prime}+\langle x\rangle$. Thus, (2) follows.

If $y\in L^{\flat\prime}\cap\boldsymbol{V}'^{\perp}$, then by Remark 2.52(3), we have

$$\mathcal{N}(L^{\flat\prime})^{\circ}.{}^{\mathrm{K}}\mathcal{N}(x)=\mathcal{N}'(L^{\flat\prime}\cap\boldsymbol{V}')^{\circ}.{}^{\mathrm{K}}\mathcal{N}'(x')=\mathrm{length}_{O_{\breve{E}}}\mathcal{N}'(L^{\flat\prime}\cap\boldsymbol{V}')^{\circ}\cap\mathcal{N}'(x').$$

By Corollary 2.46, we have

$$\mathrm{length}_{O_{\breve{E}}}\mathcal{N}'(L^{\flat\prime}\cap\boldsymbol{V}')^{\circ}\cap\mathcal{N}'(x')=2\sum_{\substack{L'\subseteq L'^{\vee}(\subseteq\boldsymbol{V}')\\ L'\cap(V_{L^\flat}\cap\boldsymbol{V}')=L^{\flat\prime}\cap\boldsymbol{V}'}}\mathbb{1}_{L'}(x')=2\sum_{\substack{L\subseteq L^{\vee}\\ L\cap V_{L^\flat}=L^{\flat\prime}}}\mathbb{1}_{L}(x).$$

Thus, (2) follows. ☐

Lemma 2.54. *Let $L^\flat$ be an element of $\flat(\boldsymbol{V})$ (Definition 2.8). We have*

$$\mathcal{N}(L^\flat)^{\mathrm{h}}=\bigcup_{\substack{L^\flat\subseteq L^{\flat\prime}\subseteq(L^{\flat\prime})^{\vee}\\ t(L^{\flat\prime})=1}}\mathcal{N}(L^{\flat\prime})^{\circ}$$

as closed formal subschemes of $\mathcal{N}$, and the identity

$${}^{\mathrm{K}}\mathcal{N}(L^\flat)^{\mathrm{h}}=\sum_{\substack{L^\flat\subseteq L^{\flat\prime}\subseteq(L^{\flat\prime})^{\vee}\\ t(L^{\flat\prime})=1}}\mathcal{N}(L^{\flat\prime})^{\circ}$$

in $\mathrm{F}^{n-1}\mathrm{K}_0(\mathcal{N})/\mathrm{F}^{n}\mathrm{K}_0(\mathcal{N})$, where $\mathcal{N}(L^{\flat\prime})^{\circ}$ is introduced in Lemma 2.53(2).

Proof. This lemma can be proved by the same way as for [LZa, Theorem 4.2.1], as long as we establish the following claim replacing [LZa, Lemma 4.5.1] in the case where E/F is ramified.

- Let L be a self-dual hermitian O_E-module of rank n and $L^\flat$ a hermitian O_E-module contained in L. If $L/L^\flat$ is free, then $L^\flat$ is integral with $t(L^\flat)=1$.

However, this is just a special case of Lemma 2.23(2). ☐

Lemma 2.55. *Let Λ be a vertex O_E-lattice of $\boldsymbol{V}$ with $t(\Lambda)=4$. Take an arbitrary connected component $\mathcal{V}_\Lambda^+$ of the smooth projective curve $\mathcal{V}_\Lambda$ from Proposition 2.28, regarded as an element in $\mathrm{F}^{n-1}\mathrm{K}_0(\mathcal{N})$. For every $x\in\boldsymbol{V}\setminus\{0\}$, put $\mathrm{Int}_{\mathcal{V}_\Lambda^+}(x):=\mathcal{V}_\Lambda^+.{}^{\mathrm{K}}\mathcal{N}(x)$. Then $\mathrm{Int}_{\mathcal{V}_\Lambda^+}$ extends (uniquely) to a compactly supported locally constant function on $\boldsymbol{V}$, which we still denote by $\mathrm{Int}_{\mathcal{V}_\Lambda^+}$. Moreover, we have*

$$\widehat{\mathrm{Int}_{\mathcal{V}_\Lambda^+}}=-\mathrm{Int}_{\mathcal{V}_\Lambda^+}.$$

Proof. Since $t(\Lambda)=4$, we may choose a 4-dimensional (nonsplit) hermitian subspace $\boldsymbol{V}'$ of $\boldsymbol{V}$ such that $\Lambda\cap\boldsymbol{V}'^{\perp}$ is a self-dual O_E-lattice of $\boldsymbol{V}'^{\perp}$. We adopt the construction in Remark 2.52. Write $x=y+x'$ with respect to the orthogonal decomposition $\boldsymbol{V}=\boldsymbol{V}'^{\perp}\oplus\boldsymbol{V}'$. Put $\Lambda':=\Lambda\cap\boldsymbol{V}'$. By Remark 2.52(2) and Definition 2.27(2), $\mathcal{V}_\Lambda$ coincides with $\mathcal{V}_{\Lambda'}$ under the natural morphism $\mathcal{N}'\to\mathcal{N}$. Denote by $\mathcal{V}_{\Lambda'}^+$ the connected component of $\mathcal{V}_{\Lambda'}$ that corresponds to $\mathcal{V}_\Lambda^+$. By Remark 2.52(3), we have

$$\mathcal{V}_\Lambda^+.{}^{\mathrm{K}}\mathcal{N}(x)=\begin{cases}0, & \text{if } y\notin\Lambda\cap\boldsymbol{V}'^{\perp},\\ \mathcal{V}_{\Lambda'}^+.{}^{\mathrm{K}}\mathcal{N}'(x'), & \text{if } y\in\Lambda\cap\boldsymbol{V}'^{\perp}.\end{cases}$$

In other words, we have $\mathrm{Int}_{\mathcal{V}_\Lambda^+}=\mathbb{1}_{\Lambda\cap\boldsymbol{V}'^{\perp}}\otimes\mathrm{Int}_{\mathcal{V}_{\Lambda'}^+}$. Therefore, it suffices to consider the case where $n=4$.

We now give an explicit formula for $\mathrm{Int}_{\mathcal{V}_\Lambda^+}(x)$ when $n=4$. Let $\mathcal{N}^+$ be the connected component of $\mathcal{N}$ that contains $\mathcal{V}_\Lambda^+$, and put $Z^+\coloneqq Z\cap\mathcal{N}^+$ for every formal subscheme Z of $\mathcal{N}$. Put $\Lambda(x)\coloneqq\Lambda+\langle x\rangle$. There are three cases.

(1) Suppose that $\Lambda(x)$ is not integral. By Corollary 2.30, $\mathcal{V}_\Lambda$ has empty intersection with $\mathcal{N}(x)$. Thus, we have $\mathrm{Int}_{\mathcal{V}_\Lambda^+}(x)=0$.

(2) Suppose that $\Lambda(x)$ is integral but $x\notin\Lambda$. Then $\Lambda(x)$ has fundamental invariants $(0,0,1,1)$. By Corollary 2.30, $\mathcal{V}_\Lambda^+\cap\mathcal{N}(x)_{\mathrm{red}}=\mathcal{V}_{\Lambda(x)}^+$, which is a $\overline{k}$-point. Thus, we have $\mathrm{Int}_{\mathcal{V}_\Lambda^+}(x)\geqslant 1$. Choose a normal basis (Definition 2.11) $\{x_1,x_2,x_3,x_4\}$ of Λ and write $x=\lambda_1x_1+\lambda_2x_2+\lambda_3x_3+\lambda_4x_4$ with $\lambda_i\in E$. Without lost of generality, we may assume $\lambda_4\notin O_E$. Since $ux\in\Lambda$, we have $\Lambda(x)=\langle x_1,x_2,x_3,x\rangle$. By Corollary 2.30, $\mathcal{N}(x_1)\cap\mathcal{N}(x_2)\cap\mathcal{N}(x_3)$ contains $\mathcal{V}_\Lambda$ as a closed subscheme. By Remark 2.52 and Proposition 2.44 applied to $\boldsymbol{V}'$ spanned by x_3 and x_4, $\mathcal{N}(\Lambda(x))$ is a 0-dimensional scheme and $\mathrm{Int}(\Lambda(x))=2$. It follows that

$$\mathrm{Int}_{\mathcal{V}_\Lambda^+}(x)\leqslant\mathrm{length}_{O_{\breve{E}}}\left(\mathcal{N}(x_1)\cap\mathcal{N}(x_2)\cap\mathcal{N}(x_3)\right)\cap\mathcal{N}(x)^+=\mathrm{Int}^+(\Lambda(x))=1$$

by Lemma 2.56 below. Thus, we obtain $\mathrm{Int}^+(\Lambda(x))=1$, hence $\mathrm{Int}_{\mathcal{V}_\Lambda^+}(x)=1$.

(3) Suppose that $x\in\Lambda$. Then $\mathcal{V}_\Lambda^+$ is a closed subscheme of $\mathcal{N}(x)$, which implies

$$\mathscr{O}_{\mathcal{V}_{\Lambda^+}}\overset{\mathbb{L}}{\otimes}_{\mathscr{O}_\mathcal{N}}\mathscr{O}_{\mathcal{N}(x)}=\left(\mathscr{O}_{\mathcal{V}_{\Lambda^+}}\overset{\mathbb{L}}{\otimes}_{\mathscr{O}_{\mathcal{N}(x)}}\mathscr{O}_{\mathcal{N}(x)}\right)\overset{\mathbb{L}}{\otimes}_{\mathscr{O}_\mathcal{N}}\mathscr{O}_{\mathcal{N}(x)}=\mathscr{O}_{\mathcal{V}_{\Lambda^+}}\overset{\mathbb{L}}{\otimes}_{\mathscr{O}_{\mathcal{N}(x)}}\left(\mathscr{O}_{\mathcal{N}(x)}\overset{\mathbb{L}}{\otimes}_{\mathscr{O}_\mathcal{N}}\mathscr{O}_{\mathcal{N}(x)}\right).$$

However, by Corollary 2.34, we have $\mathscr{O}_{\mathcal{N}(x)}\overset{\mathbb{L}}{\otimes}_{\mathscr{O}_\mathcal{N}}\mathscr{O}_{\mathcal{N}(x)}=\mathscr{O}_{\mathcal{N}(x)}\otimes_{\mathscr{O}_\mathcal{N}}C(0)$ in $\mathrm{K}_0(\mathcal{N})$, where $C(0)$ is the complex (2.14). Thus, we obtain

$$\mathrm{Int}_{\mathcal{V}_\Lambda^+}(x)=\chi\left(C(0)|_{\mathcal{V}_\Lambda^+}\right)=\deg\left(\mathscr{O}_{\mathcal{V}_\Lambda^+}\right)-\deg\left(\omega|_{\mathcal{V}_\Lambda^+}\right)=-\deg\left(\omega|_{\mathcal{V}_\Lambda^+}\right)=1-q$$

by Lemma 2.42.

Since there are exactly q^2+1 vertex O_E-lattices of $\boldsymbol{V}$ properly containing Λ, combining (1–3), we obtain

$$\mathrm{Int}_{\mathcal{V}_\Lambda^+}=-q(1+q)\mathbb{1}_\Lambda+\sum_{\Lambda\subsetneq\Lambda'\subseteq\Lambda'^\vee}\mathbb{1}_{\Lambda'}.$$

It follows that

$$\widehat{\mathrm{Int}_{\mathcal{V}_\Lambda^+}}=-\frac{1+q}{q}\mathbb{1}_{\Lambda^\vee}+\frac{1}{q}\sum_{\Lambda\subsetneq\Lambda'\subseteq\Lambda'^\vee}\mathbb{1}_{\Lambda'^\vee}.\tag{2.18}$$

- If $x\in\Lambda$, then $\widehat{\mathrm{Int}_{\mathcal{V}_\Lambda^+}}(x)=-\frac{1+q}{q}+\frac{q^2+1}{q}=q-1$.
- If $\Lambda(x)$ is integral but $x\notin\Lambda$, then the number of Λ' in the summation of (2.18) such that $x\in\Lambda'^\vee$ is exactly 1 (namely, $\Lambda(x)$ itself). Thus, we have $\widehat{\mathrm{Int}_{\mathcal{V}_\Lambda^+}}(x)=-\frac{1+q}{q}+\frac{1}{q}=-1$.
- If $\Lambda(x)$ is not integral but $x\in\Lambda^\vee$, then the set of Λ' in the summation of (2.18) satisfying $x\in\Lambda'^\vee$ is bijective to the set of isotropic lines in $\Lambda^\vee/\Lambda$ perpendicular to x. Now since $\Lambda(x)$ is not integral, x is anisotropic in $\Lambda^\vee/\Lambda$, which implies that the previous set has cardinality $q+1$. Thus, we have $\widehat{\mathrm{Int}_{\mathcal{V}_\Lambda^+}}(x)=-\frac{1+q}{q}+\frac{q+1}{q}=0$.
- If $x\notin\Lambda^\vee$, then $\widehat{\mathrm{Int}_{\mathcal{V}_\Lambda^+}}(x)=0$.

Therefore, we have $\widehat{\mathrm{Int}_{\mathcal{V}_\Lambda^+}} = -\mathrm{Int}_{\mathcal{V}_\Lambda^+}$. The lemma is proved. □

Lemma 2.56. *Denote the two connected components of $\mathcal{N}$ by $\mathcal{N}^+$ and $\mathcal{N}^-$, and $\mathrm{Int}^\pm(\boldsymbol{L})$ the intersection multiplicity in Definition 2.6 on $\mathcal{N}^\pm$. Then*

$$\mathrm{Int}^+(\boldsymbol{L}) = \mathrm{Int}^-(\boldsymbol{L}) = \tfrac{1}{2}\mathrm{Int}(\boldsymbol{L}).$$

Proof. Choose a normal basis (Definition 2.11) $\{x_1, \dots, x_n\}$ of $\boldsymbol{L}$. Since $\boldsymbol{V}$ is nonsplit, there exists an anisotropic element in the basis, say x_n. Let θ the unique element in $\mathrm{U}(\boldsymbol{V})(F)$ satisfying $\theta(x_i) = 1$ for $1 \leqslant i \leqslant n-1$ and $\theta(x_n) = -x_n$. Then θ induces an automorphism of $\mathcal{N}$, preserving $\mathcal{N}(x_i)$ for $1 \leqslant i \leqslant n$, but switching $\mathcal{N}^+$ and $\mathcal{N}^-$ as $\det\theta = -1$. Thus, we have $\mathrm{Int}^+(\boldsymbol{L}) = \mathrm{Int}^-(\boldsymbol{L})$. Since $\mathrm{Int}(\boldsymbol{L}) = \mathrm{Int}^+(\boldsymbol{L}) + \mathrm{Int}^-(\boldsymbol{L})$, the lemma follows. □

Proof of Proposition 2.51. We first consider (1). By Lemma 2.54, we have for $x \in \boldsymbol{V} \setminus V_{L^\flat}$,

$$\mathrm{Int}^{\mathrm{h}}_{L^\flat}(x) = \sum_{\substack{L^\flat \subseteq L^{\flat\prime} \subseteq (L^{\flat\prime})^\vee \\ t(L^{\flat\prime})=1}} \mathcal{N}(L^{\flat\prime})^\circ . {}^{\mathrm{K}}\mathcal{N}(x),$$

which, by Lemma 2.53, equals

$$2 \sum_{\substack{L^\flat \subseteq L^{\flat\prime} \subseteq (L^{\flat\prime})^\vee \\ t(L^{\flat\prime})=1}} \sum_{\substack{L \subseteq L^\vee \\ L \cap V_{L^\flat} = L^{\flat\prime}}} \mathbb{1}_L(x) = 2 \sum_{\substack{L^\flat \subseteq L \subseteq L^\vee \\ t(L \cap V_{L^\flat})=1}} \mathbb{1}_L(x).$$

Thus, Proposition 2.51(1) follows from Definition 2.20.

We first consider (2). We may assume $r \geqslant 2$ since otherwise $\mathrm{Int}^{\mathrm{v}}_{L^\flat} \equiv 0$, hence (2) is trivial. We write $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$ for the two connected components. For every vertex O_E-lattice Λ of $\boldsymbol{V}$, we put $\mathcal{V}_\Lambda^\pm \coloneqq \mathcal{V}_\Lambda \cap \mathcal{N}^\pm$. Since the natural map $\mathrm{F}^{\frac{t(\Lambda)}{2}-2}\mathrm{K}_0(\mathcal{V}_\Lambda) \to \mathrm{F}^{n-1}\mathrm{K}_0^{\mathcal{V}_\Lambda}(\mathcal{N})$ is an isomorphism, by Lemma 2.49(3) and Lemma 2.32, there exist rational numbers $c_\Lambda^\pm$ for vertex O_E-lattices Λ of $\boldsymbol{V}$ with $t(\Lambda) = 4$, of which all but finitely many are zero, such that

$${}^{\mathrm{K}}\mathcal{N}(L^\flat)^{\mathrm{v}} - \left(\sum_\Lambda c_\Lambda^+ \cdot \mathcal{V}_\Lambda^+ + c_\Lambda^- \cdot \mathcal{V}_\Lambda^-\right)$$

has zero intersection with $\mathrm{F}^1\mathrm{K}_0(\mathcal{N})$. Thus, Proposition 2.51(2) follows from Lemma 2.55. □

2.7. **Proof of Theorem 2.7.** Let the setup be as in Subsection 2.1. In this subsection, for an element $L^\flat \in \flat(\boldsymbol{V})$ (Definition 2.8), we set $\mathrm{val}(L^\flat) = -1$ if $L^\flat$ is not integral.

Lemma 2.57. *Suppose that $r \geqslant 2$ and take an integral element $L^\flat \in \flat(\boldsymbol{V})$ whose fundamental invariants $(a_1, \dots, a_{n-2}, a_{n-1})$ satisfy $a_{n-2} < a_{n-1}$ (in particular, a_{n-1} is odd). Then the number of integral O_E-lattices of $\boldsymbol{V}$ containing $L^\flat$ with fundamental invariants $(a_1, \dots, a_{n-2}, a_{n-1}-1, a_{n-1}-1)$ is either 0 or 2. When the number is 2 and those lattices are denoted by $L^{\flat+}$ and $L^{\flat-}$, we have*

(1) $L^{\flat\pm} \cap V_{L^\flat} = L^\flat$;
(2) $a_{n-1} \geqslant 3$;
(3) there are orthogonal decompositions $L^\flat = L^\flat_\leftarrow \oplus L^\flat_\rightarrow$ and $L^{\flat\pm} = L^\flat_\leftarrow \oplus L^{\flat\pm}_\rightarrow$, in which $L^\flat_\leftarrow$, $L^\flat_\rightarrow$, and $L^{\flat\pm}_\rightarrow$ are integral hermitian O_E-modules with fundamental invariants $(a_1, \dots, a_{n-2})$, (a_{n-1}), and $(a_{n-1}-1, a_{n-1}-1)$, respectively.

Proof. Let L be an integral O_E-lattice L of $\boldsymbol{V}$ containing $L^\flat$ with fundamental invariants $(a_1, \dots, a_{n-2}, a_{n-1}-1, a_{n-1}-1)$.

We first claim that (1) must hold. We have $\mathrm{val}(L \cap V_{L^\flat}) \geqslant a_1 + \cdots + a_{n-2} + a_{n-1} - 1$ by Lemma 2.23(1). Since $L \cap V_{L^\flat}$ contains $L^\flat$ and $\mathrm{val}(L \cap V_{L^\flat})$ is odd, we must have $L \cap V_{L^\flat} = L^\flat$.

Choose a normal basis $(e_1, \dots, e_{n-1})$ of $L^\flat$ (Definition 2.11), and rearrange them such that for every $1 \leqslant i \leqslant n-1$, exactly one of the following three happens:

(a) $(e_i, e_i)_{\boldsymbol{V}} = \beta_i u^{a_i-1}$ for some $\beta_i \in O_F^\times$;
(b) $(e_i, e_{i+1})_{\boldsymbol{V}} = u^{a_i-1}$;
(c) $(e_i, e_{i-1})_{\boldsymbol{V}} = -u^{a_i-1}$.

By the claim on (1), we may write $L = L^\flat + \langle x \rangle$ in which

$$x = \lambda_1 e_1 + \cdots + \lambda_{n-1} e_{n-1} + x_n$$

for some $\lambda_i \in (E \setminus O_E) \cup \{0\}$ and $0 \neq x_n \in V_{L^\flat}^\perp$. Let T be the moment matrix with respect to the basis $\{e_1, \dots, e_{n-1}, x\}$ of L.

We show by induction that for $1 \leqslant i \leqslant n-2$, $\lambda_i = 0$. Suppose we know $\lambda_1 = \cdots \lambda_{i-1} = 0$. For λ_i (with $1 \leqslant i \leqslant n-2$), there are three cases.

- If e_i is in the situation (a) above, then applying Lemma 2.23(1) to the i-by-i minor of T consisting of rows $\{1, \dots, i\}$ and columns $\{1, \dots, i-1, n\}$, we obtain $\mathrm{val}_E(\lambda_i \beta_i u^{a_i-1}) \geqslant a_i - 1$, which implies $\lambda_i = 0$.
- If e_i is in the situation (b) above, then applying Lemma 2.23(1) to the i-by-i minor of T consisting of rows $\{1, \dots, i-1, i+1\}$ and columns $\{1, \dots, i-1, n\}$, we obtain $\mathrm{val}_E(-\lambda_i u^{a_i-1}) \geqslant a_i - 1$, which implies $\lambda_i = 0$.
- If e_i is in the situation (c) above, then applying Lemma 2.23(1) to the i-by-i minor of T consisting of rows $\{1, \dots, i\}$ and columns $\{1, \dots, i-1, n\}$, we obtain $\mathrm{val}_E(\lambda_i u^{a_i-1}) \geqslant a_i - 1$, which implies $\lambda_i = 0$.

Note that e_{n-1} is in the situation (a). Applying Lemma 2.23(1) to the $(n-1)$-by-$(n-1)$ minor of T consisting of rows $\{1, \dots, n-1\}$ and columns $\{1, \dots, n-2, n\}$, we obtain $\mathrm{val}_E(\lambda_{n-1}\beta_{n-1}u^{a_{n-1}-1}) \geqslant a_{n-1} - 2$, which implies $\lambda_{n-1} \in u^{-1}O_E$. On the other hand, $\lambda_{n-1} \neq 0$ since otherwise a_{n-1} will appear in the fundamental invariants of L, which is a contradiction. Thus, we have $\lambda_{n-1} \in u^{-1}O_E \setminus O_E$. After rescaling by an element in $O_E^\times$, we may assume $\lambda_{n-1} = u^{-1}$. Applying Lemma 2.23(1) to the $(n-1)$-by-$(n-1)$ minor of T consisting of rows $\{1, \dots, n-2, n\}$ and columns $\{1, \dots, n-2, n\}$, we obtain

$$\mathrm{val}_E\left((x_n, x_n)_{\boldsymbol{V}} - u^{-2}\beta_{n-1}u^{a_{n-1}-1}\right) \geqslant a_{n-1} - 2. \tag{2.19}$$

We note the following facts.

- The set of $x_n \in V_{L^\flat}^\perp$ satisfying (2.19) is stable under the multiplication by $1 + uO_E$.
- The set of orbits of such x_n under the multiplication by $1 + uO_E$ is bijective to the set of L.
- The number of orbits is either 0 or 2.
- If the number is 2, then $a_{n-1} \geqslant 3$, since $\boldsymbol{V}$ is nonsplit.

Thus, the main part of the lemma is proved, with the properties (1) and (2) included. For (3), we simply take $L^\flat_{\leftarrow} = \langle e_1, \dots, e_{n-2} \rangle$ with $L^\flat_{\rightarrow}$ and $L^{\flat\pm}_{\rightarrow}$ uniquely determined.

The lemma is proved. □

In the rest of subsection, we say that $L^\flat$ is *special* if $L^\flat$ is like in Lemma 2.57 for which the number is 2. We now define an open compact subset $S_{L^\flat}$ of $\boldsymbol{V}$ for an integral element

$L^\flat \in \flat(\boldsymbol{V})$ in the following way:

$$S_{L^\flat} := \begin{cases} L^{\flat+} \cup L^{\flat-}, & \text{if } L^\flat \text{ is special,} \\ L^\flat + (V_{L^\flat}^\perp)^{\mathrm{int}}, & \text{if } L^\flat \text{ is not special.} \end{cases}$$

Lemma 2.58. *Take an integral element $L^\flat \in \flat(\boldsymbol{V})$. Then for every $x \in \boldsymbol{V} \setminus (V_{L^\flat} \cup S_{L^\flat})$, we may write*

$$L^\flat + \langle x \rangle = L^{\flat\prime} + \langle x' \rangle$$

for some $L^{\flat\prime} \in \flat(\boldsymbol{V})$ satisfying $\mathrm{val}(L^{\flat\prime}) < \mathrm{val}(L^\flat)$.

Proof. Take an element $x \in \boldsymbol{V} \setminus (V_{L^\flat} \cup S_{L^\flat})$. Put $L := L^\flat + \langle x \rangle$. If L is not integral, then by Lemma 2.12, we may write $L = L^{\flat\prime} + \langle x' \rangle$ with $L^{\flat\prime} \in \flat(\boldsymbol{V})$ that is not integral, hence the lemma follows.

In what follows, we assume L integral and write its fundamental invariants as $(a'_1, \dots, a'_n)$. By Lemma 2.12, it suffices to show that $a'_1 + \cdots + a'_{n-1} \leqslant a_1 + \cdots + a_{n-1} - 2$.

Choose a normal basis $(e_1, \dots, e_{n-1})$ of $L^\flat$ (Definition 2.11), and rearrange them such that for every $1 \leqslant i \leqslant n-1$, exactly one of the following three happens:

(a) $(e_i, e_i)_{\boldsymbol{V}} = \beta_i u^{a_i - 1}$ for some $\beta_i \in O_F^\times$;
(b) $(e_i, e_{i+1})_{\boldsymbol{V}} = u^{a_i - 1}$;
(c) $(e_i, e_{i-1})_{\boldsymbol{V}} = -u^{a_i - 1}$.

Write $x = \lambda_1 e_1 + \cdots + \lambda_{n-1} e_{n-1} + x_n$ for some $\lambda_i \in (E \setminus O_E) \cup \{0\}$ and $0 \neq x_n \in V_{L^\flat}^\perp$. Let T be the moment matrix with respect to the basis $\{e_1, \dots, e_{n-1}, x\}$ of L.

If $\lambda_1 = \cdots = \lambda_{n-1} = 0$, then since $x \notin S_{L^\flat}$, we have either $\langle x \rangle$ is not integral, or $\mathrm{val}(x) \leqslant a_{n-1} - 2$ (only possible when $L^\flat$ is special), which implies $a'_1 + \cdots + a'_{n-1} \leqslant a_1 + \cdots + a_{n-1} - 2$.

If $\lambda_i \neq 0$ for some $1 \leqslant i \leqslant n-1$ such that e_i is in the situation (b) or (c), then applying Lemma 2.23(1) to the $(n-1)$-by-$(n-1)$ minor of T deleting the i-th row and the i-th column, we obtain $a'_1 + \cdots + a'_{n-1} \leqslant a_1 + \cdots + a_{n-1} - 2$.

If $\lambda_i \notin u^{-1} O_E$ for some $1 \leqslant i \leqslant n-1$ such that e_i is in the situation (a), then applying Lemma 2.23(1) to the $(n-1)$-by-$(n-1)$ minor of T deleting the i-th row and the n-th column, we obtain $a'_1 + \cdots + a'_{n-1} \leqslant a_1 + \cdots + a_{n-1} - 2$.

If $\lambda_i \neq 0$ and $\lambda_j \neq 0$ for $1 \leqslant i < j \leqslant n-1$ such that both e_i and e_j are in the situation (a), then applying Lemma 2.23(1) to the $(n-1)$-by-$(n-1)$ minor of T deleting the i-th row and the j-th column, we obtain $a'_1 + \cdots + a'_{n-1} \leqslant a_1 + \cdots + a_{n-1} - 2$.

The remaining case is that $\lambda_i \in u^{-1} O_E \setminus O_E$ for a unique element $1 \leqslant i \leqslant n-1$ such that e_i is in the situation (a). Then $L^\flat + \langle x \rangle$ is the orthogonal sum of $\langle e_1, \dots, \widehat{e_i}, \dots, e_{n-1} \rangle$ and $\langle e_i, x \rangle$. In particular, if we write the fundamental invariants of $\langle e_i, x \rangle$ as (b_1, b_2), then the fundamental invariants of $L^\flat + \langle x \rangle$ is the nondecreasing rearrangement of $(a_1, \dots, \widehat{a_i}, \dots, a_{n-1}, b_1, b_2)$. We have two cases:

- If $(x, x)_{\boldsymbol{V}} \in u^{e_i - 1} O_F$, then $(b_1, b_2) = (a_i - 1, a_i - 1)$. Thus, we have either $a'_1 + \cdots + a'_{n-1} \leqslant a_1 + \cdots + a_{n-1} - 2$, or $i = n-1$, $a_{n-2} < a_{n-1}$, and $L^\flat + \langle x \rangle$ has fundamental invariants $(a_1, \dots, a_{n-2}, a_{n-1} - 1, a_{n-1} - 1)$ (hence $L^\flat$ is special). The latter case is not possible as $x \notin S_{L^\flat}$.
- If $(x, x)_{\boldsymbol{V}} \notin u^{e_i - 1} O_F$, then $b_1 \leqslant a_i - 2$. Thus we have $a'_1 + \cdots + a'_{n-1} \leqslant a_1 + \cdots + a_{n-1} - 2$.

The lemma is proved. □

Proof of Theorem 2.7. For every element $L^\flat \in \flat(\boldsymbol{V})$, we define a function

$$\Phi_{L^\flat} := \partial\mathrm{Den}^{\mathrm{v}}_{L^\flat} - \mathrm{Int}^{\mathrm{v}}_{L^\flat},$$

which is a compactly supported locally constant function on $\boldsymbol{V}$ by Proposition 2.22 and Proposition 2.51(2). It enjoys the following properties:

(1) For $x \in \boldsymbol{V} \setminus V_{L^\flat}$, we have $\Phi_{L^\flat}(x) = \partial\mathrm{Den}_{L^\flat}(x) - \mathrm{Int}_{L^\flat}(x)$ by Proposition 2.51(1).

(2) $\Phi_{L^\flat}$ is invariant under the translation by $L^\flat$, which follows from (1) and the similar properties for $\partial\mathrm{Den}_{L^\flat}$ and $\mathrm{Int}_{L^\flat}$.

(3) The support of $\widehat{\Phi_{L^\flat}}$ is contained in $\boldsymbol{V}^{\mathrm{int}}$, by Proposition 2.22 and Proposition 2.51(2).

We prove by induction on $\mathrm{val}(L^\flat)$ that $\Phi_{L^\flat} \equiv 0$.

The initial case is that $\mathrm{val}(L^\flat) = -1$, that is, $L^\flat$ is not integral. Then we have $\partial\mathrm{Den}_{L^\flat} = \mathrm{Int}_{L^\flat} = 0$, hence $\Phi_{L^\flat} \equiv 0$ by (1).

Now consider $L^\flat$ that is integral, and assume $\Phi_{L^{\flat\prime}} \equiv 0$ for every $L^{\flat\prime} \in \flat(\boldsymbol{V})$ satisfying $\mathrm{val}(L^{\flat\prime}) < \mathrm{val}(L^\flat)$. For every $x \in \boldsymbol{V} \setminus (V_{L^\flat} \cup S_{L^\flat})$, by Lemma 2.58, we may write $L^\flat + \langle x\rangle = L^{\flat\prime} + \langle x'\rangle$ with some $L^{\flat\prime} \in \flat(\boldsymbol{V})$ satisfying $\mathrm{val}(L^{\flat\prime}) < \mathrm{val}(L^\flat)$; and we have

$$\begin{aligned}
\Phi_{L^\flat}(x) &= \partial\mathrm{Den}_{L^\flat}(x) - \mathrm{Int}_{L^\flat}(x) \\
&= \partial\mathrm{Den}(L^\flat + \langle x\rangle) - \mathrm{Int}(L^\flat + \langle x\rangle) \\
&= \partial\mathrm{Den}(L^{\flat\prime} + \langle x'\rangle) - \mathrm{Int}(L^{\flat\prime} + \langle x'\rangle) \\
&= \Phi_{L^{\flat\prime}}(x') = 0
\end{aligned}$$

by the induction hypothesis. Thus, the support of $\Phi_{L^\flat}$ is contained in $S_{L^\flat}$. There are two cases.

Suppose that $L^\flat$ is not special. By (2), we may write $\Phi_{L^\flat} = \mathbb{1}_{L^\flat} \otimes \phi$ for a locally constant function ϕ on $V_{L^\flat}^\perp$ supported on $(V_{L^\flat}^\perp)^{\mathrm{int}}$. Then $\widehat{\Phi_{L^\flat}} = C \cdot \mathbb{1}_{(L^\flat)^\vee} \otimes \widehat{\phi}$ for some $C \in \mathbb{Q}^\times$. Now since $\widehat{\phi}$ is invariant under the translation by $u^{-1}(V_{L^\flat}^\perp)^{\mathrm{int}}$, we must have $\widehat{\phi} = 0$ by (3), that is, $\Phi_{L^\flat} \equiv 0$.

Suppose that $L^\flat$ is special. We fix the orthogonal decompositions $L^\flat = L^\flat_\leftarrow \oplus L^\flat_\rightarrow$ and $L^{\flat\pm} = L^\flat_\leftarrow \oplus L^{\flat\pm}_\rightarrow$ from Lemma 2.57. Put $V_\leftarrow := L^\flat_\leftarrow \otimes_{O_F} F$ and denote by $V_\rightarrow$ the orthogonal complement of $V_\leftarrow$ in $\boldsymbol{V}$. Then both $L^{\flat+}_\rightarrow$ and $L^{\flat-}_\rightarrow$ are integral O_E-lattices of $V_\rightarrow$ with fundamental invariants $(a_{n-1}-1, a_{n-1}-1)$. Moreover, we have $S_{L^\flat} = L^\flat_\leftarrow \times (L^{\flat+}_\rightarrow \cup L^{\flat-}_\rightarrow)$. Thus, by (2), we may write $\Phi_{L^\flat} = \mathbb{1}_{L^\flat_\leftarrow} \otimes \phi$ for a locally constant function ϕ on $V_\rightarrow$ supported on $L^{\flat+}_\rightarrow \cup L^{\flat-}_\rightarrow$. Since $a_{n-1} \geqslant 3$ by Lemma 2.57, we have $L^{\flat+}_\rightarrow \cup L^{\flat-}_\rightarrow \subseteq uV^{\mathrm{int}}_\rightarrow$, which implies that the support of ϕ is contained in $uV^{\mathrm{int}}_\rightarrow$. On the other hand, by (3), the support of $\widehat{\phi}$ is contained in $V^{\mathrm{int}}_\rightarrow$. Together, we must have $\phi = 0$ by the Uncertainty Principle [LZa, Proposition 8.1.6], that is, $\Phi_{L^\flat} \equiv 0$.

By (1), we have $\partial\mathrm{Den}_{L^\flat}(x) = \mathrm{Int}_{L^\flat}(x)$ for every $x \in \boldsymbol{V} \setminus V_{L^\flat}$. In particular, Theorem 2.7 follows as every O_E-lattice $\boldsymbol{L}$ of $\boldsymbol{V}$ is of the form $L^\flat + \langle x\rangle$ for some $L^\flat \in \flat(\boldsymbol{V})$. ☐

2.8. **Comparison with absolute Rapoport–Zink spaces.** Let the setup be as in Subsection 2.1. In this subsection, we compare $\mathcal{N}$ to certain (absolute) Rapoport–Zink space under the assumption that F is *unramified* over $\mathbb{Q}_p$. Put $f := [F : \mathbb{Q}_p]$, hence $q = p^f$. This subsection is redundant if $f = 1$.

To begin with, we fix a subset Φ of $\mathrm{Hom}(E, \mathbb{C}_p) = \mathrm{Hom}(E, \breve{E})$ containing φ_0 and satisfying $\mathrm{Hom}(E, \breve{E}) = \Phi \coprod \Phi^{\mathtt{c}}$. Recall that we have regarded E as a subfield of $\breve{E}$ via φ_0. We introduce more notation.

- For every ring R, we denote by $\mathsf{W}(R)$ the p-typical Witt ring of R, with F, V, $[\ ]$, and $\mathsf{I}(R)$ its (p-typical) Frobenius, the Verschiebung, the Teichmüller lift, and the augmentation ideal, respectively. For an F^i-linear map $\mathsf{f}\colon \mathsf{P}\to\mathsf{Q}$ between $\mathsf{W}(R)$-modules with $i\geqslant 1$, we denote by $\mathsf{f}^\natural\colon \mathsf{W}(R)\otimes_{\mathsf{F}^i,\mathsf{W}(R)}\mathsf{P}\to\mathsf{Q}$ its induced $\mathsf{W}(R)$-linear map.
- For $i\in\mathbb{Z}/f\mathbb{Z}$, put $\psi_i\coloneqq\mathsf{F}^i\colon O_F\to O_F$, define $\hat{\psi}_i\colon O_F\to\mathsf{W}(O_F)$ to be the composition of ψ_i with the Cartier homomorphism $O_F\to\mathsf{W}(O_F)$, and denote by φ_i the unique element in Φ above ψ_i.
- For $i\in\mathbb{Z}/f\mathbb{Z}$, let ϵ_i be the unique unit in $\mathsf{W}(O_F)$ satisfying ${}^{\mathsf{V}}\epsilon_i=[\psi_i(u^2)]-\hat{\psi}_i(u^2)$, which exists by [ACZ16, Lemma 2.24]. We then fix a unit μ_u in $\mathsf{W}(O_{\breve{F}})$, where $\breve{F}$ denotes the complete maximal unramified extension of F in $\breve{E}$, such that

$$\frac{{}^{\mathsf{F}^f}\mu_u}{\mu_u}=\prod_{i=1}^{f-1}{}^{\mathsf{F}^{f-1-i}}\epsilon_i, \tag{2.20}$$

  which is possible since the right-hand side is a unit in $\mathsf{W}(O_F)$.
- For a p-divisible group X over an object S of $\mathrm{Sch}^{\mathrm{v}}_{/O_{\breve{E}}}$ with an action by O_F, we have a decomposition

$$\mathrm{Lie}(X)=\bigoplus_{i=0}^{f-1}\mathrm{Lie}_{\psi_i}(X)$$

  of $\mathscr{O}_S$-modules according to the action of O_F on $\mathrm{Lie}(X)$.

Definition 2.59. Let S be an object of $\mathrm{Sch}_{/O_{\breve{E}}}$. We define a category $\mathrm{Exo}^{\Phi}_{(n-1,1)}(S)$ whose objects are triples (X,ι_X,λ_X) in which

- X is a p-divisible group over S of dimension nf and height $2nf$;
- $\iota_X\colon O_E\to\mathrm{End}(X)$ is an action of O_E on X satisfying:
  - (Kottwitz condition): the characteristic polynomial of $\iota_X(u)$ on the $\mathscr{O}_S$-module $\mathrm{Lie}_{\psi_0}(X)$ is $(T-u)^{n-1}(T+u)\in\mathscr{O}_S[T]$,
  - (Wedge condition): we have
  $$\bigwedge^2(\iota_X(u)-u\mid\mathrm{Lie}_{\psi_0}(X))=0,$$
  - (Spin condition): for every geometric point s of S, the action of $\iota_X(u)$ on $\mathrm{Lie}_{\psi_0}(X_s)$ is nonzero;
  - (Banal condition): for $1\leqslant i\leqslant f-1$, O_E acts on $\mathrm{Lie}_{\psi_i}(X)$ via φ_i;
- $\lambda_X\colon X\to X^\vee$ is a ι_X-compatible polarization such that $\ker(\lambda_X)=X[\iota_X(u)]$.

A morphism (resp. quasi-morphism) from (X,ι_X,λ_X) to (Y,ι_Y,λ_Y) is an O_E-linear isomorphism (resp. quasi-isogeny) $\rho\colon X\to Y$ of height zero such that $\rho^*\lambda_Y=\lambda_X$.

When S belongs to $\mathrm{Sch}^{\mathrm{v}}_{/O_{\breve{E}}}$, we denote by $\mathrm{Exo}^{\Phi,\mathrm{b}}_{(n-1,1)}(S)$ the subcategory of $\mathrm{Exo}^{\Phi}_{(n-1,1)}(S)$ consisting of (X,ι_X,λ_X) in which X is supersingular.

Note that both $\mathrm{Exo}^{\mathrm{b}}_{(n-1,1)}$ and $\mathrm{Exo}^{\Phi,\mathrm{b}}_{(n-1,1)}$ are prestacks (that is, presheaves valued in groupoids) on $\mathrm{Sch}^{\mathrm{v}}_{/O_{\breve{E}}}$. Now we construct a morphism

$$-^{\mathrm{rel}}\colon \mathrm{Exo}^{\Phi,\mathrm{b}}_{(n-1,1)}\to\mathrm{Exo}^{\mathrm{b}}_{(n-1,1)} \tag{2.21}$$

of prestacks on $\mathrm{Sch}^{\mathrm{v}}_{/O_{\breve{E}}}$. We will use the theory of displays [Zin02, Lau08] and O_F-displays [ACZ16].

Let $S = \operatorname{Spec} R$ be an affine scheme in $\mathrm{Sch}^{\mathrm{v}}_{/O_{\breve{E}}}$. Take an object (X, ι_X, λ_X) of $\mathrm{Exo}^{\Phi,\mathrm{b}}_{(n-1,1)}(S)$. Write $(\mathsf{P}, \mathsf{Q}, \mathsf{F}, \dot{\mathsf{F}})$ for the display of X (as a formal p-divisible group). The action of O_F on P induces decompositions

$$\mathsf{P} = \bigoplus_{i=0}^{f-1} \mathsf{P}_i, \quad \mathsf{Q} = \bigoplus_{i=0}^{f-1} \mathsf{Q}_i, \quad \mathsf{F} = \sum_{i=0}^{f-1} \mathsf{F}_i, \quad \dot{\mathsf{F}} = \sum_{i=0}^{f-1} \dot{\mathsf{F}}_i,$$

where P_i is the $\mathsf{W}(R)$-submodule on which O_F acts via $\hat{\psi}_i$, and $\mathsf{Q}_i = \mathsf{Q} \cap \mathsf{P}_i$. It is clear that the above decomposition is O_E-linear, and P_i is a projective $O_E \otimes_{O_F, \hat{\psi}_i} \mathsf{W}(R)$-module of rank n.

Lemma 2.60. *For $1 \leqslant i \leqslant f-1$, we have*

$$\mathsf{Q}_i = (u \otimes 1 - 1 \otimes [\varphi_i(u)])\mathsf{P}_i + \mathsf{I}(R)\mathsf{P}_i,$$

and that the map

$$\mathsf{F}'_i := \dot{\mathsf{F}}_i \circ (u \otimes 1 - 1 \otimes [\varphi_i(u)]) \cdot \colon \mathsf{P}_i \to \mathsf{P}_{i+1}$$

is a Frobenius linear epimorphism, hence isomorphism.

Proof. The Banal condition in Definition 2.59 implies that for $1 \leqslant i \leqslant f-1$,

$$(u \otimes 1 - 1 \otimes [\varphi_i(u)])\mathsf{P}_i + \mathsf{I}(R)\mathsf{P}_i \subseteq \mathsf{Q}_i.$$

To show the reverse inclusion, it suffices to show that the image of $(u \otimes 1 - 1 \otimes [\varphi_i(u)])\mathsf{P}_i$ in $\mathsf{P}_i/\mathsf{I}(R)\mathsf{P}_i = \mathsf{P}_i \otimes_{\mathsf{W}(R)} R$ is projective of rank n. But the image is $(u \otimes 1 - 1 \otimes \varphi_i(u))\mathsf{P}_i \otimes_{\mathsf{W}(R)} R$, which has rank n since P_i is projective over $O_E \otimes_{O_F, \hat{\psi}_i} \mathsf{W}(R)$ of rank n.

Now we show that $(\mathsf{F}'_i)^\natural$ is surjective. It suffices to show that $\operatorname{coker}(\mathsf{F}'_i)^\natural \otimes_{\mathsf{W}(R)} \kappa$ vanishes for every homomorphism $\mathsf{W}(R) \to \kappa$ with κ a perfect field of characteristic p. Since $\mathsf{W}(R) \to \kappa$ necessarily vanishes on $\mathsf{I}(R)$, it lifts to a homomorphism $\mathsf{W}(R) \to \mathsf{W}(\kappa)$. Thus, we may just assume that R is a perfect field of characteristic p. Since

$$(u \otimes 1 - 1 \otimes [\varphi_i(u)])(-u \otimes 1 - 1 \otimes [\varphi_i(u)]) = [\psi_i(u^2)] - \hat{\psi}_i(u^2) = {}^{\mathsf{V}}\epsilon_i$$

in which ϵ_i is a unit in $\mathsf{W}(O_F)$, the image of the map

$$(u \otimes 1 - 1 \otimes [\varphi_i(u)]) \cdot \colon \mathsf{P}_i \to \mathsf{P}_i \tag{2.22}$$

contains $(u \otimes 1 - 1 \otimes [\varphi_i(u)])\mathsf{P}_i + \mathsf{W}(R)\,{}^{\mathsf{V}}\epsilon_i \cdot \mathsf{P}_i$. As R is a perfect field of characteristic p, we have $\mathsf{W}(R)\,{}^{\mathsf{V}}\epsilon_i = \mathsf{I}(R)$, hence (2.22) is surjective. Thus, F'_i is a Frobenius linear epimorphism as F_i is.

The lemma is proved. □

Now we put

$$\mathsf{P}^{\mathrm{rel}} := \mathsf{P}_0, \quad \mathsf{Q}^{\mathrm{rel}} := \mathsf{Q}_0, \quad \mathsf{F}^{\mathrm{rel}} := \mathsf{F}'_{f-1} \circ \cdots \circ \mathsf{F}'_1 \circ \mathsf{F}_0, \quad \dot{\mathsf{F}}^{\mathrm{rel}} := \mathsf{F}'_{f-1} \circ \cdots \circ \mathsf{F}'_1 \circ \dot{\mathsf{F}}_0.$$

Then $(\mathsf{P}^{\mathrm{rel}}, \mathsf{Q}^{\mathrm{rel}}, \mathsf{F}^{\mathrm{rel}}, \dot{\mathsf{F}}^{\mathrm{rel}})$ defines an $f(\text{-}\mathbb{Z}_p)$-display in the sense of [ACZ16, Definition 2.1] with an O_E-action, for which the Kottwitz condition, the Wedge condition, and the Spin condition are obviously inherited. It remains to construct the polarization $\lambda_{X^{\mathrm{rel}}}$. By Remark 2.61 below, we have the collection of perfect symmetric $\mathsf{W}(R)$-bilinear pairings $\{(\ ,\ )_i \mid i \in \mathbb{Z}/f\mathbb{Z}\}$

coming from λ_X. For $x, y \in \mathsf{P}_0$, put $x_i := (\mathsf{F}'_{i-1}\circ\cdots\circ\mathsf{F}'_1\circ\dot{\mathsf{F}}_0)(x)$ and $y_i := (\mathsf{F}'_{i-1}\circ\cdots\circ\mathsf{F}'_1\circ\dot{\mathsf{F}}_0)(y)$ for $1 \leqslant i \leqslant f$, and we have

$$\begin{aligned}(\dot{\mathsf{F}}^{\mathrm{rel}}x, \dot{\mathsf{F}}^{\mathrm{rel}}y)_0 &= (\mathsf{F}'_{f-1}x_{f-1}, \mathsf{F}'_{f-1}y_{f-1})_0 \\ &= (\dot{\mathsf{F}}_{f-1}((u\otimes 1 - 1\otimes[\varphi_{f-1}(u)])x_{f-1}), \dot{\mathsf{F}}_{f-1}((u\otimes 1 - 1\otimes[\varphi_{f-1}(u)])y_{f-1}))_0 \\ &= {}^{\mathsf{V}^{-1}}((u\otimes 1 - 1\otimes[\varphi_{f-1}(u)])x_{f-1}, (u\otimes 1 - 1\otimes[\varphi_{f-1}(u)])y_{f-1})_{f-1} \\ &= {}^{\mathsf{V}^{-1}}\left(([\psi_{f-1}(u^2)] - \hat{\psi}_{f-1}(u^2))\cdot(x_{f-1}, y_{f-1})_{f-1}\right) \\ &= {}^{\mathsf{V}^{-1}}\left({}^{\mathsf{V}}\epsilon_{f-1}\cdot(x_{f-1}, y_{f-1})_{f-1}\right) \\ &= \epsilon_{f-1}\cdot {}^{\mathsf{F}}(x_{f-1}, y_{f-1})_{f-1} \\ &= \cdots = \left(\prod_{i=1}^{f-1} {}^{\mathsf{F}^{f-1-i}}\epsilon_i\right)\cdot {}^{\mathsf{F}^{f-1}}(x_1, y_1)_1 \\ &= \left(\prod_{i=1}^{f-1} {}^{\mathsf{F}^{f-1-i}}\epsilon_i\right)\cdot {}^{\mathsf{F}^{f-1}\mathsf{V}^{-1}}(x, y)_0.\end{aligned}$$

Put $(\ ,\ )^{\mathrm{rel}} := \mu_u(\ ,\ )_0$, which satisfies $(\dot{\mathsf{F}}^{\mathrm{rel}}x, \dot{\mathsf{F}}^{\mathrm{rel}}y)^{\mathrm{rel}} = {}^{\mathsf{F}^{f-1}\mathsf{V}^{-1}}(x,y)^{\mathrm{rel}}$ by (2.20). Then the $f(\text{-}\mathbb{Z}_p)$-display $(\mathsf{P}^{\mathrm{rel}}, \mathsf{Q}^{\mathrm{rel}}, \mathsf{F}^{\mathrm{rel}}, \dot{\mathsf{F}}^{\mathrm{rel}})$ with O_E-action together with the pairing $(\ ,\ )^{\mathrm{rel}}$ define an object $(X, \iota_X, \lambda_X)^{\mathrm{rel}}$ of $\mathrm{Exo}^{\mathrm{b}}_{(n-1,1)}(S)$, as explained in the proof of [Mih20, Proposition 3.4] and Remark 2.61 below. It is clear that the construction is functorial in S.

Remark 2.61. For an object (X, ι_X, λ_X) of $\mathrm{Exo}^{\Phi,\mathrm{b}}_{(n-1,1)}(S)$ with $(\mathsf{P}, \mathsf{Q}, \mathsf{F}, \dot{\mathsf{F}})$ the display of X, we have a similar claim as in Remark 2.2 concerning the polarization λ_X. In particular, as discussed in [Mih20, Section 11.1], the polarization λ_X, or rather its symmetrization, is equivalent to a collection of perfect symmetric $\mathsf{W}(R)$-bilinear pairings

$$\{(\ ,\ )_i \colon \mathsf{P}_i \times \mathsf{P}_i \to \mathsf{W}(R) \mid i \in \mathbb{Z}/f\mathbb{Z}\},$$

satisfying $(\iota_X(\alpha)x, y)_i = (x, \iota_X(\alpha^{\mathrm{c}})y)_i$ for every $\alpha \in O_E$ and $(\dot{\mathsf{F}}_i x, \dot{\mathsf{F}}_i y)_{i+1} = {}^{\mathsf{V}^{-1}}(x,y)_i$ for $i \in \mathbb{Z}/f\mathbb{Z}$.

Similarly, for an object $(X', \iota_{X'}, \lambda_{X'})$ of $\mathrm{Exo}^{\mathrm{b}}_{(n-1,1)}(S)$ with $(\mathsf{P}', \mathsf{Q}', \mathsf{F}', \dot{\mathsf{F}}')$ the $f(\text{-}\mathbb{Z}_p)$-display of X', the polarization $\lambda_{X'}$ is equivalent to a perfect symmetric $\mathsf{W}(R)$-bilinear pairing

$$(\ ,\ )' \colon \mathsf{P}' \times \mathsf{P}' \to \mathsf{W}(R),$$

satisfying $(\iota_{X'}(\alpha)x, y)' = (x, \iota_{X'}(\alpha^{\mathrm{c}})y)'$ for every $\alpha \in O_E$ and $(\dot{\mathsf{F}}'x, \dot{\mathsf{F}}'y)' = {}^{\mathsf{F}^{f-1}\mathsf{V}^{-1}}(x,y)'$.

Proposition 2.62. *The morphism* (2.21) *is an isomorphism.*

Proof. It suffices to show that for every affine scheme $S = \operatorname{Spec} R$ in $\mathrm{Sch}^{\mathrm{v}}_{/O_{\breve{E}}}$, the functor $-^{\mathrm{rel}}(S)$ is fully faithful and essentially surjective.

We first show that $-^{\mathrm{rel}}(S)$ is fully faithful. Take an object (X, ι_X, λ_X) of $\mathrm{Exo}^{\Phi,\mathrm{b}}_{(n-1,1)}(S)$. It suffices to show that the natural map $\mathrm{Aut}((X, \iota_X, \lambda_X)) \to \mathrm{Aut}((X, \iota_X, \lambda_X)^{\mathrm{rel}})$ is an isomorphism, which follows from a stronger statement that the natural map $\mathrm{End}_{O_E}(X) \to \mathrm{End}_{O_E}(X^{\mathrm{rel}})$ is an isomorphism, where X^{rel} denotes the first entry of $(X, \iota_X, \lambda_X)^{\mathrm{rel}}$, which is an O_F-divisible group. For the latter, it amounts to showing that the natural map

$$\mathrm{End}_{O_E}((\mathsf{P}, \mathsf{Q}, \mathsf{F}, \dot{\mathsf{F}})) \to \mathrm{End}_{O_E}((\mathsf{P}^{\mathrm{rel}}, \mathsf{Q}^{\mathrm{rel}}, \mathsf{F}^{\mathrm{rel}}, \dot{\mathsf{F}}^{\mathrm{rel}})) \tag{2.23}$$

is an isomorphism. For the injectivity, let f be an element in the source, which decomposes as $\mathsf{f}=\sum_{i=0}^{f-1}\mathsf{f}_i$ for endomorphisms $\mathsf{f}_i\colon \mathsf{P}_i\to\mathsf{P}_i$ preserving Q_i and commuting with F and $\dot{\mathsf{F}}$. Since for every $i\in\mathbb{Z}/f\mathbb{Z}$, $\dot{\mathsf{F}}_i$ is a Frobenius linear surjective map from Q_i to P_{i+1}, the map f is determined by f_0. Thus, (2.23) is injective. For the surjectivity, let $\mathsf{f}^{\mathrm{rel}}$ be an element in the target. Put $\mathsf{f}_0\coloneqq\mathsf{f}^{\mathrm{rel}}\colon\mathsf{P}_0\to\mathsf{P}_0$. By Lemma 2.63(2) below, there is a unique endomorphism f_1 of P_1 rendering the following diagram

$$\begin{CD}
\mathsf{W}(R)\otimes_{\mathsf{F},\mathsf{W}(R)}\mathsf{Q}_0 @>{\dot{\mathsf{F}}_0^\natural}>> \mathsf{P}_1\\
@V{1\otimes(\mathsf{f}_0|_{\mathsf{Q}_0})}VV @VV{\mathsf{f}_1}V\\
\mathsf{W}(R)\otimes_{\mathsf{F},\mathsf{W}(R)}\mathsf{Q}_0 @>{\dot{\mathsf{F}}_0^\natural}>> \mathsf{P}_1
\end{CD}$$

commute. For $2\leqslant i\leqslant f-1$, we define f_i to be the unique endomorphism of P_i satisfying that

$$\mathsf{f}_i\circ(\mathsf{F}'_{i-1}\circ\cdots\circ\mathsf{F}'_1)^\natural=(\mathsf{F}'_{i-1}\circ\cdots\circ\mathsf{F}'_1)^\natural\circ(1\otimes\mathsf{f}_i).$$

Then $\mathsf{f}\coloneqq\sum_{i=0}^{f-1}\mathsf{f}_i$ is an O_E-linear endomorphism of P, which commutes with $\dot{\mathsf{F}}$ and hence F. It remains to check that $\mathsf{f}(\mathsf{Q})\subseteq\mathsf{Q}$, which follows from Lemma 2.60.

We then show that $-^{\mathrm{rel}}(S)$ is essentially surjective. Take an object $(X',\iota_{X'},\lambda_{X'})$ of $\mathrm{Exo}^{\mathrm{b}}_{(n-1,1)}(S)$ in which X' is given by an $f(\text{-}\mathbb{Z}_p)$-display $(\mathsf{P}',\mathsf{Q}',\mathsf{F}',\dot{\mathsf{F}}')$. For $0\leqslant i\leqslant f-1$, put $\mathsf{P}_i\coloneqq\mathsf{W}(R)\otimes_{\mathsf{F}^i,\mathsf{W}(R)}\mathsf{P}'$. Denote by $\mathsf{u}_0\colon\mathsf{P}_0\to\mathsf{P}_0$ the endomorphism given by the action of $u\in O_E$ on P'. Put $\mathsf{Q}_0=\mathsf{Q}'$ and for $1\leqslant i\leqslant f-1$, put

$$\mathsf{Q}_i\coloneqq((1\otimes\mathsf{u}_0)\otimes1-(1\otimes1)\otimes[\varphi_i(u)])\mathsf{P}_i+\mathsf{I}(R)\mathsf{P}_i.$$

Fix a normal decomposition $\mathsf{P}'=\mathsf{L}'\oplus\mathsf{T}'$ for Q' and let $\ddot{\mathsf{F}}'\coloneqq\dot{\mathsf{F}}'|_{\mathsf{L}'}+\mathsf{F}'|_{\mathsf{T}'}\colon\mathsf{P}'\to\mathsf{P}'$ be the corresponding F^f-linear isomorphism. For $0\leqslant i<f-1$, let $\ddot{\mathsf{F}}_i\colon\mathsf{P}_i\to\mathsf{P}_{i+1}$ be the Frobenius linear isomorphism induced by the identity map on P'; and finally let $\ddot{\mathsf{F}}_{f-1}\colon\mathsf{P}_{f-1}\to\mathsf{P}_0$ be the Frobenius linear isomorphism induced by $\ddot{\mathsf{F}}'$. Let $\dot{\mathsf{F}}_0\colon\mathsf{Q}_0\to\mathsf{P}_1$ be the map defined by the formula $\dot{\mathsf{F}}_0(l+{}^{\mathsf{V}}w\cdot t)=\ddot{\mathsf{F}}_0(l)+w\ddot{\mathsf{F}}_0(t)$ for $l\in\mathsf{L}'$, $t\in\mathsf{T}'$, and $w\in\mathsf{W}(R)$, which is a Frobenius linear epimorphism. By Lemma 2.63(2) below, there is a unique endomorphism u_1 of P_1 rendering the following diagram

$$\begin{CD}
\mathsf{W}(R)\otimes_{\mathsf{F},\mathsf{W}(R)}\mathsf{Q}_0 @>{\dot{\mathsf{F}}_0^\natural}>> \mathsf{P}_1\\
@V{1\otimes(\mathsf{u}_0|_{\mathsf{Q}_0})}VV @VV{\mathsf{u}_1}V\\
\mathsf{W}(R)\otimes_{\mathsf{F},\mathsf{W}(R)}\mathsf{Q}_0 @>{\dot{\mathsf{F}}_0^\natural}>> \mathsf{P}_1
\end{CD}$$

commute.[14] For $2\leqslant i\leqslant f-1$, we define u_i to be the unique endomorphism of P_i satisfying that

$$\mathsf{u}_i\circ(\ddot{\mathsf{F}}_{i-1}\circ\cdots\circ\ddot{\mathsf{F}}_1)^\natural=(\ddot{\mathsf{F}}_{i-1}\circ\cdots\circ\ddot{\mathsf{F}}_1)^\natural\circ(1\otimes\mathsf{u}_1),$$

[14]We warn the readers that the endomorphism u_1 might be different from $1\otimes\mathsf{u}_0$ as u does not necessarily preserve the normal decomposition. However, the image of $\mathsf{u}_1-1\otimes\mathsf{u}_0$ is contained in $\mathsf{I}(R)\mathsf{P}_1$.

and define a map $\dot{\mathsf{F}}_i\colon \mathsf{Q}_i \to \mathsf{P}_{i+1}$ by the following (compatible) formulae

$$\begin{cases} \dot{\mathsf{F}}_i((\mathsf{u}_i \otimes 1 - 1 \otimes [\varphi_i(u)])x) = \ddot{\mathsf{F}}_i(x), \\ \dot{\mathsf{F}}_i({}^{\mathsf{V}}w \cdot x) = \dfrac{w}{\epsilon_i} \cdot (\mathsf{u}_{i+1} \otimes 1 + 1 \otimes {}^{\mathsf{F}}[\varphi_i(u)])\ddot{\mathsf{F}}_i(x), \end{cases}$$

for $x \in \mathsf{P}_i$ and $w \in \mathsf{W}(R)$, which is a Frobenius linear epimorphism. Put

$$\mathsf{P} \coloneqq \bigoplus_{i=0}^{f-1} \mathsf{P}_i, \quad \mathsf{Q} \coloneqq \bigoplus_{i=0}^{f-1} \mathsf{Q}_i, \quad \dot{\mathsf{F}} \coloneqq \sum_{i=0}^{f-1} \dot{\mathsf{F}}_i, \quad \mathsf{u} \coloneqq \sum_{i=0}^{f-1} \mathsf{u}_i.$$

Then it is straightforward to check that $(\mathsf{P}, \mathsf{Q}, \mathsf{F}, \dot{\mathsf{F}})$ is a display with an action by O_E for which u acts by u, where F is determined by $\dot{\mathsf{F}}$ in the usual way. Now we construct a collection of perfect symmetric $\mathsf{W}(R)$-bilinear pairings $\{(\ ,\ )_i \mid i \in \mathbb{Z}/f\mathbb{Z}\}$ as in Remark 2.61. Put $(\ ,\ )_0 \coloneqq \mu_u^{-1}(\ ,\ )'$, where $(\ ,\ )'$ is the pairing induced by $\lambda_{X'}$. Define inductively for $1 \leqslant i \leqslant f-1$ the unique (perfect symmetric $\mathsf{W}(R)$-bilinear) pairing $(\ ,\ )_i$ satisfying $(\dot{\mathsf{F}}_{i-1}x, \dot{\mathsf{F}}_{i-1}y)_i = {}^{\mathsf{V}^{-1}}(x,y)_{i-1}$. It is clear that we also have $(\dot{\mathsf{F}}_{f-1}x, \dot{\mathsf{F}}_{f-1}y)_0 = {}^{\mathsf{V}^{-1}}(x,y)_{f-1}$. Then the display $(\mathsf{P}, \mathsf{Q}, \mathsf{F}, \dot{\mathsf{F}})$ with the O_E-action together with the collection of pairings $\{(\ ,\ )_i \mid i \in \mathbb{Z}/f\mathbb{Z}\}$ define an object $(X, \iota_X, \lambda_X) \in \mathrm{Exo}^{\Phi,\mathrm{b}}_{(n-1,1)}(S)$, which satisfies $(X, \iota_X, \lambda_X)^{\mathrm{rel}} \simeq (X', \iota_{X'}, \lambda_{X'})$ by construction.

The proposition is proved. ∎

Lemma 2.63. *Let R be a ring on which p is nilpotent. For a pair (P, Q) in which P is a projective $\mathsf{W}(R)$-module of finite rank and Q is a submodule of P containing $\mathsf{I}(R)\mathsf{P}$ such that P/Q is a projective R-module, we define $\mathsf{Q}^\star$ to be the image of $J(R)\mathsf{P}$ under the map $\mathsf{W}(R) \otimes_{\mathsf{F},\mathsf{W}(R)} \mathsf{I}(R)\mathsf{P} \to \mathsf{W}(R) \otimes_{\mathsf{F},\mathsf{W}(R)} \mathsf{Q}$ that is the base change of the inclusion map $\mathsf{I}(R)\mathsf{P} \to \mathsf{Q}$, where $J(R)$ denotes the kernel of $(\mathsf{V}^{-1})^\natural\colon \mathsf{W}(R) \otimes_{\mathsf{F},\mathsf{W}(R)} \mathsf{I}(R) \to \mathsf{W}(R)$. Then for every Frobenius linear epimorphism $\dot{\mathsf{F}}\colon \mathsf{Q} \to \mathsf{P}'$ with P' a projective $\mathsf{W}(R)$-module of the same rank as P, we have*

(1) the kernel of $\dot{\mathsf{F}}^\natural$ coincides with $\mathsf{Q}^\star$;

(2) for every endomorphism $\mathsf{f}\colon \mathsf{P} \to \mathsf{P}$ that preserves Q, there exists a unique endomorphism $\mathsf{f}'\colon \mathsf{P}' \to \mathsf{P}'$ rendering the following diagram

$$\begin{CD} \mathsf{W}(R) \otimes_{\mathsf{F},\mathsf{W}(R)} \mathsf{Q} @>{\dot{\mathsf{F}}^\natural}>> \mathsf{P}' \\ @V{1\otimes(\mathsf{f}|_{\mathsf{Q}})}VV @VV{\mathsf{f}'}V \\ \mathsf{W}(R) \otimes_{\mathsf{F},\mathsf{W}(R)} \mathsf{Q} @>{\dot{\mathsf{F}}^\natural}>> \mathsf{P}' \end{CD}$$

commute.

Proof. We first claim that $\mathsf{J}(R)$ is contained in the kernel of the map

$$\mathsf{W}(R) \otimes_{\mathsf{F},\mathsf{W}(R)} \mathsf{I}(R) \to \mathsf{W}(R) \otimes_{\mathsf{F},\mathsf{W}(R)} \mathsf{W}(R) = \mathsf{W}(R) \tag{2.24}$$

that is the base change of the inclusion map $\mathsf{I}(R) \to \mathsf{W}(R)$. Take an element $x = \sum a_i \otimes {}^{\mathsf{V}}b_i$ in $\mathsf{W}(R) \otimes_{\mathsf{F},\mathsf{W}(R)} \mathsf{I}(R)$. If $x \in J(R)$, then $\sum a_i b_i = 0$. But the image of x under (2.24) is $\sum a_i {}^{\mathsf{FV}}b_i$, which equals $p\sum a_i b_i$. Thus, $J(R)$ is contained in the kernel of (2.24).

For (1), choose a normal decomposition $\mathsf{P} = \mathsf{L} \oplus \mathsf{T}$ of $\mathsf{W}(R)$-modules such that $\mathsf{Q} = \mathsf{L} \oplus \mathsf{I}(R)\mathsf{T}$. By (the proof of) [Lau10, Lemma 2.5], there exists a Frobenius linear automorphism

Ψ of P such that $\dot{\mathsf{F}}(l+at)=\Psi(l)+{}^{\mathsf{V}^{-1}}a\cdot\Psi(t)$ for $l\in\mathsf{L}$, $t\in\mathsf{T}$, and $a\in\mathsf{I}(R)$. Thus $\ker\dot{\mathsf{F}}^\natural$ equals the submodule $J(R)\mathsf{T}$ of $\mathsf{W}(R)\otimes_{\mathsf{F},\mathsf{W}(R)}\mathsf{Q}$. However, by the claim above, the image of $J(R)\mathsf{L}$ under the map $\mathsf{W}(R)\otimes_{\mathsf{F},\mathsf{W}(R)}\mathsf{I}(R)\mathsf{P}\to\mathsf{W}(R)\otimes_{\mathsf{F},\mathsf{W}(R)}\mathsf{Q}$ vanishes. Thus, we have $J(R)\mathsf{T}=\mathsf{Q}^\star$.

For (2), the uniqueness follows since $\dot{\mathsf{F}}^\natural$ is surjective; and the existence follows since the map $1\otimes(\mathsf{f}|_{\mathsf{Q}})$ preserves $\mathsf{Q}^\star$, which is a consequence of the definition of $\mathsf{Q}^\star$. □

To define our (absolute) Rapoport–Zink space, we fix an object $(\boldsymbol{X},\iota_{\boldsymbol{X}},\lambda_{\boldsymbol{X}})\in\mathrm{Exo}^{\Phi,\mathrm{b}}_{(n-1,1)}(\overline{k})$.

Definition 2.64. We define a functor $\mathcal{N}^\Phi:=\mathcal{N}^\Phi_{(\boldsymbol{X},\iota_{\boldsymbol{X}},\lambda_{\boldsymbol{X}})}$ on $\mathrm{Sch}^{\mathrm{v}}_{/O_{\breve{E}}}$ such that for every object S of $\mathrm{Sch}^{\mathrm{v}}_{/O_{\breve{E}}}$, $\mathcal{N}(S)$ consists of quadruples $(X,\iota_X,\lambda_X;\rho_X)$ in which

- (X,ι_X,λ_X) is an object of $\mathrm{Exo}^{\Phi,\mathrm{b}}_{(n-1,1)}(S)$;
- ρ_X is a quasi-morphism from $(X,\iota_X,\lambda_X)\times_S(S\otimes_{O_{\breve{E}}}\overline{k})$ to $(\boldsymbol{X},\iota_{\boldsymbol{X}},\lambda_{\boldsymbol{X}})\otimes_{\overline{k}}(S\otimes_{O_{\breve{E}}}\overline{k})$ in the category $\mathrm{Exo}^{\Phi,\mathrm{b}}_{(n-1,1)}(S\otimes_{O_{\breve{E}}}\overline{k})$.

Corollary 2.65. *The morphism*

$$\mathcal{N}^\Phi=\mathcal{N}^\Phi_{(\boldsymbol{X},\iota_{\boldsymbol{X}},\lambda_{\boldsymbol{X}})}\to\mathcal{N}:=\mathcal{N}_{(\boldsymbol{X},\iota_{\boldsymbol{X}},\lambda_{\boldsymbol{X}})^{\mathrm{rel}}}$$

sending $(X,\iota_X,\lambda_X;\rho_X)$ *to* $((X,\iota_X,\lambda_X)^{\mathrm{rel}};\rho_X^{\mathrm{rel}})$ *is an isomorphism.*

Proof. This follows immediately from Proposition 2.62. □

Now we study special divisors on $\mathcal{N}^\Phi$ and their relation with those on $\mathcal{N}$. Fix a triple $(X_0,\iota_{X_0},\lambda_{X_0})$ where

- X_0 is a supersingular p-divisible group over $\operatorname{Spec}O_{\breve{E}}$ of dimension f and height $2f$;
- $\iota_{X_0}\colon O_E\to\mathrm{End}(X_0)$ is an O_E-action on X_0 such that for $0\leqslant i\leqslant f-1$, the summand $\mathrm{Lie}_{\psi_i}(X)$ has rank 1 on which O_E acts via φ_i;
- $\lambda_{X_0}\colon X_0\to X_0^\vee$ is a ι_{X_0}-compatible principal polarization.

Note that ι_{X_0} induces an isomorphism $\iota_{X_0}\colon O_E\xrightarrow{\sim}\mathrm{End}_{O_E}(X_0)$. Put

$$\boldsymbol{V}:=\mathrm{Hom}_{O_E}(X_0\otimes_{O_{\breve{E}}}\overline{k},\boldsymbol{X})\otimes\mathbb{Q},$$

which is a vector space over E of dimension n, equipped with a natural hermitian form similar to (2.1). By a construction similar to (2.21), we obtain a triple $(X_0,\iota_{X_0},\lambda_{X_0})^{\mathrm{rel}}$ as in the definition of special divisors on $\mathcal{N}$ (Definition 2.5), and a canonical map

$$\mathrm{Hom}_{O_E}(X_0\otimes_{O_{\breve{E}}}\overline{k},\boldsymbol{X})\to\mathrm{Hom}_{O_E}(X_0^{\mathrm{rel}}\otimes_{O_{\breve{E}}}\overline{k},\boldsymbol{X}^{\mathrm{rel}}),$$

which induces a map

$$-^{\mathrm{rel}}\colon\boldsymbol{V}\to\boldsymbol{V}^{\mathrm{rel}}:=\mathrm{Hom}_{O_E}(X_0^{\mathrm{rel}}\otimes_{O_{\breve{E}}}\overline{k},\boldsymbol{X}^{\mathrm{rel}})\otimes\mathbb{Q}.\tag{2.25}$$

For every nonzero element $x\in\boldsymbol{V}$, we have similarly a closed formal subscheme $\mathcal{N}^\Phi(x)$ of $\mathcal{N}^\Phi$ defined similarly as in Definition 2.5.

Corollary 2.66. *The map* (2.25) *is an isomorphism of hermitian spaces. Moreover, under the isomorphism in Corollary* 2.65*, we have* $\mathcal{N}^\Phi(x)=\mathcal{N}(x^{\mathrm{rel}})$.

Proof. By the definition of $-^{\mathrm{rel}}$, the map (2.25) is clearly an isometry. Since both V and $\mathsf{V}^{\mathrm{rel}}$ have dimension n, (2.25) is an isomorphism of hermitian spaces. The second assertion follows from Corollary 2.65 and construction of $-^{\mathrm{rel}}$, parallel to [Mih20, Remark 4.4]. □

Remark 2.67. Let S be an object of $\mathrm{Sch}_{/O_{\breve{E}}}$. We have another category $\mathrm{Exo}^{\Phi}_{(n,0)}(S)$ whose objects are triples (X, ι_X, λ_X) in which

- X is a p-divisible group over S of dimension nf and height $2nf$;
- $\iota_X \colon O_E \to \mathrm{End}(X)$ is an action of O_E on X such that for $0 \leqslant i \leqslant f-1$, O_E acts on $\mathrm{Lie}_{\psi_i}(X)$ via φ_i;
- $\lambda_X \colon X \to X^\vee$ is a ι_X-compatible polarization such that $\ker(\lambda_X) = X[\iota_X(u)]$.

Morphisms are defined similarly as in Definition 2.59. The category $\mathrm{Exo}^{\Phi}_{(n,0)}(S)$ is a connected groupoid. Moreover, one can show that there is a canonical isomorphism $\mathrm{Exo}^{\Phi}_{(n,0)} \to \mathrm{Exo}_{(n,0)}$ of prestacks after restriction to $\mathrm{Sch}^{\mathrm{v}}_{/O_{\breve{E}}}$ similar to (2.21).

Remark 2.68. It is desirable to extend the results in this subsection to a general finite extension $F/\mathbb{Q}_p$. We hope to address this problem in the future.

3. Local theta lifting at ramified places

Throughout this section, we fix a *ramified* quadratic extension E/F of p-adic fields with p odd, with $\mathtt{c} \in \mathrm{Gal}(E/F)$ the Galois involution. We fix a uniformizer $u \in E$ satisfying $u^{\mathtt{c}} = -u$, and denote by q the cardinality of $O_E/(u)$. Let $n = 2r$ be an even positive integer. We fix a nontrivial additive character $\psi_F \colon F \to \mathbb{C}^\times$ of conductor O_F.

The goal of this section is to compute the doubling L-function, the doubling epsilon factor, the spherical doubling zeta integral, and the local theta lifting for a tempered admissible irreducible representation π of $G_r(F)$ that is spherical with respect to the standard special maximal compact subgroup.

3.1. Weil representation and spherical module.

We equip $W_r \coloneqq E^{2r}$ with the skew-hermitian form given by the matrix $\begin{pmatrix} & 1_r \\ -1_r & \end{pmatrix}$. We denote by $\{e_1, \dots, e_{2r}\}$ the natural basis of W_r. Denote by G_r the unitary group of W_r, which is a reductive group over F. We write elements of W_r in the row form, on which G_r acts from the right. Let $K_r \subseteq G_r(F)$ be the stabilizer of the lattice $O_E^{2r} \subseteq W_r$, which is a special maximal compact subgroup. We fix the Haar measure $\mathrm{d}g$ on $G_r(F)$ that gives K_r volume 1. Let P_r be the Borel subgroup of G_r consisting of elements of the form

$$\begin{pmatrix} a & b \\ & {}^{\mathrm{t}}a^{\mathtt{c},-1} \end{pmatrix},$$

in which a is a lower-triangular matrix in $\mathrm{Res}_{E/F}\mathrm{GL}_r$. Let P_r^0 be the maximal parabolic subgroup of G_r containing P_r with the unipotent radical N_r^0, such that the standard diagonal Levi factor M_r^0 of P_r^0 is isomorphic to $\mathrm{Res}_{E/F}\mathrm{GL}_r$.

We fix a a *split* hermitian space $(V, (\ ,\ )_V)$ over E of dimension $n = 2r$, and a self-dual lattice Λ_V of V, namely, $\Lambda_V = \Lambda_V^\vee \coloneqq \{x \in V \mid \mathrm{Tr}_{E/F}(x,y)_V \in O_F \text{ for every } y \in \Lambda_V\}$. Put $H_V \coloneqq \mathrm{U}(V)$, and let L_V be the stabilizer of Λ_V in $H_V(F)$. We fix the Haar measure $\mathrm{d}h$ on $H_V(F)$ that gives L_V volume 1.

Remark 3.1. We have

(1) There exists an isomorphism $\kappa \colon W_r \to V$ of E-vector spaces satisfying $(\kappa(e_i), \kappa(e_j))_V = 0$, $(\kappa(e_{r+i}), \kappa(e_{r+j}))_V = 0$, and $(\kappa(e_i), \kappa(e_{r+j}))_V = u^{-1}\delta_{ij}$ for $1 \leqslant i, j \leqslant r$, and such that L_V is generated by $\{\kappa(e_i) \mid 1 \leqslant i \leqslant 2r\}$ as an O_E-submodule.

(2) The double coset $K_r\backslash G_r(F)/K_r$ has representatives

$$\begin{pmatrix} u^{a_1} &&&&& \\ & \ddots &&&& \\ && u^{a_r} &&& \\ &&& (-u)^{-a_1} && \\ &&&& \ddots & \\ &&&&& (-u)^{-a_r} \end{pmatrix}$$

where $0 \leqslant a_1 \leqslant \cdots \leqslant a_r$ are integers.

We introduce two Hecke algebras:

$$\mathcal{H}_{W_r} \coloneqq \mathbb{C}[K_r\backslash G_r(F)/K_r], \qquad \mathcal{H}_V \coloneqq \mathbb{C}[L_V\backslash H_V(F)/L_V].$$

Then by the remark above, both $\mathcal{H}_{W_r}$ and $\mathcal{H}_V$ are commutative complex algebras, and are canonically isomorphic to $\mathcal{T}_r \coloneqq \mathbb{C}[T_1^{\pm 1},\dots,T_r^{\pm 1}]^{\{\pm 1\}^r \rtimes \mathfrak{S}_r}$.

Let $(\omega_{W_r,V},\mathcal{V}_{W_r,V})$ be the Weil representation of $G_r(F)\times H_V(F)$ (with respect to the additive character ψ_F and the trivial splitting character). We recall the action under the Schrödinger model $\mathcal{V}_{W_r,V}\simeq C_c^\infty(V^r)$ as follows:

- for $a\in \mathrm{GL}_r(E)$ and $\phi\in C_c^\infty(V^r)$, we have
$$\omega_{W_r,V}\left(\begin{pmatrix} a & \\ & {}^{\mathrm{t}}a^{\mathtt{c},-1}\end{pmatrix}\right)\phi(x) = |\det a|_E^r\cdot\phi(xa);$$
- for $b\in \mathrm{Herm}_r(F)$ and $\phi\in C_c^\infty(V^r)$, we have
$$\omega_{W_r,V}\left(\begin{pmatrix} 1_r & b \\ & 1_r\end{pmatrix}\right)\phi(x) = \psi_F(\mathrm{tr}\, bT(x))\cdot\phi(x)$$
where $T(x)\coloneqq ((x_i,x_j)_V)_{1\leqslant i,j\leqslant r}$ is the moment matrix of $x=(x_1,\dots,x_r)$;
- for $\phi\in C_c^\infty(V^r)$, we have
$$\omega_{W_r,V}\left(\begin{pmatrix} & 1_r \\ -1_r & \end{pmatrix}\right)\phi(x) = \widehat{\phi}(x);$$
- for $h\in H_V(F)$ and $\phi\in C_c^\infty(V^r)$, we have
$$\omega_{W_r,V}(h)\phi(x) = \phi(h^{-1}x).$$

Here, we recall the Fourier transform $C_c^\infty(V^r)\to C_c^\infty(V^r)$ sending ϕ to $\widehat{\phi}$ defined by the formula

$$\widehat{\phi}(x)\coloneqq \int_{V^r}\phi(y)\psi_F\left(\sum_{i=1}^r \mathrm{Tr}_{E/F}(x_i,y_i)_V\right)\mathrm{d}y,$$

where $\mathrm{d}y$ is the self-dual Haar measure on V^r.

Definition 3.2. We define the *spherical module* $\mathcal{S}_{W_r,V}$ to be the subspace of $\mathcal{V}_{W_r,V}$ consisting of elements that are fixed by $K_r\times L_V$, as a module over $\mathcal{H}_{W_r}\otimes_{\mathbb{C}}\mathcal{H}_V$ via the representation $\omega_{W_r,V}$. We denote by $\mathrm{Sph}(V^r)$ the corresponding subspace of $C_c^\infty(V^r)$ under the Schrödinger model.

Lemma 3.3. *The function $\mathbb{1}_{\Lambda_V^r}$ belongs to* $\mathrm{Sph}(V^r)$.

Proof. It suffices to check that

$$\omega_{W_r,V}\left(\begin{pmatrix} & 1_r \\ -1_r & \end{pmatrix}\right)\mathbb{1}_{\Lambda_V^r} = \mathbb{1}_{\Lambda_V^r},$$

which follows from the fact that $\Lambda_V^\vee=\Lambda_V$. The lemma follows. □

Proposition 3.4. *The annihilator of the $\mathcal{H}_{W_r} \otimes_{\mathbb{C}} \mathcal{H}_V$-module $\mathcal{S}_{W_r,V}$ is $\mathcal{I}_{W_r,V}$, where $\mathcal{I}_{W_r,V}$ denotes the diagonal ideal of $\mathcal{H}_{W_r} \otimes_{\mathbb{C}} \mathcal{H}_V$.*

Proof. The same proof of [Liu22, Proposition 4.4] (with $\epsilon = +$ and $d = r$) works in this case as well, using Lemma 3.3. □

In what follows, we review the construction of unramified principal series of $G_r(F)$ and $H_V(F)$.

We identify M_r, the standard diagonal Levi factor of P_r, with $(\mathrm{Res}_{E/F}\,\mathrm{GL}_1)^r$, under which we write an element of $M_r(F)$ as $a = (a_1, \dots, a_r)$ with $a_i \in E^\times$ its eigenvalue on e_i for $1 \leqslant i \leqslant r$. For every tuple $\sigma = (\sigma_1, \dots, \sigma_r) \in (\mathbb{C}/\frac{2\pi i}{\log q}\mathbb{Z})^r$, we define a character χ_r^σ of $M_r(F)$ and hence $P_r(F)$ by the formula

$$\chi_r^\sigma(a) = \prod_{i=1}^r |a_i|_E^{\sigma_i + i - 1/2}.$$

We then have the normalized principal series

$$\mathrm{I}_{W_r}^\sigma \coloneqq \{\varphi \in C^\infty(G_r(F)) \mid \varphi(ag) = \chi_r^\sigma(a)\varphi(g) \text{ for } a \in P_r(F) \text{ and } g \in G_r(F)\},$$

which is an admissible representation of $G_r(F)$ via the right translation. We denote by $\pi_{W_r}^\sigma$ the unique irreducible constituent of $\mathrm{I}_{W_r}^\sigma$ that has nonzero K_r-invariants.

For V, we fix a basis $\{v_r, \dots, v_1, v_{-1}, \dots, v_{-r}\}$ of the O_E-lattice Λ_V, satisfying $(v_i, v_j)_V = u^{-1}\delta_{i,-j}$ for every $1 \leqslant i, j \leqslant r$. We have an increasing filtration

$$\{0\} = Z_{r+1} \subseteq Z_r \subseteq \cdots \subseteq Z_1 \tag{3.1}$$

of isotropic E-subspaces of V where Z_i be the E-subspaces of V spanned by $\{v_r, \dots, v_i\}$. Let Q_V be the (minimal) parabolic subgroup of H_V that stabilizes (3.1). Let M_V be the Levi factor of Q_V stabilizing the lines spanned by v_i for every i. Then we have the canonical isomorphism $M_V = (\mathrm{Res}_{E/F}\,\mathrm{GL}_1)^r$, under which we write an element of $M_V(F)$ as $b = (b_1, \dots, b_r)$ with $b_i \in E^\times$ its eigenvalue on v_i for $1 \leqslant i \leqslant r$. For every tuple $\sigma = (\sigma_1, \dots, \sigma_r) \in (\mathbb{C}/\frac{2\pi i}{\log q}\mathbb{Z})^r$, we define a character χ_V^σ of $M_V(F)$ and hence $Q_V(F)$ by the formula

$$\chi_V^\sigma(b) = \prod_{i=1}^r |b_i|_E^{\sigma_i + i - 1/2}.$$

We then have the normalized principal series

$$\mathrm{I}_V^\sigma \coloneqq \{\varphi \in C^\infty(H_V(F)) \mid \varphi(bh) = \chi_V^\sigma(b)\varphi(h) \text{ for } b \in Q_V(F) \text{ and } h \in H_V(F)\},$$

which is an admissible representation of $H_V(F)$ via the right translation. We denote by π_V^σ the unique irreducible constituent of I_V^σ that has nonzero L_V-invariants.

3.2. **Doubling zeta integral and doubling L-factor.** In this section, we compute certain doubling zeta integrals and doubling L-factors for irreducible admissible representations π of $G_r(F)$ satisfying $\pi^{K_r} \neq \{0\}$. We will freely use notation from [Liu22, Section 5].

We have the degenerate principal series $\mathrm{I}_r^\square(s) \coloneqq \mathrm{Ind}_{P_r^\square}^{G_r^\square}(|\ |_E^s \circ \Delta)$ of $G_r^\square(F)$. Let $\mathfrak{f}_r^{(s)}$ be the unique section of $\mathrm{I}_r^\square(s)$ such that for every $g \in pK_r$ with $p \in P_r^\square(F)$,

$$\mathfrak{f}_r^{(s)}(g) = |\Delta(p)|_E^{s+r}.$$

It is a holomorphic standard, hence good section.

Remark 3.5. By definition, we have $\mathrm{I}_r^\square(s) \subseteq \mathrm{I}_{W_{2r}}^{\sigma_s^\square}$, where

$$\sigma_s^\square := (s+r-\tfrac{1}{2}, s+r-\tfrac{3}{2}, \dots, s-r+\tfrac{3}{2}, s-r+\tfrac{1}{2}) \in (\mathbb{C}/\tfrac{2\pi i}{\log q}\mathbb{Z})^{2r}.$$

Moreover, if we denote by $\varphi^{\sigma_s^\square}$ the unique section in $\mathrm{I}_{W_{2r}}^{\sigma_s^\square}$ that is fixed by K_{2r} and such that $\varphi^{\sigma_s^\square}(1_{4r}) = 1$, then $\mathfrak{f}_r^{(s)} = \varphi^{\sigma_s^\square}$.

Let π be an irreducible admissible representation of $G_r(F)$. For every element $\xi \in \pi^\vee \boxtimes \pi$, we denote by $H_\xi \in C^\infty(G_r(F))$ its associated matrix coefficient. Then for every meromorphic section $f^{(s)}$ of $\mathrm{I}_r^\square(s)$, we have the (doubling) zeta integral:

$$Z(\xi, f^{(s)}) := \int_{G_r(F)} H_\xi(g) f^{(s)}(\mathbf{w}_r(g, 1_{2r}))\,\mathrm{d}g,$$

which is absolutely convergent for $\operatorname{Re} s$ large enough and has a meromorphic continuation. We let $L(s,\pi)$ and $\varepsilon(s,\pi,\psi_F)$ be the doubling L-factor and the doubling epsilon factor of π, respectively, defined in [Yam14, Theorem 5.2].

Take an element $\sigma = (\sigma_1, \dots, \sigma_r) \in (\mathbb{C}/\frac{2\pi i}{\log q}\mathbb{Z})^r$. We define an L-factor

$$L^\sigma(s) := \prod_{i=1}^{r} \frac{1}{(1-q^{\sigma_i - s})(1-q^{-\sigma_i - s})}.$$

Let ξ^σ be a generator of the one dimensional space $((\pi_{W_r}^\sigma)^\vee)^{K_r} \boxtimes (\pi_{W_r}^\sigma)^{K_r}$, which satisfies $H_{\xi^\sigma}(1_{2r}) \neq 0$. We normalize ξ^σ such that $H_{\xi^\sigma}(1_{2r}) = 1$, which makes it unique.

Proposition 3.6. *For $\sigma \in (\mathbb{C}/\frac{2\pi i}{\log q}\mathbb{Z})^r$, we have*

$$Z(\xi^\sigma, \mathfrak{f}_r^{(s)}) = \frac{L^\sigma(s+\frac{1}{2})}{b_{2r}(s)},$$

where $b_{2r}(s) := \prod_{i=1}^r \frac{1}{1-q^{-2s-2i}}$.

Proof. We have an isomorphism $m\colon \operatorname{Res}_{E/F}\mathrm{GL}_r \to M_r^0$ sending a to $\begin{pmatrix} a & \\ & {}^{\mathrm{t}}a^{\mathrm{c},-1}\end{pmatrix}$. Let τ be the unramified constituent of the normalized induction of $\boxtimes_{i=1}^r |\ |_E^{\sigma_i}$, as a representation of $\mathrm{GL}_r(E)$. We fix vectors $v_0 \in \tau$ and $v_0^\vee \in \tau^\vee$ fixed by $M_r^0(F) \cap K_r = m(\mathrm{GL}_r(O_E))$ such that $\langle v_0^\vee, v_0\rangle_\tau = 1$.

By a similar argument in [GPSR87, Section 6] or in the proof of [Liu22, Proposition 5.6], we have

$$Z(\xi^\sigma, \mathfrak{f}_r^{(s)}) = C_{\mathbf{w}'_r}(s) \int_{\mathrm{GL}_r(E)} \varphi^{\mathbf{w}'_r\sigma_s^\square}(\mathbf{w}''_r(m(a), 1_{2r}))|\det a|_E^{-r/2} \langle \tau^\vee(a) v_0^\vee, v_0\rangle_\tau\,\mathrm{d}a, \tag{3.2}$$

where

$$C_{\mathbf{w}'_r}(s) = \prod_{i=1}^r \frac{\zeta_E(2s+2i)}{\zeta_E(2s+r+i)} \prod_{i=1}^r \frac{\zeta_F(2s+2i-1)}{\zeta_F(2s+2i)} = \prod_{i=1}^r \frac{\zeta_E(2s+2i-1)}{\zeta_E(2s+r+i)}.$$

See the proof of [Liu22, Proposition 5.6] for unexplained notation. By [GPSR87, Proposition 6.1], we have

$$\int_{\mathrm{GL}_r(E)} \varphi^{\mathbf{w}'_r\sigma_s^\square}(\mathbf{w}''_r(m(a), 1_{2r}))|\det a|_E^{-r/2} \langle \tau^\vee(a) v_0^\vee, v_0\rangle_\tau\,\mathrm{d}a = \frac{L(s+\frac{1}{2},\tau)L(s+\frac{1}{2},\tau^\vee)}{\prod_{i=1}^r \zeta_E(2s+i)}.$$

Combining with (3.2), we have

$$Z(\xi^\sigma, \mathfrak{f}_r^{(s)}) = \left(\prod_{i=1}^{r} \frac{\zeta_E(2s+2i-1)}{\zeta_E(2s+r+i)}\right) \cdot \left(\frac{L(s+\frac{1}{2},\tau)L(s+\frac{1}{2},\tau^\vee)}{\prod_{i=1}^{r}\zeta_E(2s+i)}\right)$$
$$= \frac{L(s+\frac{1}{2},\tau)L(s+\frac{1}{2},\tau^\vee)}{\prod_{i=1}^{r}\zeta_E(2s+2i)} = \frac{L^\sigma(s+\frac{1}{2})}{b_{2r}(s)}.$$

The proposition is proved. □

Proposition 3.7. *For $\sigma \in (\mathbb{C}/\frac{2\pi i}{\log q}\mathbb{Z})^r$, we have $L(s, \pi^\sigma_{W_r}) = L^\sigma(s)$ and $\varepsilon(s, \pi^\sigma_{W_r}, \psi_F) = 1$.*

Proof. It follows from the same argument for [Yam14, Proposition 7.1], using Proposition 3.6. □

Remark 3.8. It is clear that the base change $\mathrm{BC}(\pi^\sigma_{W_r})$ is well-defined, which is an unramified irreducible admissible representation of $\mathrm{GL}_n(E)$, and we have $L(s, \pi^\sigma_{W_r}) = L(s, \mathrm{BC}(\pi^\sigma_{W_r}))$ by Proposition 3.7.

For an irreducible admissible representation π of $G_r(F)$, let $\Theta(\pi, V)$ be the π-isotypic quotient of $\mathcal{V}_{W_r,V}$, which is an admissible representation of $H_V(F)$, and $\theta(\pi, V)$ its maximal semisimple quotient. By [Wal90], $\theta(\pi, V)$ is either zero, or an irreducible admissible representation of $H_V(F)$, known as the *theta lifting* of π to V (with respect to the additive character ψ_F and the trivial splitting character).

Proposition 3.9. *For an irreducible admissible representation π of $G_r(F)$ of the form $\pi^\sigma_{W_r}$ for an element $\sigma = (\sigma_1, \dots, \sigma_r) \in (i\mathbb{R}/\frac{2\pi i}{\log q}\mathbb{Z})^r$, we have $\theta(\pi, V) \simeq \pi^\sigma_V$.*

Proof. By the same argument in the proof of [Liu22, Theorem 6.2], we have $\Theta(\pi, V)^{L_V} \neq \{0\}$. By our assumption on σ, π is tempered. By (the same argument for) [GI16, Theorem 4.1(v)], $\Theta(\pi, V)$ is a semisimple representation of $H_V(F)$, hence $\Theta(\pi, V) = \theta(\pi, V)$. In particular, we have $\theta(\pi, V)^{L_V} \neq \{0\}$. By Proposition 3.4, the diagonal ideal $\mathcal{I}_{W_r,V}$ annihilates $(\pi^\sigma_{W_r})^{K_r} \boxtimes \theta(\pi, V)^{L_V}$, which implies that $\theta(\pi, V) \simeq \pi^\sigma_V$. □

4. Arithmetic inner product formula

In this section, we collect all local ingredients and deduce our main theorems, following the same line as in [LL21]. In Subsection 4.1 and 4.2, we recall the doubling method and the arithmetic theta lifting from [LL21], respectively. In Subsection 4.3, we prove the vanishing of local indices at split places, by proving the second main ingredient of this article, namely, Theorem 4.21. In Subsection 4.4, we recall the formula for local indices at inert places. In Subsection 4.5, we compute local indices at ramified places, based on the Kudla–Rapoport type formula Theorem 2.7. In Subsection 4.6, we recall the formula for local indices at archimedean places. The deduction of the main results of the article is explained in Subsection 4.7, which is a straightforward modification of [LL21, Section 11].

4.1. Recollection on doubling method.

For the readers' convenience, we copy three groups of notation from [LL21, Section 2] to here. The only difference is item (H5), which reflects the fact that we are able to study certain places in $\mathtt{V}_F^{\mathrm{ram}}$ in the current article.

Notation 4.1. Let E/F be a CM extension of number fields, so that $\mathtt{c}$ is a well-defined element in $\mathrm{Gal}(E/F)$. We continue to fix an embedding $\boldsymbol{\iota}\colon E \hookrightarrow \mathbb{C}$. We denote by $\mathbf{u}$ the (archimedean) place of E induced by $\boldsymbol{\iota}$ and regard E as a subfield of $\mathbb{C}$ via $\boldsymbol{\iota}$.

(F1) We denote by
- $\mathtt{V}_F$ and $\mathtt{V}_F^{\mathrm{fin}}$ the set of all places and non-archimedean places of F, respectively;
- $\mathtt{V}_F^{\mathrm{spl}}$, $\mathtt{V}_F^{\mathrm{int}}$, and $\mathtt{V}_F^{\mathrm{ram}}$ the subsets of $\mathtt{V}_F^{\mathrm{fin}}$ of those that are split, inert, and ramified in E, respectively;
- $\mathtt{V}_F^{(\diamond)}$ the subset of $\mathtt{V}_F$ of places above $\diamond$ for every place $\diamond$ of $\mathbb{Q}$; and
- $\mathtt{V}_E^?$ the places of E above $\mathtt{V}_F^?$.

Moreover,
- for every place $u \in \mathtt{V}_E$ of E, we denote by $\underline{u} \in \mathtt{V}_F$ the underlying place of F;
- for every $v \in \mathtt{V}_F^{\mathrm{fin}}$, we denote by $\mathfrak{p}_v$ the maximal ideal of O_{F_v}, and put $q_v := |O_{F_v}/\mathfrak{p}_v|$;
- for every $v \in \mathtt{V}_F$, we put $E_v := E \otimes_F F_v$ and denote by $|\ |_{E_v} \colon E_v^\times \to \mathbb{C}^\times$ the normalized norm character.

(F2) Let $m \geqslant 0$ be an integer.
- We denote by Herm_m the subscheme of $\mathrm{Res}_{E/F} \mathrm{Mat}_{m,m}$ of m-by-m matrices b satisfying ${}^{\mathrm{t}}b^{\mathtt{c}} = b$. Put $\mathrm{Herm}_m^\circ := \mathrm{Herm}_m \cap \mathrm{Res}_{E/F} \mathrm{GL}_m$.
- For every ordered partition $m = m_1 + \cdots + m_s$ with m_i a positive integer, we denote by $\partial_{m_1,\dots,m_s} \colon \mathrm{Herm}_m \to \mathrm{Herm}_{m_1} \times \cdots \times \mathrm{Herm}_{m_s}$ the morphism that extracts the diagonal blocks with corresponding ranks.
- We denote by $\mathrm{Herm}_m(F)^+$ (resp. $\mathrm{Herm}_m^\circ(F)^+$) the subset of $\mathrm{Herm}_m(F)$ of elements that are totally semi-positive definite (resp. totally positive definite).

(F3) For every $u \in \mathtt{V}_E^{(\infty)}$, we fix an embedding $\iota_u \colon E \hookrightarrow \mathbb{C}$ inducing u (with $\iota_{\mathbf{u}} = \boldsymbol{\iota}$), and identify E_u with $\mathbb{C}$ via ι_u.

(F4) Let $\eta := \eta_{E/F} \colon \mathbb{A}_F^\times \to \mathbb{C}^\times$ be the quadratic character associated to E/F. For every $v \in \mathtt{V}_F$ and every positive integer m, put

$$b_{m,v}(s) := \prod_{i=1}^{m} L(2s+i, \eta_v^{m-i}).$$

Put $b_m(s) := \prod_{v \in \mathtt{V}_F} b_{m,v}(s)$.

(F5) For every element $T \in \mathrm{Herm}_m(\mathbb{A}_F)$, we have the character $\psi_T \colon \mathrm{Herm}_m(\mathbb{A}_F) \to \mathbb{C}^\times$ given by the formula $\psi_T(b) := \psi_F(\mathrm{tr}\, bT)$.

(F6) Let R be a commutative F-algebra. A (skew-)hermitian space over $R \otimes_F E$ is a free $R \otimes_F E$-module V of finite rank, equipped with a (skew-)hermitian form $(\ ,\ )_V$ with respect to the involution $\mathtt{c}$ that is nondegenerate.

Notation 4.2. We fix an even positive integer $n = 2r$. Let $(V, (\ ,\ )_V)$ be a hermitian space over $\mathbb{A}_E$ of rank n that is totally positive definite.

(H1) For every commutative $\mathbb{A}_F$-algebra R and every integer $m \geqslant 0$, we denote by

$$T(x) := \left((x_i, x_j)_V\right)_{i,j} \in \mathrm{Herm}_m(R)$$

the moment matrix of an element $x = (x_1, \dots, x_m) \in V^m \otimes_{\mathbb{A}_F} R$.

(H2) For every $v \in \mathtt{V}_F$, we put $V_v := V \otimes_{\mathbb{A}_F} F_v$, which is a hermitian space over E_v, and define the local Hasse invariant of V_v to be $\epsilon(V_v) := \eta_v((-1)^r \det V_v) \in \{\pm 1\}$ which equals 1 for all but finitely many v. In what follows, we will abbreviate $\epsilon(V_v)$ as ϵ_v. Recall that V is coherent (resp. incoherent) if $\prod_{v \in \mathtt{V}_F} \epsilon_v = 1$ (resp. $\prod_{v \in \mathtt{V}_F} \epsilon_v = -1$).

(H3) Let v be a place of F and $m \geqslant 0$ an integer.

- For $T \in \mathrm{Herm}_m(F_v)$, we put $(V_v^m)_T \coloneqq \{x \in V_v^m \mid T(x) = T\}$, and
$$(V_v^m)_{\mathrm{reg}} \coloneqq \bigcup_{T \in \mathrm{Herm}_m^\circ(F_v)} (V_v^m)_T.$$
- We denote by $\mathscr{S}(V_v^m)$ the space of (complex valued) Bruhat–Schwartz functions on V_v^m. When $v \in \mathtt{V}_F^{(\infty)}$, we have the Gaussian function $\phi_v^0 \in \mathscr{S}(V_v^m)$ given by the formula $\phi_v^0(x) = \mathrm{e}^{-2\pi \operatorname{tr} T(x)}$.
- We have a Fourier transform map $\widehat{\ }\colon \mathscr{S}(V_v^m) \to \mathscr{S}(V_v^m)$ sending ϕ to $\widehat{\phi}$ defined by the formula
$$\widehat{\phi}(x) \coloneqq \int_{V_v^m} \phi(y) \psi_{E,v}\left(\sum_{i=1}^m (x_i, y_i)_V\right) \mathrm{d}y,$$
where $\mathrm{d}y$ is the self-dual Haar measure on V_v^m with respect to $\psi_{E,v}$.
- In what follows, we will always use this self-dual Haar measure on V_v^m.

(H4) Let $m \geqslant 0$ be an integer. For $T \in \mathrm{Herm}_m(F)$, we put
$$\mathrm{Diff}(T, V) \coloneqq \{v \in \mathtt{V}_F \mid (V_v^m)_T = \emptyset\},$$
which is a finite subset of $\mathtt{V}_F \setminus \mathtt{V}_F^{\mathrm{spl}}$.

(H5) Take a nonempty finite subset $\mathtt{R} \subseteq \mathtt{V}_F^{\mathrm{fin}}$ that contains
$$\{v \in \mathtt{V}_F^{\mathrm{ram}} \mid \text{either } \epsilon_v = -1, \text{ or } v \mid 2, \text{ or } v \text{ is ramified over } \mathbb{Q}\}.$$
Let $\mathtt{S}$ be the subset of $\mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{R}$ consisting of v such that $\epsilon_v = -1$, which is contained in $\mathtt{V}_F^{\mathrm{int}}$.

(H6) We fix a $\prod_{v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{R}} O_{E_v}$-lattice $\Lambda^{\mathtt{R}}$ in $V \otimes_{\mathbb{A}_F} \mathbb{A}_F^{\infty,\mathtt{R}}$ such that for every $v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{R}$, $\Lambda_v^{\mathtt{R}}$ is a subgroup of $(\Lambda_v^{\mathtt{R}})^\vee$ of index $q_v^{1-\epsilon_v}$, where
$$(\Lambda_v^{\mathtt{R}})^\vee \coloneqq \{x \in V_v \mid \psi_{E,v}((x,y)_V) = 1 \text{ for every } y \in \Lambda_v^{\mathtt{R}}\}$$
is the $\psi_{E,v}$-dual lattice of $\Lambda_v^{\mathtt{R}}$.

(H7) Put $H \coloneqq \mathrm{U}(V)$, which is a reductive group over $\mathbb{A}_F$.

(H8) Denote by $L^{\mathtt{R}} \subseteq H(\mathbb{A}_F^{\infty,\mathtt{R}})$ the stabilizer of $\Lambda^{\mathtt{R}}$, which is a special maximal subgroup. We have the (abstract) Hecke algebra away from $\mathtt{R}$
$$\mathbb{T}^{\mathtt{R}} \coloneqq \mathbb{Z}[L^{\mathtt{R}} \backslash H(\mathbb{A}_F^{\infty,\mathtt{R}}) / L^{\mathtt{R}}],$$
which is a ring with the unit $\mathbb{1}_{L^{\mathtt{R}}}$, and denote by $\mathbb{S}^{\mathtt{R}}$ the subring
$$\varinjlim_{\substack{\mathtt{T} \subseteq \mathtt{V}_F^{\mathrm{spl}} \setminus \mathtt{R} \\ |\mathtt{T}| < \infty}} \mathbb{Z}[(L^{\mathtt{R}})_{\mathtt{T}} \backslash H(F_{\mathtt{T}}) / (L^{\mathtt{R}})_{\mathtt{T}}] \otimes \mathbb{1}_{(L^{\mathtt{R}})^{\mathtt{T}}}$$
of $\mathbb{T}^{\mathtt{R}}$.

(H9) Suppose that V is *incoherent*, namely, $\prod_{v \in \mathtt{V}_F} \epsilon_v = -1$. For every $u \in \mathtt{V}_E \setminus \mathtt{V}_E^{\mathrm{spl}}$, we fix a *u-nearby space* ${}^uV$ of V, which is a hermitian space over E, and an isomorphism ${}^uV \otimes_F \mathbb{A}_F^{\underline{u}} \simeq V \otimes_{\mathbb{A}_F} \mathbb{A}_F^{\underline{u}}$. More precisely,
- if $u \in \mathtt{V}_E^{(\infty)}$, then ${}^uV$ is the hermitian space over E, unique up to isomorphism, that has signature $(n-1, 1)$ at u and satisfies ${}^uV \otimes_F \mathbb{A}_F^{\underline{u}} \simeq V \otimes_{\mathbb{A}_F} \mathbb{A}_F^{\underline{u}}$;
- if $u \in \mathtt{V}_E^{\mathrm{fin}} \setminus \mathtt{V}_E^{\mathrm{spl}}$, then ${}^uV$ is the hermitian space over E, unique up to isomorphism, that satisfies ${}^uV \otimes_F \mathbb{A}_F^{\underline{u}} \simeq V \otimes_{\mathbb{A}_F} \mathbb{A}_F^{\underline{u}}$.

Put ${}^uH \coloneqq \mathrm{U}({}^uV)$, which is a reductive group over F. Then ${}^uH(\mathbb{A}_F^{\underline{u}})$ and $H(\mathbb{A}_F^{\underline{u}})$ are identified.

Notation 4.3. Let $m \geqslant 0$ be an integer. We equip $W_m = E^{2m}$ and $\bar{W}_m = E^{2m}$ the skew-hermitian forms given by the matrices $\mathtt{w}_m$ and $-\mathtt{w}_m$, respectively.

(G1) Let G_m be the unitary group of both W_m and $\bar{W}_m$. We write elements of W_m and $\bar{W}_m$ in the row form, on which G_m acts from the right.

(G2) We denote by $\{e_1, \dots, e_{2m}\}$ and $\{\bar{e}_1, \dots, \bar{e}_{2m}\}$ the natural bases of W_m and $\bar{W}_m$, respectively.

(G3) Let $P_m \subseteq G_m$ be the parabolic subgroup stabilizing the subspace generated by $\{e_{r+1}, \dots, e_{2m}\}$, and $N_m \subseteq P_m$ its unipotent radical.

(G4) We have

- a homomorphism $m\colon \operatorname{Res}_{E/F} \mathrm{GL}_m \to P_m$ sending a to
$$m(a) \coloneqq \begin{pmatrix} a & \\ & {}^{\mathrm{t}}a^{\mathtt{c},-1} \end{pmatrix},$$
which identifies $\operatorname{Res}_{E/F} \mathrm{GL}_m$ as a Levi factor of P_m.
- a homomorphism $n\colon \mathrm{Herm}_m \to N_m$ sending b to
$$n(b) \coloneqq \begin{pmatrix} 1_m & b \\ & 1_m \end{pmatrix},$$
which is an isomorphism.

(G5) We define a maximal compact subgroup $K_m = \prod_{v \in \mathtt{V}_F} K_{m,v}$ of $G_m(\mathbb{A}_F)$ in the following way:

- for $v \in \mathtt{V}_F^{\mathrm{fin}}$, $K_{m,v}$ is the stabilizer of the lattice $O_{E_v}^{2m}$;
- for $v \in \mathtt{V}_F^{(\infty)}$, $K_{m,v}$ is the subgroup of the form
$$[k_1, k_2] \coloneqq \frac{1}{2} \begin{pmatrix} k_1 + k_2 & -ik_1 + ik_2 \\ ik_1 - ik_2 & k_1 + k_2 \end{pmatrix},$$
in which $k_i \in \mathrm{GL}_m(\mathbb{C})$ satisfies $k_i \, {}^{\mathrm{t}}k_i^{\mathtt{c}} = 1_m$ for $i = 1, 2$. Here, we have identified $G_m(F_v)$ as a subgroup of $\mathrm{GL}_{2m}(\mathbb{C})$ via the embedding ι_u with $v = \underline{u}$ in Notation 4.1(F3).

(G6) For every $v \in \mathtt{V}_F^{(\infty)}$, we have a character $\kappa_{m,v}\colon K_{m,v} \to \mathbb{C}^\times$ that sends $[k_1, k_2]$ to $\det k_1 / \det k_2$.[15]

(G7) For every $v \in \mathtt{V}_F$, we define a Haar measure $\mathrm{d}g_v$ on $G_m(F_v)$ as follows:

- for $v \in \mathtt{V}_F^{\mathrm{fin}}$, $\mathrm{d}g_v$ is the Haar measure under which $K_{m,v}$ has volume 1;
- for $v \in \mathtt{V}_F^{(\infty)}$, $\mathrm{d}g_v$ is the product of the measure on $K_{m,v}$ of total volume 1 and the standard hyperbolic measure on $G_m(F_v)/K_{m,v}$ (see, for example, [EL, Section 2.1]).

Put $\mathrm{d}g = \prod_v \mathrm{d}g_v$, which is a Haar measure on $G_m(\mathbb{A}_F)$.

(G8) We denote by $\mathcal{A}(G_m(F)\backslash G_m(\mathbb{A}_F))$ the space of both $\mathcal{Z}(\mathfrak{g}_{m,\infty})$-finite and $K_{m,\infty}$-finite automorphic forms on $G_m(\mathbb{A}_F)$, where $\mathcal{Z}(\mathfrak{g}_{m,\infty})$ denotes the center of the complexified universal enveloping algebra of the Lie algebra $\mathfrak{g}_{m,\infty}$ of $G_m \otimes_F F_\infty$. We denote by

- $\mathcal{A}^{[r]}(G_m(F)\backslash G_m(\mathbb{A}_F))$ the maximal subspace of $\mathcal{A}(G_m(F)\backslash G_m(\mathbb{A}_F))$ on which for every $v \in \mathtt{V}_F^{(\infty)}$, $K_{m,v}$ acts by the character $\kappa_{m,v}^r$,
- $\mathcal{A}^{[r]\mathtt{R}}(G_m(F)\backslash G_m(\mathbb{A}_F))$ the maximal subspace of $\mathcal{A}^{[r]}(G_m(F)\backslash G_m(\mathbb{A}_F))$ on which
  - for every $v \in \mathtt{V}_F^{\mathrm{fin}} \setminus (\mathtt{R} \cup \mathtt{S})$, $K_{m,v}$ acts trivially; and

[15] In fact, both $K_{m,v}$ and $\kappa_{m,v}$ do not depend on the choice of the embedding ι_u for $v = \underline{u} \in \mathtt{V}_F^{(\infty)}$.

– for every $v \in \mathtt{S}$, the standard Iwahori subgroup $I_{m,v}$ acts trivially and $\mathbb{C}[I_{m,v}\backslash K_{m,v}/I_{m,v}]$ acts by the character $\kappa^{-}_{m,v}$ ([Liu22, Definition 2.1]),

- $\mathcal{A}_{\mathrm{cusp}}(G_m(F)\backslash G_m(\mathbb{A}_F))$ the subspace of $\mathcal{A}(G_m(F)\backslash G_m(\mathbb{A}_F))$ of cusp forms, and by $\langle\ ,\ \rangle_{G_m}$ the hermitian form on $\mathcal{A}_{\mathrm{cusp}}(G_m(F)\backslash G_m(\mathbb{A}_F))$ given by the Petersson inner product with respect to the Haar measure $\mathrm{d}g$.

For a subspace $\mathcal{V}$ of $\mathcal{A}(G_m(F)\backslash G_m(\mathbb{A}_F))$, we denote by

- $\mathcal{V}^{[r]}$ the intersection of $\mathcal{V}$ and $\mathcal{A}^{[r]}(G_m(F)\backslash G_m(\mathbb{A}_F))$,
- $\mathcal{V}^{[r]\mathtt{R}}$ the intersection of $\mathcal{V}$ and $\mathcal{A}^{[r]\mathtt{R}}(G_m(F)\backslash G_m(\mathbb{A}_F))$,
- $\mathcal{V}^{\mathtt{c}}$ the subspace $\{\varphi^{\mathtt{c}} \mid \varphi \in \mathcal{V}\}$.

Assumption 4.4. In what follows, we will consider an irreducible automorphic subrepresentation $(\pi, \mathcal{V}_\pi)$ of $\mathcal{A}_{\mathrm{cusp}}(G_r(F)\backslash G_r(\mathbb{A}_F))$ satisfying that

(1) for every $v \in \mathtt{V}_F^{(\infty)}$, π_v is the (unique up to isomorphism) discrete series representation whose restriction to $K_{r,v}$ contains the character $\kappa^r_{r,v}$;
(2) for every $v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{R}$, π_v is unramified (resp. almost unramified) with respect to $K_{r,v}$ if $\epsilon_v = 1$ (resp. $\epsilon_v = -1$);
(3) for every $v \in \mathtt{V}_F^{\mathrm{fin}}$, π_v is tempered.

We realize the contragredient representation $\pi^\vee$ on $\mathcal{V}_\pi^{\mathtt{c}}$ via the Petersson inner product $\langle\ ,\ \rangle_{G_r}$ (Notation 4.3(G8)). By (1) and (2), we have $\mathcal{V}_\pi^{[r]\mathtt{R}} \neq \{0\}$, where $\mathcal{V}_\pi^{[r]\mathtt{R}}$ is defined in Notation 4.3(G8).

Remark 4.5. By Proposition 4.8(2) below, we know that when $\mathtt{R} \subseteq \mathtt{V}_F^{\mathrm{spl}}$, V coincides with the hermitian space over $\mathbb{A}_E$ of rank n determined by π via local theta dichotomy.

Definition 4.6. We define the L-function for π as the Euler product $L(s,\pi) \coloneqq \prod_v L(s,\pi_v)$ over all places of F, in which

(1) for $v \in \mathtt{V}_F^{\mathrm{fin}}$, $L(s,\pi_v)$ is the doubling L-function defined in [Yam14, Theorem 5.2];
(2) for $v \in \mathtt{V}_F^{(\infty)}$, $L(s,\pi_v)$ is the L-function of the standard base change $\mathrm{BC}(\pi_v)$ of π_v. By Assumption 4.4(1), $\mathrm{BC}(\pi_v)$ is the principal series representation of $\mathrm{GL}_n(\mathbb{C})$ that is the normalized induction of $\arg^{n-1} \boxtimes \arg^{n-3} \boxtimes \cdots \boxtimes \arg^{3-n} \boxtimes \arg^{1-n}$ where $\arg\colon \mathbb{C}^\times \to \mathbb{C}^\times$ is the argument character.

Remark 4.7. Let v be a place of F.

(1) For $v \in \mathtt{V}_F^{(\infty)}$, doubling L-function is only well-defined up to an entire function without zeros. However, one can show that $L(s,\pi_v)$ satisfies the requirement for the doubling L-function in [Yam14, Theorem 5.2].
(2) For $v \in \mathtt{V}_F^{\mathrm{spl}}$, the standard base change $\mathrm{BC}(\pi_v)$ is well-defined and we have $L(s,\pi_v) = L(s,\mathrm{BC}(\pi_v))$ by [Yam14, Theorem 7.2].
(3) For $v \in \mathtt{V}_F^{\mathrm{int}} \backslash \mathtt{R}$, the standard base change $\mathrm{BC}(\pi_v)$ is well-defined and we have $L(s,\pi_v) = L(s,\mathrm{BC}(\pi_v))$ by [Liu22, Remark 1.4].
(4) For $v \in \mathtt{V}_F^{\mathrm{ram}} \backslash \mathtt{R}$, the standard base change $\mathrm{BC}(\pi_v)$ is well-defined and we have $L(s,\pi_v) = L(s,\mathrm{BC}(\pi_v))$ by Remark 3.8.

In particular, when $\mathtt{R} \subseteq \mathtt{V}_F^{\mathrm{spl}}$, we have $L(s,\pi) = \prod_v L(s,\mathrm{BC}(\pi_v))$.

Recall that we have the normalized doubling integral

$$\mathfrak{Z}^\natural_{\pi_v,V_v}\colon \pi_v^\vee \otimes \pi_v \otimes \mathscr{S}(V_v^{2r}) \to \mathbb{C}$$

from [LL21, Section 3].

Proposition 4.8. *Let $(\pi, \mathcal{V}_\pi)$ be as in Assumption 4.4.*

(1) For every $v \in \mathtt{V}_F^{\mathrm{fin}}$, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_r(F_v)\times G_r(F_v)}(\mathrm{I}^{\square}_{r,v}(0), \pi_v \boxtimes \pi_v^\vee) = 1.$$

(2) For every $v \in (\mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{R}) \cup \mathtt{V}_F^{\mathrm{spl}}$, V_v is the unique hermitian space over E_v of rank $2r$, up to isomorphism, such that $\mathfrak{Z}^\natural_{\pi_v,V_v} \neq 0$.

(3) For every $v \in \mathtt{V}_F^{\mathrm{fin}}$, $\mathrm{Hom}_{G_r(F_v)}(\mathscr{S}(V_v^r), \pi_v)$ is irreducible as a representation of $H(F_v)$, and is nonzero if $v \in (\mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{R}) \cup \mathtt{V}_F^{\mathrm{spl}}$.

Proof. This is same as [LL21, Proposition 3.6] except that in (2) we have to take care of the case where $v \in \mathtt{V}_F^{\mathrm{ram}}$, which is a consequence of Proposition 3.9. □

Proposition 4.9. *Let $(\pi, \mathcal{V}_\pi)$ be as in Assumption 4.4 such that $L(\frac{1}{2}, \pi) = 0$. Take*

- *$\varphi_1 = \otimes_v \varphi_{1v} \in \mathcal{V}_\pi^{[r]\mathtt{R}}$ and $\varphi_2 = \otimes_v \varphi_{2v} \in \mathcal{V}_\pi^{[r]\mathtt{R}}$ such that $\langle \varphi_{1v}^{\mathtt{c}}, \varphi_{2v}\rangle_{\pi_v} = 1$ for $v \in \mathtt{V}_F \setminus \mathtt{R}$,[16] and*
- *$\Phi = \otimes_v \Phi_v \in \mathscr{S}(V^{2r})$ such that Φ_v is the Gaussian function (Notation 4.2(H3)) for $v \in \mathtt{V}_F^{(\infty)}$, and $\Phi_v = \mathbb{1}_{(\Lambda_v^{\mathtt{R}})^{2r}}$ for $v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{R}$.*

Then we have

$$\begin{aligned} &\int_{G_r(F)\backslash G_r(\mathbb{A}_F)} \int_{G_r(F)\backslash G_r(\mathbb{A}_F)} \varphi_2(g_2)\varphi_1^{\mathtt{c}}(g_1) E'(0, (g_1, g_2), \Phi)\, \mathrm{d}g_1\, \mathrm{d}g_2 \\ &= \frac{L'(\frac{1}{2}, \pi)}{b_{2r}(0)} \cdot C_r^{[F:\mathbb{Q}]} \cdot \prod_{v \in \mathtt{V}_F^{\mathrm{fin}}} \mathfrak{Z}^\natural_{\pi_v,V_v}(\varphi_{1v}^{\mathtt{c}}, \varphi_{2v}, \Phi_v) \\ &= \frac{L'(\frac{1}{2}, \pi)}{b_{2r}(0)} \cdot C_r^{[F:\mathbb{Q}]} \cdot \prod_{v \in \mathtt{S}} \frac{(-1)^r q_v^{r-1}(q_v+1)}{(q_v^{2r-1}+1)(q_v^{2r}-1)} \cdot \prod_{v \in \mathtt{R}} \mathfrak{Z}^\natural_{\pi_v,V_v}(\varphi_{1v}^{\mathtt{c}}, \varphi_{2v}, \Phi_v), \end{aligned}$$

where

$$C_r := (-1)^r 2^{-2r} \pi^{r^2} \frac{\Gamma(1)\cdots\Gamma(r)}{\Gamma(r+1)\cdots\Gamma(2r)},$$

and the measure on $G_r(\mathbb{A}_F)$ is the one defined in Notation 4.3(G7).

Proof. The proof is same as [LL21, Proposition 3.7], with the additional input

$$\mathfrak{Z}^\natural_{\pi_v,V_v}(\varphi_{1v}^{\mathtt{c}}, \varphi_{2v}, \Phi_v) = 1$$

for $v \in \mathtt{V}_F^{\mathrm{ram}} \setminus \mathtt{R}$ by Proposition 3.6. □

Suppose that V is incoherent. By [Liu11b, Section 2B], we have

(1) Take an element $u \in \mathtt{V}_E \setminus \mathtt{V}_E^{\mathrm{spl}}$, and ${}^u\Phi = \otimes_v {}^u\Phi_v \in \mathscr{S}({}^uV^{2r} \otimes_F \mathbb{A}_F)$, where we recall from Notation 4.2(H9) that ${}^uV$ is the u-nearby hermitian space, such that $\mathrm{supp}({}^u\Phi_v) \subseteq ({}^uV_v^{2r})_{\mathrm{reg}}$ (Notation 4.2(H3)) for v in a nonempty subset $\mathtt{R}' \subseteq \mathtt{R}$. Then for every $g \in P_r^\square(F_{\mathtt{R}'}) G_r^\square(\mathbb{A}_F^{\mathtt{R}'})$, we have

$$E(0, g, {}^u\Phi) = \sum_{T^\square \in \mathrm{Herm}^\circ_{2r}(F)} \prod_{v \in \mathtt{V}_F} W_{T^\square}(0, g_v, {}^u\Phi_v).$$

[16]Strictly speaking, what we fixed is a decomposition $\varphi_1^{\mathtt{c}} = \otimes_v (\varphi_1^{\mathtt{c}})_v$ and have abused notation by writing $\varphi_{1v}^{\mathtt{c}}$ instead of $(\varphi_1^{\mathtt{c}})_v$.

(2) Take $\Phi = \otimes_v \Phi_v \in \mathscr{S}(V^{2r})$ such that $\mathrm{supp}(\Phi_v) \subseteq (V_v^{2r})_{\mathrm{reg}}$ for v in a subset $\mathtt{R}' \subseteq \mathtt{R}$ of cardinality at least 2. Then for every $g \in P_r^\Box(F_{\mathtt{R}'})G_r^\Box(\mathbb{A}_F^{\mathtt{R}'})$, we have

$$E'(0, g, \Phi) = \sum_{w \in \mathtt{V}_F \setminus \mathtt{V}_F^{\mathrm{spl}}} \mathfrak{E}(g, \Phi)_w,$$

where

$$\mathfrak{E}(g, \Phi)_w := \sum_{\substack{T^\Box \in \mathrm{Herm}_{2r}^\circ(F) \\ \mathrm{Diff}(T^\Box, V) = \{w\}}} W'_{T^\Box}(0, g_w, \Phi_w) \prod_{v \in \mathtt{V}_F \setminus \{w\}} W_{T^\Box}(0, g_v, \Phi_v).$$

Here, $\mathrm{Diff}(T^\Box, V)$ is defined in Notation 4.2(H4).

Definition 4.10. Suppose that V is incoherent. Take an element $u \in \mathtt{V}_E \setminus \mathtt{V}_E^{\mathrm{spl}}$, and a pair (T_1, T_2) of elements in $\mathrm{Herm}_r(F)$.

(1) For ${}^u\Phi = \otimes_v {}^u\Phi_v \in \mathscr{S}({}^uV^{2r} \otimes_F \mathbb{A}_F)$, we put

$$E_{T_1,T_2}(g, {}^u\Phi) := \sum_{\substack{T^\Box \in \mathrm{Herm}_{2r}^\circ(F) \\ \partial_{r,r} T^\Box = (T_1, T_2)}} \prod_{v \in \mathtt{V}_F} W_{T^\Box}(0, g_v, {}^u\Phi_v).$$

(2) For $\Phi = \otimes_v \Phi_v \in \mathscr{S}(V^{2r})$, we put

$$\mathfrak{E}_{T_1,T_2}(g, \Phi)_u := \sum_{\substack{T^\Box \in \mathrm{Herm}_{2r}^\circ(F) \\ \mathrm{Diff}(T^\Box, V) = \{\underline{u}\} \\ \partial_{r,r} T^\Box = (T_1, T_2)}} W'_{T^\Box}(0, g_{\underline{u}}, \Phi_{\underline{u}}) \prod_{v \in \mathtt{V}_F \setminus \{\underline{u}\}} W_{T^\Box}(0, g_v, \Phi_v).$$

Here, $\partial_{r,r} \colon \mathrm{Herm}_{2r} \to \mathrm{Herm}_r \times \mathrm{Herm}_r$ is defined in Notation 4.1(F2).

4.2. **Recollection on arithmetic theta lifting.** From this moment, we will assume $F \neq \mathbb{Q}$.

Recall that we have fixed a $\mathbf{u}$-nearby space ${}^{\mathbf{u}}V$ and an isomorphism ${}^{\mathbf{u}}V \otimes_F \mathbb{A}_F^{\mathbf{u}} \simeq V \otimes_{\mathbb{A}_F} \mathbb{A}_F^{\mathbf{u}}$ from Notation 4.2(H9). For every open compact subgroup $L \subseteq H(\mathbb{A}_F^\infty)$, we have the Shimura variety X_L associated to $\mathrm{Res}_{F/\mathbb{Q}}\, {}^{\mathbf{u}}H$ of the level L, which is a smooth quasi-projective scheme over E (which is regarded as a subfield of $\mathbb{C}$ via $\boldsymbol{\iota}$) of dimension $n-1$. We remind the readers its complex uniformization

$$(X_L \otimes_E \mathbb{C})^{\mathrm{an}} \simeq {}^{\mathbf{u}}H(F) \backslash \mathfrak{D} \times H(\mathbb{A}_F)/L,$$

where $\mathfrak{D}$ denotes the complex manifold of negative lines in ${}^{\mathbf{u}}V \otimes_E \mathbb{C}$ and the Deligne homomorphism is the one adopted in [LTXZZ, Section 3.2]. In what follows, for a place $u \in \mathtt{V}_E$, we put $X_{L,u} := X_L \otimes_E E_u$ as a scheme over E_u.

For every $\phi^\infty \in \mathscr{S}(V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)^L$ and $T \in \mathrm{Herm}_m(F)$, we put

$$Z_T(\phi^\infty)_L := \sum_{\substack{x \in L \backslash V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty \\ T(x) = T}} \phi^\infty(x) Z(x)_L,$$

where $Z(x)_L$ is Kudla's special cycle recalled in [LL21, Definition 4.1]. As the above summation is finite, $Z_T(\phi^\infty)_L$ is a well-defined element in $\mathrm{CH}^m(X_L)_{\mathbb{C}}$. For every $g \in G_m(\mathbb{A}_F)$, Kudla's *generating function* is defined to be

$$Z_{\phi^\infty}(g)_L := \sum_{T \in \mathrm{Herm}_m(F)^+} \omega_{m,\infty}(g_\infty)\phi_\infty^0(T) \cdot Z_T(\omega_m^\infty(g^\infty)\phi^\infty)_L$$

as a formal sum valued in $\mathrm{CH}^m(X_L)_{\mathbb{C}}$, where

$$\omega_{m,\infty}(g_\infty)\phi^0_\infty(T) \coloneqq \prod_{v\in\mathtt{V}_F^{(\infty)}} \omega_{m,v}(g_v)\phi^0_v(T).$$

Here, we note that for $v \in \mathtt{V}_F^{(\infty)}$, the function $\omega_{m,v}(g_v)\phi^0_v$ factors through the moment map $V_v^m \to \mathrm{Herm}_m(F_v)$ (see Notation 4.2(H1)).

Hypothesis 4.11 (Modularity of generating functions of codimension m, [LL21, Hypothesis 4.5])**.** *For every open compact subgroup $L \subseteq H(\mathbb{A}_F^\infty)$, every $\phi^\infty \in \mathscr{S}(V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)^L$, and every complex linear map $l\colon \mathrm{CH}^m(X_L)_{\mathbb{C}} \to \mathbb{C}$, the assignment*

$$g \mapsto l(Z_{\phi^\infty}(g)_L)$$

is absolutely convergent, and gives an element in $\mathcal{A}^{[r]}(G_m(F)\backslash G_m(\mathbb{A}_F))$. In other words, the function $Z_{\phi^\infty}(-)_L$ defines an element in $\mathrm{Hom}_{\mathbb{C}}(\mathrm{CH}^m(X_L)^\vee_{\mathbb{C}}, \mathcal{A}^{[r]}(G_m(F)\backslash G_m(\mathbb{A}_F)))$.

Definition 4.12. Let $(\pi, \mathcal{V}_\pi)$ be as in Assumption 4.4. Assume Hypothesis 4.11 on the modularity of generating functions of codimension r. For every $\varphi \in \mathcal{V}_\pi^{[r]}$, every open compact subgroup $L \subseteq H(\mathbb{A}_F^\infty)$, and every $\phi^\infty \in \mathscr{S}(V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)^L$, we put

$$\Theta_{\phi^\infty}(\varphi)_L \coloneqq \int_{G_r(F)\backslash G_r(\mathbb{A}_F)} \varphi^{\mathtt{c}}(g) Z_{\phi^\infty}(g)_L \,\mathrm{d}g,$$

which is an element in $\mathrm{CH}^r(X_L)_{\mathbb{C}}$ by [LL21, Proposition 4.7]. It is clear that the image of $\Theta_{\phi^\infty}(\varphi)_L$ in

$$\mathrm{CH}^r(X)_{\mathbb{C}} \coloneqq \varinjlim_{L} \mathrm{CH}^r(X_L)_{\mathbb{C}}$$

depends only on φ and ϕ^∞, which we denote by $\Theta_{\phi^\infty}(\varphi)$. Finally, we define the *arithmetic theta lifting* of $(\pi, \mathcal{V}_\pi)$ to V (with respect to $\boldsymbol{\iota}$) to be the complex subspace $\Theta(\pi, V)$ of $\mathrm{CH}^r(X)_{\mathbb{C}}$ spanned by $\Theta_{\phi^\infty}(\varphi)$ for all $\varphi \in \mathcal{V}_\pi^{[r]}$ and $\phi^\infty \in \mathscr{S}(V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)$.

We recall Beilinson's height pairing for our particular use from [LL21, Section 6]. We have a map

$$\langle\ ,\ \rangle^\ell_{X_L,E}\colon \mathrm{CH}^r(X_L)^{\langle\ell\rangle}_{\mathbb{C}} \times \mathrm{CH}^r(X_L)^{\langle\ell\rangle}_{\mathbb{C}} \to \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$$

that is complex linear in the first variable, and conjugate symmetric. Here, ℓ is a rational prime such that $X_{L,u}$ has smooth projective reduction for every $u \in \mathtt{V}_E^{(\ell)}$. For a pair (c_1, c_2) of elements in $\mathrm{Z}^r(X_L)^{\langle\ell\rangle}_{\mathbb{C}} \times \mathrm{Z}^r(X_L)^{\langle\ell\rangle}_{\mathbb{C}}$ with disjoint supports, we have

$$\langle c_1, c_2\rangle^\ell_{X_L,E} = \sum_{u\in\mathtt{V}_E^{(\infty)}} 2\langle c_1, c_2\rangle_{X_{L,u},E_u} + \sum_{u\in\mathtt{V}_E^{\mathrm{fin}}} \log q_u \cdot \langle c_1, c_2\rangle^\ell_{X_{L,u},E_u},$$

in which

- q_u is the residue cardinality of E_u for $u \in \mathtt{V}_E^{\mathrm{fin}}$;
- $\langle c_1, c_2\rangle^\ell_{X_{L,u},E_u} \in \mathbb{C}\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$ is the non-archimedean local index recalled in [LL21, Section B] for $u \in \mathtt{V}_E^{\mathrm{fin}}$ (see [LL21, Remark B.11] when u is above ℓ), which equals zero for all but finitely many u;
- $\langle c_1, c_2\rangle_{X_{L,u},E_u} \in \mathbb{C}$ is the archimedean local index for $u \in \mathtt{V}_E^{(\infty)}$, recalled in [LL21, Section 10].

Definition 4.13. We say that a rational prime ℓ is $\mathtt{R}$-*good* if ℓ is unramified in E and satisfies $\mathtt{V}_F^{(\ell)} \subseteq \mathtt{V}_F^{\mathrm{fin}} \setminus (\mathtt{R}\cup\mathtt{S})$.

Definition 4.14. For every open compact subgroup $L_{\mathtt{R}}$ of $H(F_{\mathtt{R}})$ and every subfield $\mathbb{L}$ of $\mathbb{C}$, we define

(1) $(\mathbb{S}^{\mathtt{R}}_{\mathbb{L}})^0_{L_{\mathtt{R}}}$ to be the subalgebra of $\mathbb{S}^{\mathtt{R}}_{\mathbb{L}}$ (Notation 4.2(H8)) of elements that annihilate

$$\bigoplus_{i\neq 2r-1} \mathrm{H}^i_{\mathrm{dR}}(X_{L_{\mathtt{R}}L^{\mathtt{R}}}/E)\otimes_{\mathbb{Q}}\mathbb{L},$$

(2) for every rational prime ℓ, $(\mathbb{S}^{\mathtt{R}}_{\mathbb{L}})^{\langle\ell\rangle}_{L_{\mathtt{R}}}$ to be the subalgebra of $\mathbb{S}^{\mathtt{R}}_{\mathbb{L}}$ of elements that annihilate

$$\bigoplus_{u\in \mathtt{V}^{\mathrm{fin}}_E\setminus\mathtt{V}^{(\ell)}_E} \mathrm{H}^{2r}(X_{L_{\mathtt{R}}L^{\mathtt{R}},u},\mathbb{Q}_\ell(r))\otimes_{\mathbb{Q}}\mathbb{L}.$$

Here, $L^{\mathtt{R}}$ is defined in Notation 4.2(H8).

Definition 4.15. Consider a nonempty subset $\mathtt{R}'\subseteq\mathtt{R}$, an $\mathtt{R}$-good rational prime ℓ, and an open compact subgroup L of $H(\mathbb{A}^\infty_F)$ of the form $L_{\mathtt{R}}L^{\mathtt{R}}$ where $L^{\mathtt{R}}$ is defined in Notation 4.2(H8). An $(\mathtt{R},\mathtt{R}',\ell,L)$-*admissible sextuple* is a sextuple $(\phi^\infty_1,\phi^\infty_2,\mathrm{s}_1,\mathrm{s}_2,g_1,g_2)$ in which

- for $i=1,2$, $\phi^\infty_i=\otimes_v\phi^\infty_{iv}\in\mathscr{S}(V^r\otimes_{\mathbb{A}_F}\mathbb{A}^\infty_F)^L$ in which $\phi^\infty_{iv}=\mathbb{1}_{(\Lambda^{\mathtt{R}}_v)^r}$ for $v\in\mathtt{V}^{\mathrm{fin}}_F\setminus\mathtt{R}$, satisfying that $\mathrm{supp}(\phi^\infty_{1v}\otimes(\phi^\infty_{2v})^{\mathtt{c}})\subseteq(V^{2r}_v)_{\mathrm{reg}}$ for $v\in\mathtt{R}'$;
- for $i=1,2$, s_i is a product of two elements in $(\mathbb{S}^{\mathtt{R}}_{\mathbb{Q}^{\mathrm{ac}}})^{\langle\ell\rangle}_{L_{\mathtt{R}}}$;
- for $i=1,2$, g_i is an element in $G_r(\mathbb{A}^{\mathtt{R}'}_F)$.

For an $(\mathtt{R},\mathtt{R}',\ell,L)$-admissible sextuple $(\phi^\infty_1,\phi^\infty_2,\mathrm{s}_1,\mathrm{s}_2,g_1,g_2)$ and every pair (T_1,T_2) of elements in $\mathrm{Herm}^\circ_r(F)^+$, we define

(1) the global index $I_{T_1,T_2}(\phi^\infty_1,\phi^\infty_2,\mathrm{s}_1,\mathrm{s}_2,g_1,g_2)^\ell_L$ to be

$$\langle\omega_{r,\infty}(g_{1\infty})\phi^0_\infty(T_1)\cdot\mathrm{s}^*_1Z_{T_1}(\omega^\infty_r(g^\infty_1)\phi^\infty_1)_L,\omega_{r,\infty}(g_{2\infty})\phi^0_\infty(T_2)\cdot\mathrm{s}^*_2Z_{T_2}(\omega^\infty_r(g^\infty_2)\phi^\infty_2)_L\rangle^\ell_{X_L,E}$$

as an element in $\mathbb{C}\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$, where we note that for $i=1,2$, $\mathrm{s}^*_iZ_{T_i}(\omega^\infty_r(g^\infty_i)\phi^\infty_i)_L$ belongs to $\mathrm{CH}^r(X_L)^{\langle\ell\rangle}_{\mathbb{C}}$ by Definition 4.14(2);

(2) for every $u\in\mathtt{V}^{\mathrm{fin}}_E$, the local index $I_{T_1,T_2}(\phi^\infty_1,\phi^\infty_2,\mathrm{s}_1,\mathrm{s}_2,g_1,g_2)^\ell_{L,u}$ to be

$$\langle\omega_{r,\infty}(g_{1\infty})\phi^0_\infty(T_1)\cdot\mathrm{s}^*_1Z_{T_1}(\omega^\infty_r(g^\infty_1)\phi^\infty_1)_L,\omega_{r,\infty}(g_{2\infty})\phi^0_\infty(T_2)\cdot\mathrm{s}^*_2Z_{T_2}(\omega^\infty_r(g^\infty_2)\phi^\infty_2)_L\rangle^\ell_{X_{L,u},E_u}$$

as an element in $\mathbb{C}\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$;

(3) for every $u\in\mathtt{V}^{(\infty)}_E$, the local index $I_{T_1,T_2}(\phi^\infty_1,\phi^\infty_2,\mathrm{s}_1,\mathrm{s}_2,g_1,g_2)_{L,u}$ to be

$$\langle\omega_{r,\infty}(g_{1\infty})\phi^0_\infty(T_1)\cdot\mathrm{s}^*_1Z_{T_1}(\omega^\infty_r(g^\infty_1)\phi^\infty_1)_L,\omega_{r,\infty}(g_{2\infty})\phi^0_\infty(T_2)\cdot\mathrm{s}^*_2Z_{T_2}(\omega^\infty_r(g^\infty_2)\phi^\infty_2)_L\rangle_{X_{L,u},E_u}$$

as an element in $\mathbb{C}$.

Let $(\pi,\mathcal{V}_\pi)$ be as in Assumption 4.4, and assume Hypothesis 4.11 on the modularity of generating functions of codimension r.

Remark 4.16. In the situation of Definition 4.12 (and suppose that $F\neq\mathbb{Q}$), suppose that L has the form $L_{\mathtt{R}}L^{\mathtt{R}}$ where $L^{\mathtt{R}}$ is defined in Notation 4.2(H8). We have, from [LL21, Proposition 6.10], that for every elements $\varphi\in\mathcal{V}^{[r]\mathtt{R}}_\pi$ and $\phi^\infty\in\mathscr{S}(V^r\otimes_{\mathbb{A}_F}\mathbb{A}^\infty_F)^L$,

(1) $\mathrm{s}^*\Theta_{\phi^\infty}(\varphi)_L=\chi^{\mathtt{R}}_\pi(\mathrm{s})^{\mathtt{c}}\cdot\Theta_{\phi^\infty}(\varphi)_L$ for every $\mathrm{s}\in\mathbb{S}^{\mathtt{R}}_{\mathbb{Q}^{\mathrm{ac}}}$;
(2) $\Theta_{\phi^\infty}(\varphi)_L\in\mathrm{CH}^r(X_L)^0_{\mathbb{C}}$;
(3) under [LL21, Hypothesis 6.6], $\Theta_{\phi^\infty}(\varphi)_L\in\mathrm{CH}^r(X_L)^{\langle\ell\rangle}_{\mathbb{C}}$ for every $\mathtt{R}$-good rational prime ℓ.

We recall the *normalized height pairing* between the cycles $\Theta_{\phi^\infty}(\varphi)$ in Definition 4.12, under [LL21, Hypothesis 6.6].

Definition 4.17. Under [LL21, Hypothesis 6.6], for every elements $\varphi_1, \varphi_2 \in \mathcal{V}_\pi^{[r]}$ and $\phi_1^\infty, \phi_2^\infty \in \mathscr{S}(V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)$, we define the *normalized height pairing*

$$\langle \Theta_{\phi_1^\infty}(\varphi_1), \Theta_{\phi_2^\infty}(\varphi_2) \rangle_{X,E}^\natural \in \mathbb{C} \otimes_\mathbb{Q} \mathbb{Q}_\ell$$

to be the unique element such that for every $L = L_\mathtt{R} L^\mathtt{R}$ as in Remark 4.16 (with $\mathtt{R}$ possibly enlarged) satisfying $\varphi_1, \varphi_2 \in \mathcal{V}_\pi^{[r]\mathtt{R}}$, $\phi_1^\infty, \phi_2^\infty \in \mathscr{S}(V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)^L$, and that ℓ is $\mathtt{R}$-good, we have

$$\langle \Theta_{\phi_1^\infty}(\varphi_1), \Theta_{\phi_2^\infty}(\varphi_2) \rangle_{X,E}^\natural = \mathrm{vol}^\natural(L) \cdot \langle \Theta_{\phi_1^\infty}(\varphi_1)_L, \Theta_{\phi_2^\infty}(\varphi_2)_L \rangle_{X_L,E}^\ell,$$

where $\mathrm{vol}^\natural(L)$ is introduced in [LL21, Definition 3.8] and $\langle \Theta_{\phi_1^\infty}(\varphi_1)_L, \Theta_{\phi_2^\infty}(\varphi_2)_L \rangle_{X_L,E}^\ell$ is well-defined by Remark 4.16(3). Note that by the projection formula, the right-hand side of the above formula is independent of L.

Finally, we review the auxiliary Shimura variety that will *only* be used in the computation of local indices $I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)_{L,u}$.

Notation 4.18. We denote by T_0 the torus over $\mathbb{Q}$ such that for every commutative $\mathbb{Q}$-algebra R, we have $\mathrm{T}_0(R) = \{a \in E \otimes_\mathbb{Q} R \mid \mathrm{Nm}_{E/F}\, a \in R^\times\}$.

We choose a CM type Φ of E containing $\boldsymbol{\iota}$ and denote by E' the subfield of $\mathbb{C}$ generated by E and the reflex field of Φ. We also choose a skew hermitian space W over E of rank 1, whose group of rational similitude is canonically T_0. For a (sufficiently small) open compact subgroup L_0 of $\mathrm{T}_0(\mathbb{A}^\infty)$, we have the PEL type moduli scheme Y of CM abelian varieties with CM type Φ and level L_0, which is a smooth projective scheme over E' of dimension 0 (see, [Kot92], for example). In what follows, when we invoke this construction, the data Φ, W, and L_0 will be fixed, hence will not be carried into the notation E' and Y. For every open compact subgroup $L \subseteq H(\mathbb{A}_F^\infty)$, we put

$$X'_L \coloneqq X_L \otimes_E Y$$

as a scheme over E'.

The following notation is parallel to [LL21, Notation 5.6].

Notation 4.19. In Subsections 4.3, 4.4, and 4.5, we will consider a place $u \in \mathtt{V}_E^{\mathrm{fin}} \setminus \mathtt{V}_F^\heartsuit$ (Definition 1.1). Let p be the underlying rational prime of u. We will fix an isomorphism $\mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ under which $\boldsymbol{\iota}$ induces the place u. In particular, we may identify Φ as a subset of $\mathrm{Hom}(E, \overline{\mathbb{Q}}_p)$.

We further require that Φ in Notation 4.18 is *admissible* in the following sense: if $\Phi_v \subseteq \Phi$ denotes the subset inducing the place v for every $v \in \mathtt{V}_F^{(p)}$, then it satisfies

(1) when $v \in \mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{spl}}$, Φ_v induces the same place of E above v;
(2) when $v \in \mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{int}}$, Φ_v is the pullback of a CM type of the maximal subfield of E_v unramified over $\mathbb{Q}_p$;
(3) when $v \in \mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{ram}}$, the subfield of $\overline{\mathbb{Q}}_p$ generated by E_u and the reflex field of Φ_v is unramified over E_u.

To release the burden of notation, we denote by K the subfield of $\overline{\mathbb{Q}}_p$ generated by E_u and the reflex field of Φ, by k its residue field, and by $\breve{K}$ the completion of the maximal unramified extension of K in $\overline{\mathbb{Q}}_p$ with the residue field $\overline{\mathbb{F}}_p$. It is clear that admissible CM type always exists for $u \in \mathtt{V}_E^{\mathrm{fin}} \setminus \mathtt{V}_F^\heartsuit$, and that K is unramified over E_u.

We also choose a (sufficiently small) open compact subgroup L_0 of $\mathrm{T}_0(\mathbb{A}^\infty)$ such that $L_{0,p}$ is maximal compact. We denote by $\mathcal{Y}$ the integral model of Y over O_K such that for every $S \in \mathrm{Sch}'_{/O_K}$, $\mathcal{Y}(S)$ is the set of equivalence classes of quadruples $(A_0, \iota_{A_0}, \lambda_{A_0}, \eta^p_{A_0})$ where

- $(A_0, \iota_{A_0}, \lambda_{A_0})$ is a unitary O_E-abelian scheme over S of signature type Φ (see [LTXZZ, Definition 3.4.2 & Definition 3.4.3])[17] such that λ_{A_0} is p-principal;
- $\eta^p_{A_0}$ is an L^p_0-level structure (see [LTXZZ, Definition 4.2.2] for more details).

By [How12, Proposition 3.1.2], $\mathcal{Y}$ is finite and étale over O_K.

4.3. **Local indices at split places.** In this subsection, we compute local indices at almost all places in $\mathtt{V}_E^{\mathrm{spl}}$. Our goal is to prove the following proposition.

Proposition 4.20. *Let $\mathtt{R}$, $\mathtt{R}'$, ℓ, and L be as in Definition 4.15 such that the cardinality of $\mathtt{R}'$ is at least 2. Let $(\pi, \mathcal{V}_\pi)$ be as in Assumption 4.4. For every $u \in \mathtt{V}_E^{\mathrm{spl}}$ satisfying $\underline{u} \not\in \mathtt{R} \setminus \mathtt{V}_F^{\heartsuit}$ and $\mathtt{V}_F^{(p)} \cap \mathtt{R} \subseteq \mathtt{V}_F^{\mathrm{spl}}$ where p is the underlying rational prime of u, there exist elements $\mathrm{s}_1^u, \mathrm{s}_2^u \in \mathbb{S}^{\mathtt{R}}_{\mathbb{Q}^{\mathrm{ac}}} \setminus \mathfrak{m}^{\mathtt{R}}_\pi$ such that*

$$I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1^u\mathrm{s}_1, \mathrm{s}_2^u\mathrm{s}_2, g_1, g_2)^\ell_{L,u} = 0$$

for every $(\mathtt{R}, \mathtt{R}', \ell, L)$-admissible sextuple $(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)$ and every pair (T_1, T_2) in $\mathrm{Herm}^\circ_r(F)^+$. Moreover, we may take $\mathrm{s}_1^u = \mathrm{s}_2^u = 1$ if $\underline{u} \not\in \mathtt{R}$.

Proof. This is simply [LL21, Proposition 7.1] but without the assumption that $\pi_{\underline{u}}$ is a (tempered) principal series and without relying on [LL21, Hypothesis 6.6]. The proof is same, after we slightly generalize the construction of the integral model $\mathcal{X}_m$ to take care of places in $\mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{ram}}$, and use Theorem 4.21 below which generalizes [LL21, Lemma 7.3]. □

From now to the end of this section, we assume $\mathtt{V}_F^{(p)} \cap \mathtt{R} \subseteq \mathtt{V}_F^{\mathrm{spl}}$. We also assume $\underline{u} \in \mathtt{V}_F^{\heartsuit}$ and when we need $m \geqslant 1$ below. We invoke Notation 4.18 together with Notation 4.19. The isomorphism $\mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ in Notation 4.19 identifies $\mathrm{Hom}(E, \mathbb{C})$ with $\mathrm{Hom}(E, \mathbb{C}_p)$. For every $v \in \mathtt{V}_F^{(p)}$, let Φ_v be the subset of Φ, regarded as a subset of $\mathrm{Hom}(E, \mathbb{C}_p)$, of elements that induce the place v of F.

For every integer $m \geqslant 0$, we define a moduli functor $\mathcal{X}_m$ over O_K as follows: For every $S \in \mathrm{Sch}'_{/O_K}$, $\mathcal{X}_m(S)$ is the set of equivalence classes of tuples

$$(A_0, \iota_{A_0}, \lambda_{A_0}, \eta^p_{A_0}; A, \iota_A, \lambda_A, \eta^p_A, \{\eta_{A,v}\}_{v \in \mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{spl}} \setminus \{\underline{u}\}}, \eta_{A,u,m})$$

where

- $(A_0, \iota_{A_0}, \lambda_{A_0}, \eta^p_{A_0})$ is an element in $\mathcal{Y}(S)$;
- (A, ι_A, λ_A) is a unitary O_E-abelian scheme of signature type $n\Phi - \iota_w + \iota_w^{\mathtt{c}}$ over S, such that
  - for every $v \in \mathtt{V}_F^{(p)} \setminus \mathtt{V}_F^{\mathrm{ram}}$, $\lambda_A[v^\infty]$ is an isogeny whose kernel has order $q_v^{1-\epsilon_v}$;
  - $\mathrm{Lie}(A[u^{\mathtt{c},\infty}])$ is of rank 1 on which the action of O_E is given by the embedding $\iota_w^{\mathtt{c}}$;
  - for every $v \in \mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{ram}}$, the triple $(A_0[v^\infty], \iota_{A_0}[v^\infty], \lambda_{A_0}[v^\infty]) \otimes_{O_K} O_{\breve{K}}$ is an object of $\mathrm{Exo}^{\Phi_v}_{(n,0)}(S \otimes_{O_K} O_{\breve{K}})$ (Remark 2.67, with $E = E_v$, $F = F_v$, and $\breve{E} = \breve{K}$);[18]
- η^p_A is an L^p-level structure;
- for every $v \in \mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{spl}} \setminus \{\underline{u}\}$, $\eta_{A,v}$ is an L_v-level structure;

[17] Here, our notation on objects is slightly different from [LTXZZ] or [LL21] as we, in particular, retrieve the O_E-action ι_{A_0}.

[18] The sign condition is redundant in our case by [RSZ20, Remark 5.1(i)].

- $\eta_{A,u,m}$ is a Drinfeld level-m structure.

See [LL21, Section 7] for more details for the last three items. By [RSZ20, Theorem 4.5], for every $m \geqslant 0$, $\mathcal{X}_m$ is a regular scheme, flat (smooth, if $m = 0$) and projective over O_K, and admits a canonical isomorphism

$$\mathcal{X}_m \otimes_{O_K} K \simeq X'_{L_{\underline{u},m}L^{\underline{u}}} \otimes_{E'} K$$

of schemes over K. Note that for every integer $m \geqslant 0$, $\mathbb{S}^{\mathtt{R}\cup\mathtt{V}_F^{(p)}}$ naturally gives a ring of étale correspondences of $\mathcal{X}_m$.[19]

The following theorem confirms the conjecture proposed in [LL21, Remark 7.4], and the rest of this subsection will be devoted to its proof. It is worth mentioning that even in the situation of [LL21, Lemma 7.3], the argument below is slightly improved so that [LL21, Hypothesis 6.6] is not relied on anymore.

Theorem 4.21. *Let the situation be as in Proposition 4.20 and assume $\underline{u} \in \mathtt{V}_F^{\heartsuit}$ and $p \neq \ell$. For every integer $m \geqslant 0$,*

$$\left(\mathrm{H}^{2r}(\mathcal{X}_m, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{Q}^{\mathrm{ac}}\right)_{\mathfrak{m}} = 0$$

holds, where $\mathfrak{m} \coloneqq \mathfrak{m}_\pi^{\mathtt{R}} \cap \mathbb{S}_{\mathbb{Q}^{\mathrm{ac}}}^{\mathtt{R}\cup\mathtt{V}_F^{(p)}}$.

We temporarily allow n to be an arbitrary positive integer, not necessarily even. Put $Y_m \coloneqq \mathcal{X}_m \otimes_{O_K} k$. For every point $x \in Y_m(\overline{\mathbb{F}}_p)$, we know that $A_x[u^{\mathtt{c},\infty}]$ is a one-dimensional $O_{F_{\underline{u}}}$-divisible group of (relative) height n, and we let $0 \leqslant h(x) \leqslant n-1$ be the height of its étale part. For $0 \leqslant h \leqslant n-1$, let $Y_m^{[h]}$ be locus where $h(x) \leqslant h$, which is Zariski closed, hence will be endowed with the reduced induced scheme structure, and put $Y_m^{(h)} \coloneqq Y_m^{[h]} - Y_m^{[h-1]}$ ($Y_m^{[-1]} = \emptyset$). It is known that $Y_m^{(h)}$ is smooth over k of pure dimension h.

Now we suppose that $m \geqslant 1$. Let $\mathfrak{S}_m^h$ be the set of free $O_{F_{\underline{u}}}/\mathfrak{p}_{\underline{u}}^m$-submodules of $(\mathfrak{p}_{\underline{u}}^{-m}/O_{F_{\underline{u}}})^n$ of rank $n-h$, and put $\mathfrak{S}_m \coloneqq \bigcup_{h=0}^{n-1} \mathfrak{S}_m^h$. For every $M \in \mathfrak{S}_m^h$, we denote by $Y_m^{(M)} \subseteq Y_m^{(h)}$ the (open and closed) locus where the kernel of the Drinfeld level-m structure is M. Then we have

$$Y_m^{(h)} = \coprod_{M \in \mathfrak{S}_m^h} Y_m^{(M)}$$

for every $0 \leqslant h \leqslant n-1$. Let $Y_m^{[M]}$ be the scheme-theoretic closure of $Y_m^{(M)}$ inside Y_m. Then we have

$$Y_m^{[M]} = \bigcup_{\substack{M' \in \mathfrak{S}_m \\ M \subseteq M'}} Y_m^{(M')} \tag{4.1}$$

as a disjoint union of strata. Note that Hecke operators away from $\underline{u}$ (of level $L^{\underline{u}}$) preserve $Y_m^{(M)}$ and hence $Y_m^{[M]}$ for every $M \in \mathfrak{S}_m$.

We need some general notation. For a sequence $(g_1, \dots, g_t)$ of nonnegative integers with $g = g_1 + \cdots + g_t$, we denote by $\mathrm{P}_{g_1,\dots,g_t}$ the standard upper triangular parabolic subgroup of GL_g of block sizes $g_1, \dots, g_t$, and $\mathrm{M}_{g_1,\dots,g_t}$ its standard diagonal Levi subgroup. Moreover, we denote by $C_m^{g_1,\dots,g_t}$ the cardinality of

$$\mathrm{GL}_g(O_{F_{\underline{u}}}/\mathfrak{p}_{\underline{u}}^m)/\mathrm{P}_{g_1,\dots,g_t}(O_{F_{\underline{u}}}/\mathfrak{p}_{\underline{u}}^m),$$

[19] When $m = 0$, we do not need $\underline{u} \in \mathtt{V}_F^{\heartsuit}$ as the same holds even when K is ramified over E_u.

which depends only on the partition $g=g_1+\cdots+g_t$. We also put

$$L^g_{\underline{u},m}\coloneqq \ker\left(\mathrm{GL}_g(O_{F_{\underline{u}}})\to \mathrm{GL}_g(O_{F_{\underline{u}}}/\mathfrak{p}^m_{\underline{u}})\right).$$

Lemma 4.22. *For $(g_1,\dots,g_t)$ with $g=g_1+\cdots+g_t$ as above and another integer $g'\geqslant g$, we have*

$$C^{g'-g,g}_m C^{g_1,\dots,g_t}_m = C^{g'-g+g_1,g_2,\dots,g_t}_m.$$

Proof. It follows from the isomorphism

$$\mathrm{P}_{g'-g,g}(O_{F_{\underline{u}}}/\mathfrak{p}^m_{\underline{u}})/\mathrm{P}_{g'-g+g_1,g_2,\dots,g_t}(O_{F_{\underline{u}}}/\mathfrak{p}^m_{\underline{u}})\simeq \mathrm{GL}_g(O_{F_{\underline{u}}}/\mathfrak{p}^m_{\underline{u}})/\mathrm{P}_{g_1,\dots,g_t}(O_{F_{\underline{u}}}/\mathfrak{p}^m_{\underline{u}}).$$

□

Lemma 4.23. *Suppose that $m\geqslant 1$. Take a sequence $(g_1,\dots,g_t)$ of nonnegative integers with $g=g_1+\cdots+g_t$. Let $\pi_1\boxtimes\cdots\boxtimes\pi_t$ be an admissible representation of $\mathrm{M}_{g_1,\dots,g_t}(F_{\underline{u}})$. Then we have*

$$\dim\left(\mathrm{Ind}^{\mathrm{GL}_g(F_{\underline{u}})}_{\mathrm{P}_{g_1,\dots,g_t}(F_{\underline{u}})}\pi_1\boxtimes\cdots\boxtimes\pi_t\right)^{L^g_{\underline{u},m}} = C^{g_1,\dots,g_t}_m\prod_{i=1}^t\dim\pi_i^{L^{g_i}_{\underline{u},m}}.$$

Proof. Pick a set X of representatives of the double coset

$$\mathrm{P}_{g_1,\dots,g_t}(F_{\underline{u}})\backslash\mathrm{GL}_g(F_{\underline{u}})/L^g_{\underline{u},m}$$

contained in $\mathrm{GL}_g(O_{F_{\underline{u}}})$, which is possible by the Iwasawa decomposition. Then an element

$$f\in\left(\mathrm{Ind}^{\mathrm{GL}_g(F_{\underline{u}})}_{\mathrm{P}_{g_1,\dots,g_t}(F_{\underline{u}})}\pi_1\boxtimes\cdots\boxtimes\pi_t\right)^{L^g_{\underline{u},m}}$$

is determined by $f|_X$. Since $\mathrm{GL}_g(O_{F_{\underline{u}}})$ normalizes $L^g_{\underline{u},m}$, a function f' on X is of the form $f'=f|_X$ if and only if f' takes values in $\bigotimes_{i=1}^t\pi_i^{L^{g_i}_{\underline{u},m}}$. As $|X|=C^{g_1,\dots,g_t}_m$, the lemma follows. □

For an irreducible supercuspidal representation π of $\mathrm{GL}_g(F_{\underline{u}})$ and a positive integer s, we have the representation $\mathrm{Sp}_s(\pi)$ of $\mathrm{GL}_{sg}(F_{\underline{u}})$ defined in [HT01, Section I.3]. In particular, when ϕ is an unramified character of $F^\times_{\underline{u}}$, $\mathrm{Sp}_s(\phi)$ is the Steinberg representation of $\mathrm{GL}_s(F_{\underline{u}})$ twisted by $\phi|\ |_{\underline{u}}^{\frac{s-1}{2}}$.

Lemma 4.24. *Suppose that $m\geqslant 1$. For every positive integer g and every unramified character ϕ of $F^\times_{\underline{u}}$, we have*

$$\sum_{h=0}^g(-1)^h C^{g-h,h}_m\dim\mathrm{Sp}_h(\phi)^{L^h_{\underline{u},m}}=0.$$

Proof. We claim the identity

$$\sum_{h=0}^g(-1)^h\left[\mathrm{Ind}^{\mathrm{GL}_g(F_{\underline{u}})}_{\mathrm{P}_{h,g-h}(F_{\underline{u}})}\mathrm{Sp}_h(\phi)\boxtimes\left(\phi|\ |_{\underline{u}}^{\frac{g+h-1}{2}}\circ\det{}_{g-h}\right)\right]=0 \tag{4.2}$$

in $\mathrm{Groth}(\mathrm{GL}_g(F_{\underline{u}}))$. Assuming it, we have

$$\sum_{h=0}^g(-1)^h\dim\left(\mathrm{Ind}^{\mathrm{GL}_g(F_{\underline{u}})}_{\mathrm{P}_{h,g-h}(F_{\underline{u}})}\mathrm{Sp}_h(\phi)\boxtimes\left(\phi|\ |_{\underline{u}}^{\frac{g+h-1}{2}}\circ\det{}_{g-h}\right)\right)^{L^g_{\underline{u},m}}=0.$$

By Lemma 4.23, the lemma follows.

For the claim, put

$$\mathrm{I}(\phi) := \mathrm{Ind}_{\mathrm{P}_{1,\dots,1}(F_{\underline{u}})}^{\mathrm{GL}_g(F_{\underline{u}})} \phi \boxtimes \phi|\ |_{\underline{u}} \boxtimes \cdots \boxtimes \phi|\ |_{\underline{u}}^{g-1}.$$

By the transitivity of (normalized) parabolic induction, every irreducible constituent of

$$\mathrm{I}(\phi)^{h,g-h} := \mathrm{Ind}_{\mathrm{P}_{h,g-h}(F_{\underline{u}})}^{\mathrm{GL}_g(F_{\underline{u}})} \mathrm{Sp}_h(\phi) \boxtimes \left(\phi|\ |_{\underline{u}}^{\frac{g+h-1}{2}} \circ \det{}_{g-h}\right)$$

is a constituent of $\mathrm{I}(\phi)$. By [Zel80], there is a bijection between the set of irreducible subquotients of $\mathrm{I}(\phi)$ and the set of sequences of signs of length $g-1$. For such a sequence σ, we denote by $\mathrm{I}(\phi)_\sigma$ the corresponding irreducible subquotient. For $0 \leqslant h \leqslant g-1$, we denote by $\sigma(i)$ the sequence starting from h negative signs followed by $g-1-h$ positive signs. In particular,

$$\mathrm{I}(\phi)_{\sigma(g-1)} = \mathrm{Sp}_g(\phi) = \mathrm{I}(\phi)^{g,0}, \qquad \mathrm{I}(\phi)_{\sigma(0)} = \phi|\ |_{\underline{u}}^{\frac{2g-1}{2}} \circ \det{}_g = \mathrm{I}(\phi)^{0,g}.$$

By [HT01, Lemma I.3.2], we have

$$[\mathrm{I}(\phi)^{h,g-h}] = [\mathrm{I}(\phi)_{\sigma(h)}] + [\mathrm{I}(\phi)_{\sigma(h-1)}]$$

in $\mathrm{Groth}(\mathrm{GL}_g(F_{\underline{u}}))$ for $0 < h < g$. Thus, (4.2) follows. □

Proposition 4.25. *Fix an isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. Suppose that $m \geqslant 1$. For every $0 \leqslant h \leqslant n-1$ and $M \in \mathfrak{S}_m^h$, we have*

$$\mathrm{H}^j(Y_m^{[M]} \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell)_{\mathfrak{m}} = 0$$

for every $j \neq h$.

This is an extension of [TY07, Proposition 4.4]. However, we allow arbitrary principal level structure at $\underline{u}$ and our case involves endoscopy.

Proof. In what follows, h will always denote an integer satisfying $0 \leqslant h \leqslant n-1$. Denote by D_{n-h} the division algebra over $F_{\underline{u}}$ of Hasse invariant $\frac{1}{n-h}$, with the maximal order $O_{\mathrm{D}_{n-h}}$.

For a $\mathfrak{T}$-scheme Y of finite type over k, and a (finite) character $\chi\colon \mathrm{T}_0(\mathbb{Q})\backslash\mathrm{T}_0(\mathbb{A}^\infty)/L_0 \to \overline{\mathbb{Q}}_\ell^\times$, we put

$$[\mathrm{H}_{?,\chi}(Y, \overline{\mathbb{Q}}_\ell)] := \sum_{j\in\mathbb{Z}} (-1)^j \mathrm{H}^j_?(Y \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell)[\chi]$$

as an element in $\mathrm{Groth}(\mathrm{Gal}(\overline{\mathbb{F}}_p/k))$ for $? \in \{\ , c\}$.

Let I_m^h be the Igusa variety (of the first kind) introduced in [HT01, Section IV.1] so that I_m^h is isomorphic to $Y_m^{(M)}$ for every $M \in \mathfrak{S}_m^h$ as schemes over k (but not as schemes over $Y_0^{(h)}$). Combining with (4.1), we obtain the identity

$$\begin{aligned}
(4.3)\qquad [\mathrm{H}_\chi(Y_m^{[M]}, \overline{\mathbb{Q}}_\ell)] &= \sum_{h'=0}^{h} \sum_{\substack{M'\in\mathfrak{S}_m^{h'} \\ M\subseteq M'}} (-1)^{h-h'} [\mathrm{H}_{c,\chi}(Y_m^{(M')}, \overline{\mathbb{Q}}_\ell)] \\
&= \sum_{h'=0}^{h} (-1)^{h-h'} \cdot \left|\{M' \in \mathfrak{S}_m^{h'} \mid M \subseteq M'\}\right| \cdot [\mathrm{H}_{c,\chi}(I_m^{h'}, \overline{\mathbb{Q}}_\ell)] \\
&= \sum_{h'=0}^{h} (-1)^{h-h'} C_m^{h-h',h'} \cdot [\mathrm{H}_{c,\chi}(I_m^{h'}, \overline{\mathbb{Q}}_\ell)]
\end{aligned}$$

in $\mathrm{Groth}(\mathrm{Gal}(\overline{\mathbb{F}}_p/k))$.

Now to compute $[\mathrm{H}_\chi(I^{h'}_m,\overline{\mathbb{Q}}_\ell)]$, we use [CS17, Lemma 5.5.1] in which the corresponding $J_b(\mathbb{Q}_p)$ is $\mathrm{D}_{n-h'}\times\mathrm{GL}_{h'}(F_{\underline{u}})$, and we take $\phi=\phi^{\underline{u}}\phi_{\underline{u}}$ where $\phi^{\underline{u}}$ is the characteristic function of $L^{\underline{u}}$ and $\phi_{\underline{u}}$ is the characteristic function of $O^\times_{\mathrm{D}_{n-h'}}\times L^{h'}_{\underline{u},m}$. Then we have the identity

$$[\mathrm{H}_{c,\chi}(I^{h'}_m,\overline{\mathbb{Q}}_\ell)]=\sum_{\boldsymbol{n}}\sum_{\Pi^{\boldsymbol{n}}}c(\boldsymbol{n},\Pi^{\boldsymbol{n}})\cdot\mathrm{Red}^{h'}_{\boldsymbol{n}}(\pi^{\boldsymbol{n}}_{\underline{u}})^{O^\times_{\mathrm{D}_{n-h'}}\times L^{h'}_{\underline{u},m}}\tag{4.4}$$

in $\mathrm{Groth}(\mathrm{D}^\times_{n-h'}/O^\times_{\mathrm{D}_{n-h'}})$, where

- $\boldsymbol{n}$ runs through *ordered* pairs (n_1,n_2) of nonnegative integers such that $n_1+n_2=n$, which gives an elliptic endoscopic group $G_{\boldsymbol{n}}$ of $\mathrm{U}(^{\mathbf{u}}V)$;
- $\Pi^{\boldsymbol{n}}$ runs through a finite set of certain isobaric irreducible cohomological (with respect to the trivial algebraic representation) automorphic representations of $\mathbb{G}_{\boldsymbol{n}}(\mathbb{A}_F)$, with $\pi^{\boldsymbol{n}}_{\underline{u}}$ the descent of $\Pi^{\boldsymbol{n}}_{\underline{u}}$ to $G_{\boldsymbol{n}}(F_{\underline{u}})\simeq\mathrm{M}_{n_1,n_2}(F_{\underline{u}})$;
- $c(\boldsymbol{n},\Pi^{\boldsymbol{n}})$ is a constant depending only on $\boldsymbol{n}$ and $\Pi^{\boldsymbol{n}}$ but not on h';
- $\mathrm{Red}^{h'}_{\boldsymbol{n}}\colon\mathrm{Groth}(\mathrm{M}_{n_1,n_2}(F_{\underline{u}}))\to\mathrm{Groth}(\mathrm{D}^\times_{n-h'}\times\mathrm{GL}_{h'}(F_{\underline{u}}))$ is the zero map if $h'<n_2$, and otherwise is the composition of
  - the normalized Jacquet functor
  $$\mathrm{Groth}(\mathrm{M}_{n_1,n_2}(F_{\underline{u}}))\to\mathrm{Groth}(\mathrm{M}_{n-h',h'-n_2,n_2}(F_{\underline{u}})),$$
  - the normalized parabolic induction
  $$\mathrm{Groth}(\mathrm{M}_{n-h',h'-n_2,n_2}(F_{\underline{u}}))\to\mathrm{Groth}(\mathrm{M}_{n-h',h'}(F_{\underline{u}})),$$
  - the Langlands–Jacquet map (on the first factor)
  $$\mathrm{Groth}(\mathrm{M}_{n-h',h'}(F_{\underline{u}}))\to\mathrm{Groth}(\mathrm{D}^\times_{n-h'}\times\mathrm{GL}_{h'}(F_{\underline{u}})).$$

The image of $[\mathrm{H}_{c,\chi}(I^{h'}_m,\overline{\mathbb{Q}}_\ell)]$ in $\mathrm{Groth}(\mathrm{Gal}(\overline{\mathbb{F}}_p/k))$ is given by the map

$$\mathrm{Groth}(\mathrm{D}^\times_{n-h'}/O^\times_{\mathrm{D}_{n-h'}})\to\mathrm{Groth}(\mathrm{Gal}(\overline{\mathbb{F}}_p/k))$$

sending an (unramified) character $\phi\circ\mathrm{Nm}_{\mathrm{D}^\times_{n-h'}}$ to $\mathrm{rec}(\phi^{-1})\cdot\breve{\chi}$, where $\breve{\chi}$ is a finite character of $\mathrm{Gal}(\overline{\mathbb{F}}_p/k)$ determined by χ. In what follows, we will regard

$$\mathrm{Red}^{h'}_{\boldsymbol{n}}(\pi^{\boldsymbol{n}}_{\underline{u}})^{O^\times_{\mathrm{D}_{n-h'}}\times L^{h'}_{\underline{u},m}}$$

as an element of $\mathrm{Groth}(\mathrm{Gal}(\overline{\mathbb{F}}_p/k))$ via the above map.

Now let us compute for each $\boldsymbol{n}=(n_1,n_2)$,

$$\sum_{h'=0}^{h}(-1)^{h-h'}C^{h-h',h'}_m\cdot\mathrm{Red}^{h'}_{\boldsymbol{n}}(\pi^{\boldsymbol{n}}_{\underline{u}})^{O^\times_{\mathrm{D}_{n-h'}}\times L^{h'}_{\underline{u},m}}\tag{4.5}$$

in $\mathrm{Groth}(\mathrm{Gal}(\overline{\mathbb{F}}_p/k))$, when $\pi^{\boldsymbol{n}}_{\underline{u}}$ is tempered. Write $\pi^{\boldsymbol{n}}_{\underline{u}}=\pi^1\boxtimes\pi^2$ where π^α is an tempered irreducible admissible representation of $\mathrm{GL}_{n_\alpha}(F_{\underline{u}})$. In particular, π^1 is a full induction of the form

$$\mathrm{Ind}^{\mathrm{GL}_{n_1}(F_{\underline{u}})}_{\mathrm{P}_{s_1g_1,\dots,s_tg_t}(F_{\underline{u}})}\mathrm{Sp}_{s_1}(\pi^1_1)\boxtimes\cdots\boxtimes\mathrm{Sp}_{s_t}(\pi^1_t),$$

where $s_1,\dots,s_t$ and $g_1,\dots,g_t$ are positive integers satisfying $s_1g_1+\cdots+s_tg_t=n_1$; and for $1\leqslant i\leqslant t$, π^1_i is an irreducible supercuspidal representation of $\mathrm{GL}_{g_i}(F_{\underline{u}})$ such that $\mathrm{Sp}_{s_i}(\pi^1_i)$

is unitary. Let $\mathbb{I}$ be the subset of $\{1,\dots,t\}$ such that π^1_i is an unramified character (hence $g_i=1$) and $s_i\geqslant n-h$. Then we have for $h'\geqslant n_2$,

$$\begin{aligned}(4.6)\qquad &\mathrm{Red}^{h'}_{\boldsymbol{n}}(\pi^{\boldsymbol{n}}_{\underline{u}})^{O^\times_{\mathrm{D}_{n-h'}}\times L^{h'}_{\underline{u},m}}\\ &=\sum_{\substack{i\in\mathbb{I}\\ s_i\geqslant n-h'}}\dim\left(\mathrm{Ind}^{\mathrm{GL}_{h'}(F_{\underline{u}})}_{\mathrm{P}_?(F_{\underline{u}})}\mathrm{Sp}_{s_i+h'-n}(\pi^1_i)\boxtimes\left(\boxtimes_{j\neq i}\mathrm{Sp}_{s_j}(\pi^1_j)\right)\boxtimes\pi^2\right)^{L^{h'}_{\underline{u},m}}\\ &\qquad\cdot[\mathrm{rec}((\pi^1_i)^{-1}|\ |^{\frac{1-n}{2}}_{\underline{u}})\cdot\breve{\chi}]\end{aligned}$$

in which the suppressed subscript in $\mathrm{P}_?$ is $(s_i+h'-n,s_1g_1,\dots,\widehat{s_ig_i},\dots,s_tg_t,n_2)$.

We claim that for each $i\in\mathbb{I}$,

(4.7)
$$\sum_{h'=n-s_i}^{h}(-1)^{h-h'}C^{h-h',h'}_m\dim\left(\mathrm{Ind}^{\mathrm{GL}_{h'}(F_{\underline{u}})}_{\mathrm{P}_?(F_{\underline{u}})}\mathrm{Sp}_{s_i+h'-n}(\pi^1_i)\boxtimes\left(\boxtimes_{j\neq i}\mathrm{Sp}_{s_j}(\pi^1_j)\right)\boxtimes\pi^2\right)^{L^{h'}_{\underline{u},m}}=0$$

if $s_i>n-h$. In fact, by Lemma 4.23, there is a nonnegative integer D independent of h' such that the left-hand side of (4.7) equals

$$\begin{aligned}&\sum_{h'=n-s_i}^{h}(-1)^{h-h'}C^{h-h',h'}_m\cdot C^{s_i+h'-n,s_1g_1,\dots,\widehat{s_ig_i},\dots,s_tg_t,n_2}_m\cdot D\cdot\dim\mathrm{Sp}_{s_i+h'-n}(\pi^1_i)^{L^{s_i+h'-n}_{\underline{u},m}}\\ &=\sum_{h'=n-s_i}^{h}(-1)^{h-h'}C^{h-h',s_i+h'-n,s_1g_1,\dots,\widehat{s_ig_i},\dots,s_tg_t,n_2}_m\cdot D\cdot\dim\mathrm{Sp}_{s_i+h'-n}(\pi^1_i)^{L^{s_i+h'-n}_{\underline{u},m}}\\ &=\sum_{h'=0}^{h+s_i-n}(-1)^{h-h'}C^{h+s_i-n-h',h',s_1g_1,\dots,\widehat{s_ig_i},\dots,s_tg_t,n_2}_m\cdot D\cdot\dim\mathrm{Sp}_{h'}(\pi^1_i)^{L^{h'}_{\underline{u},m}}\\ &=(-1)^hC^{h+s_i-n,s_1g_1,\dots,\widehat{s_ig_i},\dots,s_tg_t,n_2}_m\cdot D\sum_{h'=0}^{h+s_i-n}(-1)^{h'}C^{h+s_i-n-h',h'}_m\cdot\dim\mathrm{Sp}_{h'}(\pi^1_i)^{L^{h'}_{\underline{u},m}}\end{aligned}$$

in which the last summation vanishes by applying Lemma 4.24 with $g=h+s_i-n>0$. Here, we have used Lemma 4.22 twice.

By (4.6) and (4.7), we know that (4.5) is a linear combination of $[\mathrm{rec}((\pi^1_i)^{-1}|\ |^{\frac{1-n}{2}}_{\underline{u}})\cdot\breve{\chi}]$ with $i\in\mathbb{I}$ satisfying $s_i=n-h$. Thus, (4.5) is strictly pure of weight h since $\mathrm{Sp}_{s_i}(\pi^1_i)$ is unitary. By (4.3), (4.4), and the fact that localization at $\mathfrak{m}$ annihilates all terms in (4.4) with $\pi^{\boldsymbol{n}}_{\underline{u}}$ not tempered, we know that $[\mathrm{H}_\chi(Y^{[M]}_m,\overline{\mathbb{Q}}_\ell)]_{\mathfrak{m}}$ is strictly pure of weight h. Finally, by [Man08, Proposition 12], we know that $Y^{[M]}_m$ is smooth over k of pure dimension h. Since $Y^{[M]}_m$ is also proper, we have

$$\mathrm{H}^j(Y^{[M]}_m\otimes_k\overline{\mathbb{F}}_p,\overline{\mathbb{Q}}_\ell)[\chi]_{\mathfrak{m}}=0$$

for every $j\neq h$ and every character $\chi\colon\mathrm{T}_0(\mathbb{Q})\backslash\mathrm{T}_0(\mathbb{A}^\infty)/L_0\to\overline{\mathbb{Q}}^\times_\ell$ from the Weil conjecture. Then the proposition follows. ☐

Proof of Theorem 4.21. Recall that $n=2r$ is even. We may assume $m\geqslant 1$ since the morphism $\mathcal{X}_m\to\mathcal{X}_0$ is finite and flat. In what follows, h is always an integer satisfying $0\leqslant h\leqslant n-1=2r-1$. For a subset $\Sigma\subset\mathfrak{S}^h_m$, we put

$$Y^{(\Sigma)}_m:=\bigcup_{M\in\Sigma}Y^{(M)}_m,\qquad Y^{[\Sigma]}_m:=\bigcup_{M\in\Sigma}Y^{[M]}_m$$

in which the first union is disjoint. If $h \geqslant 1$, we also denote by $\Sigma^\dagger$ the subset of $\mathfrak{S}_m^{h-1}$ consisting of M' that contains an element in Σ.

Fix an arbitrary isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. We claim

($*$) For every $0 \leqslant h \leqslant 2r-1$ and every $\Sigma \subset \mathfrak{S}_m^h$,

$$\mathrm{H}_c^j(Y_m^{(\Sigma)} \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell)_\mathfrak{m} = \mathrm{H}^j(Y_m^{[\Sigma]} \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell)_\mathfrak{m} = 0$$

holds when $j > h$.

Assuming the claim, we prove $\mathrm{H}^{2r}(\mathcal{X}_m, \overline{\mathbb{Q}}_\ell(r))_\mathfrak{m} = 0$. By the proper base change theorem and the fact that taking global sections on $\operatorname{Spec} O_K$ is the same as restricting to $\operatorname{Spec} k$ and then taking global sections, the natural map $\mathrm{H}^{2r}(\mathcal{X}_m, \overline{\mathbb{Q}}_\ell(r)) \to \mathrm{H}^{2r}(Y_m, \overline{\mathbb{Q}}_\ell(r))$ is an isomorphism. Thus, it suffices to show that

$$\mathrm{H}^0(k, \mathrm{H}^{2r}(Y_m \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell(r)))_\mathfrak{m} = \mathrm{H}^1(k, \mathrm{H}^{2r-1}(Y_m \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell(r)))_\mathfrak{m} = 0.$$

The vanishing of $\mathrm{H}^0(k, \mathrm{H}^{2r}(Y_m \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell(r)))_\mathfrak{m}$ already follows from ($*$) as $Y_m = Y_m^{[2r-1]}$. Now we consider $\mathrm{H}^1(k, \mathrm{H}^{2r-1}(Y_m \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell(r)))_\mathfrak{m}$. By ($*$), $\mathrm{H}^{2r-1}(Y_m^{[2r-2]} \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell)_\mathfrak{m} = 0$, hence the natural map

$$\mathrm{H}_c^{2r-1}(Y_m^{(2r-1)} \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell)_\mathfrak{m} \to \mathrm{H}^{2r-1}(Y_m \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell)_\mathfrak{m}$$

is surjective. It suffices to show that $\mathrm{H}^1(k, \mathrm{H}_c^{2r-1}(Y_m^{(2r-1)} \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell(r)))_\mathfrak{m} = 0$. Now we prove by induction on $0 \leqslant h \leqslant 2r-1$ that for every $M \in \mathfrak{S}_m^h$, $\mathrm{H}^1(k, \mathrm{H}_c^h(Y_m^{(M)} \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell(r)))_\mathfrak{m} = 0$.

The case $h = 0$ is trivial. Consider $h > 0$ and $M \in \mathfrak{S}_m^h$. Since $Y_m^{[M]}$ is proper smooth over k by [Man08, Proposition 12], we have $\mathrm{H}^1(k, \mathrm{H}^h(Y_m^{[M]} \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell(r)))_\mathfrak{m} = 0$ by the Weil conjecture. By ($*$), we have $\mathrm{H}^h(Y_m^{[\{M\}^\dagger]} \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell)_\mathfrak{m} = 0$. Thus, it suffices to show that $\mathrm{H}^1(k, \mathrm{H}^{h-1}(Y_m^{[\{M\}^\dagger]} \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell(r)))_\mathfrak{m} = 0$. By ($*$) again, we have $\mathrm{H}^{h-1}(Y_m^{[\{M\}^{\dagger\dagger}]} \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell)_\mathfrak{m} = 0$. Thus, the desired vanishing property follows from

$$\mathrm{H}^1(k, \mathrm{H}_c^{h-1}(Y_m^{(\{M\}^\dagger)} \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell(r)))_\mathfrak{m} = \bigoplus_{M' \in \{M\}^\dagger} \mathrm{H}^1(k, \mathrm{H}_c^{h-1}(Y_m^{(M')} \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell(r)))_\mathfrak{m} = 0,$$

which holds by the induction hypothesis. We have now proved $\mathrm{H}^{2r}(\mathcal{X}_m, \overline{\mathbb{Q}}_\ell(r))_\mathfrak{m} = 0$ assuming ($*$).

To show the claim ($*$), we use induction on h. To ease notation, we simply write $\mathrm{H}_?^\bullet(-)$ for $\mathrm{H}_?^\bullet(- \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell)_\mathfrak{m}$ for $? \in \{\ , c\}$. The case for $h = 0$ is trivial. Suppose that we know ($*$) for $h - 1$ for some $h \geqslant 1$. For every $M \in \mathfrak{S}_m^h$, we have the exact sequence

$$\cdots \to \mathrm{H}^{j-1}(Y_m^{[\{M\}^\dagger]}) \to \mathrm{H}_c^j(Y_m^{(M)}) \to \mathrm{H}^j(Y_m^{[M]}) \to \cdots$$

By Proposition 4.25 and the induction hypothesis, we have $\mathrm{H}_c^j(Y_m^{(M)}) = 0$ for $j > h$. Now take a subset Σ of $\mathfrak{S}_m^h$. Then we have $\mathrm{H}_c^j(Y_m^{(\Sigma)}) = \bigoplus_{M \in \Sigma} \mathrm{H}_c^j(Y_m^{(M)}) = 0$ for $j > h$. By the exact sequence

$$\cdots \to \mathrm{H}_c^j(Y_m^{(\Sigma)}) \to \mathrm{H}^j(Y_m^{[\Sigma]}) \to \mathrm{H}^j(Y_m^{[\Sigma^\dagger]}) \to \cdots$$

and the induction hypothesis, we have $\mathrm{H}^j(Y_m^{[\Sigma]}) = 0$ for $j > h$. Thus, ($*$) holds for h.

The theorem is proved. □

Remark 4.26. In fact, our proof of Theorem 4.21 shows that for general n (not necessarily even),

$$\left(\mathrm{H}^{n'}(\mathcal{X}_m, \mathbb{Q}_\ell(r')) \otimes_\mathbb{Q} \mathbb{Q}^{\mathrm{ac}}\right)_\mathfrak{m} = 0$$

as long as $n \leqslant n' \leqslant 2r'$, where $\mathfrak{m}$ is the maximal ideal of a suitable spherical Hecke algebra associated to a tempered cuspidal automorphic representation of the corresponding unitary group.

4.4. **Local indices at inert places.** In this subsection, we compute local indices at places in $\mathtt{V}_E^{\mathrm{int}}$ not above $\mathtt{R}$.

Proposition 4.27. *Let $\mathtt{R}$, $\mathtt{R}'$, ℓ, and L be as in Definition 4.15. Take an element $u \in \mathtt{V}_E^{\mathrm{int}}$ such that its underlying rational prime p is odd and satisfies $\mathtt{V}_F^{(p)} \cap \mathtt{R} \subseteq \mathtt{V}_F^{\mathrm{spl}}$.*

(1) Suppose that $\underline{u} \not\in \mathtt{S}$. Then we have

$$\log q_u \cdot \mathrm{vol}^\natural(L) \cdot I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)_{L,u}^\ell = \mathfrak{E}_{T_1,T_2}((g_1,g_2), \Phi_\infty^0 \otimes (\mathrm{s}_1\phi_1^\infty \otimes (\mathrm{s}_2\phi_2^\infty)^\mathtt{c}))_u$$

for every $(\mathtt{R},\mathtt{R}',\ell,L)$-admissible sextuple $(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)$ and every pair (T_1,T_2) in $\mathrm{Herm}_r^\circ(F)^+$.

(2) Suppose that $\underline{u} \in \mathtt{S} \cap \mathtt{V}_F^\heartsuit$ and is unramified over $\mathbb{Q}$. Recall that we have fixed a u-nearby space ${}^uV$ and an isomorphism ${}^uV \otimes_F \mathbb{A}_F^{\underline{u}} \simeq V \otimes_{\mathbb{A}_F} \mathbb{A}_F^{\underline{u}}$ from Notation 4.2(H9). We also fix a $\psi_{E,\underline{u}}$-self-dual lattice $\Lambda_{\underline{u}}^\star$ of ${}^uV_{\underline{u}}$. Then there exist elements $\mathrm{s}_1^u, \mathrm{s}_2^u \in \mathbb{S}^{\mathtt{R}}_{\mathbb{Q}^{\mathrm{ac}}} \setminus \mathfrak{m}_\pi^{\mathtt{R}}$ such that

$$\begin{aligned} &\log q_u \cdot \mathrm{vol}^\natural(L) \cdot I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1^u\mathrm{s}_1, \mathrm{s}_2^u\mathrm{s}_2, g_1, g_2)_{L,u}^\ell \\ &= \mathfrak{E}_{T_1,T_2}((g_1,g_2), \Phi_\infty^0 \otimes (\mathrm{s}_1^u\mathrm{s}_1\phi_1^\infty \otimes (\mathrm{s}_2^u\mathrm{s}_2\phi_2^\infty)^\mathtt{c}))_u \\ &\qquad - \frac{\log q_u}{q_u^r - 1} E_{T_1,T_2}((g_1,g_2), \Phi_\infty^0 \otimes (\mathrm{s}_1^u\mathrm{s}_1\phi_1^{\infty,\underline{u}} \otimes (\mathrm{s}_2^u\mathrm{s}_2\phi_2^{\infty,\underline{u}})^\mathtt{c}) \otimes \mathbb{1}_{(\Lambda_{\underline{u}}^\star)^{2r}}) \end{aligned}$$

for every $(\mathtt{R},\mathtt{R}',\ell,L)$-admissible sextuple $(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)$ and every pair (T_1,T_2) in $\mathrm{Herm}_r^\circ(F)^+$.

In both cases, the right-hand side is defined in Definition 4.10 with the Gaussian function $\Phi_\infty^0 \in \mathscr{S}(V^{2r} \otimes_{\mathbb{A}_F} F_\infty)$ (Notation 4.2(H3)), and $\mathrm{vol}^\natural(L)$ is defined in [LL21, Definition 3.8].

Proof. Part (1) is proved in the same way as [LL21, Proposition 8.1]. Part (2) is proved in the same way as [LL21, Proposition 9.1]. Note that we need to extend the definition of the integral model due to the presence of places in $\mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{ram}}$, as we do in the previous subsection. The requirement that $\underline{u} \in \mathtt{V}_F^\heartsuit$ in (2) is to ensure that K is unramified over E_u (see Notation 4.19). □

4.5. **Local indices at ramified places.** In this subsection, we compute local indices at places in $\mathtt{V}_E^{\mathrm{ram}}$ not above $\mathtt{R}$.

Proposition 4.28. *Let $\mathtt{R}$, $\mathtt{R}'$, ℓ, and L be as in Definition 4.15. Take an element $u \in \mathtt{V}_E^{\mathrm{ram}}$ such that its underlying rational prime p satisfies $\mathtt{V}_F^{(p)} \cap \mathtt{R} \subseteq \mathtt{V}_F^{\mathrm{spl}}$. Then we have*

$$\log q_u \cdot \mathrm{vol}^\natural(L) \cdot I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)_{L,u}^\ell = \mathfrak{E}_{T_1,T_2}((g_1,g_2), \Phi_\infty^0 \otimes (\mathrm{s}_1\phi_1^\infty \otimes (\mathrm{s}_2\phi_2^\infty)^\mathtt{c}))_u$$

for every $(\mathtt{R},\mathtt{R}',\ell,L)$-admissible sextuple $(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)$ and every pair (T_1,T_2) in $\mathrm{Herm}_r^\circ(F)^+$, where the right-hand side is defined in Definition 4.10 with the Gaussian function $\Phi_\infty^0 \in \mathscr{S}(V^{2r} \otimes_{\mathbb{A}_F} F_\infty)$ (Notation 4.2(H3)), and $\mathrm{vol}^\natural(L)$ is defined in [LL21, Definition 3.8].

Proof. The proof of the proposition follows the same line as in [LL21, Proposition 8.1], as long as we accomplish the following three tasks. We invoke Notation 4.18 together with Notation 4.19.

(1) Construct a good integral model $\mathcal{X}_{\tilde{L}}$ for $X_{\tilde{L}}$ over O_K for open compact subgroups $\tilde{L} \subseteq L$ satisfying $\tilde{L}_v = L_v$ for $v \in \mathtt{V}_F^{(p)} \setminus \mathtt{V}_F^{\mathrm{spl}}$, which is provided after the proof.
(2) Establish the nonarchimedean uniformization of $\mathcal{X}_{\tilde{L}}$ along the supersingular locus using the relative Rapoport–Zink space $\mathcal{N}$ from Definition 2.3, analogous to [LL21, (8.2)], and compare special divisors. This is done in Proposition 4.30 below.
(3) Show that for $x = (x_1, \dots, x_{2r}) \in {}^{\underline{u}}V^{2r}$ with $T(x) \in \mathrm{Herm}_{2r}^{\circ}(F_{\underline{u}})$, we have

$$\chi\left(\mathscr{O}_{\mathcal{N}(x_1)} \overset{\mathbb{L}}{\otimes}_{\mathscr{O}_{\mathcal{N}}} \cdots \overset{\mathbb{L}}{\otimes}_{\mathscr{O}_{\mathcal{N}}} \mathscr{O}_{\mathcal{N}(x_{2r})}\right) = \frac{b_{2r,\underline{u}}(0)}{\log q_u} W'_{T^\square}(0, 1_{4r}, \mathbb{1}_{(\Lambda^{\mathtt{R}}_{\underline{u}})^{2r}})$$

if $T(x) = T^\square$. In fact, this follows from Theorem 2.7, Remark 2.18, and the identity

$$b_{2r,\underline{u}}(0) = \prod_{i=1}^{r} (1 - q_u^{-2i}).$$

The proposition is proved. ∎

Let the situation be as in Proposition 4.28. The isomorphism $\mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ in Notation 4.19 identifies $\mathrm{Hom}(E, \mathbb{C})$ with $\mathrm{Hom}(E, \mathbb{C}_p)$. For every $v \in \mathtt{V}_F^{(p)}$, let Φ_v be the subset of Φ, regarded as a subset of $\mathrm{Hom}(E, \mathbb{C}_p)$, of elements that induce the place v of F.

To ease notation, put

$$\mathtt{U} \coloneqq \{v \in \mathtt{V}_F^{(p)} \setminus \mathtt{V}_F^{\mathrm{spl}} \mid v \neq \underline{u}\}.$$

In particular, $\mathtt{U} \cap \mathtt{R} = \emptyset$.

There is a projective system $\{\mathcal{X}_{\tilde{L}}\}$, for open compact subgroups $\tilde{L} \subseteq L$ satisfying $\tilde{L}_v = L_v$ for $v \in \mathtt{V}_F^{(p)} \setminus \mathtt{V}_F^{\mathrm{spl}}$, of smooth projective schemes over O_K (see [RSZ20, Theorem 4.7, AT type (2)]) with

$$\mathcal{X}_{\tilde{L}} \otimes_{O_K} K = X'_{\tilde{L}} \otimes_{E'} K = (X_{\tilde{L}} \otimes_E Y) \otimes_{E'} K,$$

and finite étale transition morphisms, such that for every $S \in \mathrm{Sch}'_{/O_K}$, $\mathcal{X}_{\tilde{L}}(S)$ is the set of equivalence classes of tuples

$$(A_0, \iota_{A_0}, \lambda_{A_0}, \eta^p_{A_0}; A, \iota_A, \lambda_A, \eta^p_A, \{\eta_{A,v}\}_{v \in \mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{spl}}})$$

where

- $(A_0, \iota_{A_0}, \lambda_{A_0}, \eta^p_{A_0})$ is an element in $\mathcal{Y}(S)$;
- (A, ι_A, λ_A) is a unitary O_E-abelian scheme of signature type $n\Phi - \iota_w + \iota_w^{\mathtt{c}}$ over S, such that
  - for every $v \in \mathtt{V}_F^{(p)} \setminus \mathtt{V}_F^{\mathrm{ram}}$, $\lambda_A[v^\infty]$ is an isogeny whose kernel has order $q_v^{1-\epsilon_v}$;
  - for every $v \in \mathtt{U} \cap \mathtt{V}_F^{\mathrm{ram}}$, the triple $(A_0[v^\infty], \iota_{A_0}[v^\infty], \lambda_{A_0}[v^\infty]) \otimes_{O_K} O_{\breve{K}}$ is an object of $\mathrm{Exo}^{\Phi_v}_{(n,0)}(S \otimes_{O_K} O_{\breve{K}})$ (Remark 2.67, with $E = E_v$, $F = F_v$, and $\breve{E} = \breve{K}$);
  - for $v = \underline{u}$, $(A_0[v^\infty], \iota_{A_0}[v^\infty], \lambda_{A_0}[v^\infty]) \otimes_{O_K} O_{\breve{K}}$ is an object of $\mathrm{Exo}^{\Phi_v}_{(n-1,1)}(S \otimes_{O_K} O_{\breve{K}})$ (Definition 2.59, with $E = E_v$, $F = F_v$, and $\breve{E} = \breve{K}$);
- η^p_A is an $\tilde{L}^p$-level structure;
- for every $v \in \mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{spl}}$, $\eta_{A,v}$ is an $\tilde{L}_v$-level structure.

In particular, $\mathbb{S}^{\mathtt{R}}$ is naturally a ring of étale correspondences of $\mathcal{X}_L$.

Let $\phi^\infty \in \mathscr{S}(V \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)^{\tilde{L}}$ be a p-basic element [LL21, Definition 6.5]. For every element $t \in F$ that is totally positive, we have a cycle $\mathcal{Z}_t(\phi^\infty)_{\tilde{L}} \in \mathrm{Z}^1(\mathcal{X}_{\tilde{L}})$ extending the restriction of $Z_t(\phi^\infty)$ to $X'_{\tilde{L}}$, defined similarly as in [LZa, Section 13.3].

Now we study the nonarchimedean uniformization of $\mathcal{X}_{\tilde{L}}$ along the supersingular locus. Fix a point $P_0 := (A_0, \iota_{A_0}, \lambda_{A_0}, \eta^p_{A_0}) \in \mathcal{Y}(O_{\breve{K}})$. Put

$$\mathcal{X} := \varprojlim_{\tilde{L}} \mathcal{X}_{\tilde{L}}$$

and denote by $\mathcal{X}_0$ the fiber of P_0 along the natural projection $\mathcal{X} \to \mathcal{Y}$. Let $\mathcal{X}_0^\wedge$ be the completion along the (closed) locus where $A[u^\infty]$ is supersingular, as a formal scheme over $\operatorname{Spf} O_K$. Also fix a point $\boldsymbol{P} \in \mathcal{X}_0^\wedge(\overline{\mathbb{F}}_p)$ represented by $(P_0 \otimes_{O_{\breve{K}}} \overline{\mathbb{F}}_p; \boldsymbol{A}, \iota_{\boldsymbol{A}}, \lambda_{\boldsymbol{A}}, \eta^p_{\boldsymbol{A}}, \{\eta_{\boldsymbol{A},v}\}_{v \in \mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{spl}}})$.

Put $\boldsymbol{V} := \operatorname{Hom}_{O_E}(A_0 \otimes_{O_{\breve{E}}} \overline{\mathbb{F}}_p, \boldsymbol{A}) \otimes \mathbb{Q}$. Fixing an element $\varpi \in O_F$ that has valuation 0 (resp. 1) at places in $\mathtt{U} \cap \mathtt{V}_F^{\mathrm{int}}$ (resp., $\mathtt{U} \cap \mathtt{V}_F^{\mathrm{ram}}$), we have a pairing

$$(\ ,\ )_{\boldsymbol{V}} \colon \boldsymbol{V} \times \boldsymbol{V} \to E$$

sending $(x, y) \in \boldsymbol{V}^2$ to the composition of quasi-homomorphisms

$$A_0 \xrightarrow{x} \boldsymbol{X} \xrightarrow{\lambda_{\boldsymbol{A}}} \boldsymbol{A}^\vee \xrightarrow{y^\vee} A_0^\vee \xrightarrow{\varpi^{-1}\lambda_{A_0}^{-1}} A_0$$

as an element in $\operatorname{End}_{O_E}(A_0) \otimes \mathbb{Q}$, hence in E via $\iota_{A_0}^{-1}$. We have the following properties concerning $\boldsymbol{V}$:

- $\boldsymbol{V}, (\ ,\ )_{\boldsymbol{V}}$ is a totally positive definite hermitian space over E of rank n;
- for every $v \in \mathtt{V}_F^{\mathrm{fin}} \setminus (\mathtt{V}_F^{(p)} \setminus \mathtt{V}_F^{\mathrm{spl}})$, we have a canonical isometry $\boldsymbol{V} \otimes_F F_v \simeq V \otimes_F F_v$ of hermitian spaces;
- for every $v \in \mathtt{U}$, the O_{E_v}-lattice $\boldsymbol{\Lambda}_v := \operatorname{Hom}_{O_E}(A_0 \otimes_{O_{\breve{E}}} \overline{\mathbb{F}}_p, \boldsymbol{A}) \otimes_{O_F} O_{F_v}$ is
  - self-dual if $v \in \mathtt{U} \cap \mathtt{V}_F^{\mathrm{int}}$ and $\epsilon_v = 1$,
  - almost self-dual if $v \in \mathtt{U} \cap \mathtt{V}_F^{\mathrm{int}}$ and $\epsilon_v = -1$,
  - self-dual if $v \in \mathtt{U} \cap \mathtt{V}_F^{\mathrm{ram}}$;
- $\boldsymbol{V} \otimes_F F_{\underline{u}}$ is nonsplit, and we have a canonical isomorphism

  $$\boldsymbol{V} \otimes_F F_{\underline{u}} \simeq \operatorname{Hom}_{O_{E_u}}(A_0[u^\infty] \otimes_{O_{\breve{K}}} \overline{\mathbb{F}}_p, \boldsymbol{A}[u^\infty]) \otimes \mathbb{Q}$$

  of hermitian spaces over E_u.

We have a Rapoport–Zink space $\mathcal{N}$ (Definition 2.3, with $E = E_u$, $F = F_{\underline{u}}$, $\breve{E} = \breve{K}$, and φ_0 the natural embedding) with respect to the object

$$(\boldsymbol{X}, \iota_{\boldsymbol{X}}, \lambda_{\boldsymbol{X}}) := (\boldsymbol{A}[u^\infty], \iota_{\boldsymbol{A}}[u^\infty], \lambda_{\boldsymbol{A}}[u^\infty])^{\mathrm{rel}} \in \operatorname{Exo}^{\mathrm{b}}_{(n-1,1)}(\overline{\mathbb{F}}_p),$$

where $-^{\mathrm{rel}}$ is the morphism (2.21). We now construct a morphism

$$\Upsilon^{\mathrm{rel}} \colon \mathcal{X}_0^\wedge \to \mathrm{U}(\boldsymbol{V})(F) \backslash \left(\mathcal{N} \times \mathrm{U}(\boldsymbol{V})(\mathbb{A}_F^{\infty,\underline{u}}) / \prod_{v \in \mathtt{U}} \boldsymbol{L}_v \right) \tag{4.8}$$

of formal schemes over $\operatorname{Spf} O_{\breve{K}}$, where $\boldsymbol{L}_v$ is the stabilizer of $\boldsymbol{\Lambda}_v$ in $\mathrm{U}(\boldsymbol{V})(F_v)$, as follows.

We have the Rapoport–Zink space $\mathcal{N}^{\Phi_u} = \mathcal{N}^{\Phi_u}_{(\boldsymbol{A}[u^\infty], \iota_{\boldsymbol{A}}[u^\infty], \lambda_{\boldsymbol{A}}[u^\infty])}$ from Definition 2.64. We first define a morphism

$$\Upsilon \colon \mathcal{X}_0^\wedge \to \mathrm{U}(\boldsymbol{V})(F) \backslash \left(\mathcal{N}^{\Phi_u} \times \mathrm{U}(\boldsymbol{V})(\mathbb{A}_F^{\infty,\underline{u}}) / \prod_{v \in \mathtt{U}} \boldsymbol{L}_v \right),$$

and then define Υ^{rel} as the composition of Υ with the morphism in Corollary 2.65. To construct Υ, we take a point

$$P = (P_0 \otimes_{O_{\breve{K}}} S; A, \iota_A, \lambda_A, \eta^p_A, \{\eta_{A,v}\}_{v \in \mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{spl}}}) \in \mathcal{X}_0^\wedge(S)$$

for a connected scheme S in $\mathrm{Sch}'_{/O_{\breve{K}}} \cap \mathrm{Sch}^{\mathrm{v}}_{/O_{\breve{K}}}$ with a geometric point s. In particular, $A[p^\infty]$ is supersingular. By [RZ96, Proposition 6.29], we can choose an O_E-linear quasi-isogeny

$$\rho\colon A \times_S (S \otimes_{O_{\breve{K}}} \overline{\mathbb{F}}_p) \to \boldsymbol{A} \otimes_{\overline{\mathbb{F}}_p} (S \otimes_{O_{\breve{K}}} \overline{\mathbb{F}}_p)$$

of height zero such that $\rho^*\lambda_{\boldsymbol{A}} \otimes_{\overline{\mathbb{F}}_p} (S \otimes_{O_{\breve{K}}} \overline{\mathbb{F}}_p) = \lambda_A \times_S (S \otimes_{O_{\breve{K}}} \overline{\mathbb{F}}_p)$. We have

- $(A[u^\infty], \iota_A[u^\infty], \lambda_A[u^\infty]; \rho[u^\infty])$ is an element in $\mathcal{N}^{\Phi_u}(S)$;
- the composite map

$$\begin{aligned}\boldsymbol{V} \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} &\xrightarrow{\sim} V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \xrightarrow{\eta^p_A} \mathrm{Hom}_{E\otimes_{\mathbb{Q}}\mathbb{A}^{\infty,p}}(\mathrm{H}_1(A_{0,s}, \mathbb{A}^{\infty,p}), \mathrm{H}_1(A_s, \mathbb{A}^{\infty,p})) \\ &\xrightarrow{\rho_{s*}\circ} \mathrm{Hom}_{E\otimes_{\mathbb{Q}}\mathbb{A}^{\infty,p}}(\mathrm{H}_1(A_{0,s}, \mathbb{A}^{\infty,p}), \mathrm{H}_1(\boldsymbol{A}_s, \mathbb{A}^{\infty,p})) = \boldsymbol{V} \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}\end{aligned}$$

  is an isometry, which gives rise to an element $h^p \in \mathrm{U}(\boldsymbol{V})(\mathbb{A}_F^{\infty,p})$;
- the same process as above will produce an element $h_p^{\mathrm{spl}} \in \prod_{v\in\mathtt{V}_F^{(p)}\cap\mathtt{V}_F^{\mathrm{spl}}} \mathrm{U}(\boldsymbol{V})(F_v)$;
- for every $v \in \mathtt{U}$, the image of the map

$$\rho_{s*}\circ\colon \mathrm{Hom}_{O_{E_v}}(A_{0,s}[v^\infty], A_s[v^\infty]) \to \mathrm{Hom}_{O_{E_v}}(A_{0,s}[v^\infty], \boldsymbol{A}_s[v^\infty]) \otimes \mathbb{Q} = \boldsymbol{V} \otimes_F F_v$$

  is an O_{E_v}-lattice in the same $\mathrm{U}(\boldsymbol{V})(F_v)$-orbit of $\boldsymbol{\Lambda}_v$, which gives rise to an element $h_v \in \mathrm{U}(\boldsymbol{V})(F_v)/\boldsymbol{L}_v$.

Together, we obtain an element

$$\left((A[u^\infty], \iota_A[u^\infty], \lambda_A[u^\infty]; \rho[u^\infty]), (h^p, h_p^{\mathrm{spl}}, \{h_v\}_{v\in\mathtt{U}})\right) \in \mathcal{N}^{\Phi_u}(S) \times \mathrm{U}(\boldsymbol{V})(\mathbb{A}_F^{\infty,\underline{u}})/\prod_{v\in\mathtt{U}} \boldsymbol{L}_v,$$

and we define $\Upsilon(P)$ to be its image in the quotient, which is independent of the choice of ρ.

Remark 4.29. Both $\boldsymbol{V}$ and Υ^{rel} depend on the choice of $\boldsymbol{P}$, while the isometry class of $\boldsymbol{V}$ does not.

Proposition 4.30. *The morphism Υ^{rel} (4.8) is an isomorphism. Moreover, for every p-basic element $\phi^\infty \in \mathscr{S}(V \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)^{\tilde{L}}$ and every $t \in F$ that is totally positive, we have*

$$\Upsilon^{\mathrm{rel}}\left(\mathcal{Z}_t(\phi^\infty)_{\tilde{L}}|_{\mathcal{X}_0^\wedge}\right) = \sum_{\substack{x\in\mathrm{U}(\boldsymbol{V})(F)\backslash\boldsymbol{V} \\ (x,x)_{\boldsymbol{V}}=t}} \sum_{h\in\mathrm{U}(\boldsymbol{V}^x)(F)\backslash\mathrm{U}(\boldsymbol{V})(\mathbb{A}_F^{\infty,\underline{u}})/\prod_{v\in\mathtt{U}}\boldsymbol{L}_v} \boldsymbol{\phi}(h^{-1}x)\cdot(\mathcal{N}(x^{\mathrm{rel}}), h), \tag{4.9}$$

where

- *$\boldsymbol{V}^x$ denotes the orthogonal complement of x in $\boldsymbol{V}$;*
- *$\boldsymbol{\phi}$ is a Schwartz function on $\boldsymbol{V} \otimes_F \mathbb{A}_F^{\infty,\underline{u}}$ such that $\boldsymbol{\phi}_v = \phi_v^\infty$ for $v \in \mathtt{V}_F^{\mathrm{fin}} \setminus (\mathtt{V}_F^{(p)} \setminus \mathtt{V}_F^{\mathrm{spl}})$ and $\boldsymbol{\phi}_v = \mathbb{1}_{\boldsymbol{\Lambda}_v}$ for $v \in \mathtt{U}$;*
- *x^{rel} is defined in (2.25); and*
- *$(\mathcal{N}(x^{\mathrm{rel}}), h)$ denotes the corresponding double coset in (4.8).*

Proof. By a similar argument for [RZ96, Theorem 6.30], the morphism Υ is an isomorphism. Thus, Υ^{rel} is an isomorphism as well by Corollary 2.65.

For (4.9), by a similar argument for [Liu21, Theorem 5.22], the identity holds with $\mathcal{N}(x^{\mathrm{rel}})$ replaced by $\mathcal{N}^{\Phi_u}(x)$. Then it follows by Corollary 2.66.

The proposition is proved. □

4.6. **Local indices at archimedean places.** In this subsection, we compute local indices at places in $\mathtt{V}_E^{(\infty)}$.

Proposition 4.31. *Let* $\mathtt{R}$, $\mathtt{R}'$, ℓ, *and* L *be as in Definition 4.15. Let* $(\pi, \mathcal{V}_\pi)$ *be as in Assumption 4.4. Take an element* $u \in \mathtt{V}_E^{(\infty)}$. *Consider an* $(\mathtt{R}, \mathtt{R}', \ell, L)$*-admissible sextuple* $(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)$ *and an element* $\varphi_1 \in \mathcal{V}_\pi^{[r]\mathtt{R}}$. *Let* $K_1 \subseteq G_r(\mathbb{A}_F^\infty)$ *be an open compact subgroup that fixes both* ϕ_1^∞ *and* φ_1, *and* $\mathfrak{F}_1 \subseteq G_r(F_\infty)$ *a Siegel fundamental domain for the congruence subgroup* $G_r(F) \cap g_1^\infty K_1 (g_1^\infty)^{-1}$. *Then for every* $T_2 \in \mathrm{Herm}_r^\circ(F)^+$, *we have*

$$\begin{aligned}&\mathrm{vol}^\natural(L) \cdot \int_{\mathfrak{F}_1} \varphi^{\mathtt{c}}(\tau_1 g_1) \sum_{T_1 \in \mathrm{Herm}_r^\circ(F)^+} I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, \tau_1 g_1, g_2)_{L,u}\, \mathrm{d}\tau_1 \\ &= \frac{1}{2} \int_{\mathfrak{F}_1} \varphi^{\mathtt{c}}(\tau_1 g_1) \sum_{T_1 \in \mathrm{Herm}_r^\circ(F)^+} \mathfrak{E}_{T_1,T_2}((\tau_1 g_1, g_2), \Phi_\infty^0 \otimes (\mathrm{s}_1\phi_1^\infty \otimes (\mathrm{s}_2\phi_2^\infty)^{\mathtt{c}}))_u\, \mathrm{d}\tau_1,\end{aligned}$$

in which both sides are absolutely convergent. Here, the term $\mathfrak{E}_{T_1,T_2}$ *is defined in Definition 4.10 with the Gaussian function* $\Phi_\infty^0 \in \mathscr{S}(V^{2r} \otimes_{\mathbb{A}_F} F_\infty)$ *(Notation 4.2(H3)), and* $\mathrm{vol}^\natural(L)$ *is defined in* [LL21, Definition 3.8].

Proof. This is simply [LL21, Proposition 10.1]. □

4.7. **Proof of main results.** The proofs of Theorem 1.4, Theorem 1.5, and Corollary 1.7 follow from the same lines as for [LL21, Theorem 1.5], [LL21, Theorem 1.7], and [LL21, Corollary 1.9], respectively, written in [LL21, Section 11]. However, we need to take $\mathtt{R}$ to be a finite subset of $\mathtt{V}_F^{\mathrm{spl}} \cap \mathtt{V}_F^\heartsuit$ containing $\mathtt{R}_\pi$ and of cardinality at least 2, and modify the reference according to the table below.

This article	[LL21]
Proposition 4.8	Proposition 3.6
Proposition 4.9	Proposition 3.7
Proposition 4.20	Proposition 7.1
Proposition 4.27	Proposition 8.1 & Proposition 9.1
Proposition 4.28	(not available)
Proposition 4.31	Proposition 10.1

Remark 4.32. When $\mathtt{S}_\pi = \emptyset$, Theorem 1.4, Theorem 1.5, and Corollary 1.7 can all be proved without [LL21, Hypothesis 6.6]. In fact, besides Proposition 4.27(2) (which we do not need as $\mathtt{S}_\pi = \emptyset$), the only place where [LL21, Hypothesis 6.6] is used is [LL21, Proposition 6.9(2)]. However, we can slightly modify the definition of $(\mathbb{S}_{\mathbb{L}}^{\mathtt{R}})_{L_{\mathtt{R}}}^{\langle \ell \rangle}$ in Definition 4.14(2) such that it is the ideal of $\mathbb{S}_{\mathbb{L}}^{\mathtt{R}}$ of elements that annihilate

$$\bigoplus_{u \in \mathtt{V}_E^{\mathrm{fin}} \setminus \mathtt{V}_E^{(\ell)}} \mathrm{H}_\dagger^{2r}(X_{L_{\mathtt{R}}L^{\mathtt{R}},u}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L},$$

where $\mathrm{H}_\dagger^{2r}(X_{L_{\mathtt{R}}L^{\mathtt{R}},u}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L}$ is the $\mathbb{Q}_\ell \otimes_{\mathbb{Q}} \mathbb{L}$-submodule of $\mathrm{H}^{2r}(X_{L_{\mathtt{R}}L^{\mathtt{R}},u}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L}$ generated by the image of the cycle class map $\mathrm{CH}^r(X_{L_{\mathtt{R}}L^{\mathtt{R}},u}) \to \mathrm{H}^{2r}(X_{L_{\mathtt{R}}L^{\mathtt{R}},u}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L}$.

Theorem 4.21 implies that when u satisfies $\underline{u} \in \mathtt{R} \cap \mathtt{V}_F^{\mathrm{spl}} \cap \mathtt{V}_F^\heartsuit$ and $\mathtt{V}_F^{(p)} \cap \mathtt{R} \subseteq \mathtt{V}_F^{\mathrm{spl}}$ where p is the underlying rational prime of u, there exists an element in $(\mathbb{S}_{\mathbb{Q}^{\mathrm{ac}}}^{\mathtt{R}})_{L_{\mathtt{R}}}^{\langle \ell \rangle} \setminus \mathfrak{m}_\pi^{\mathtt{R}}$ that annihilates $\mathrm{H}_\dagger^{2r}(X_{L_{\mathtt{R}}L^{\mathtt{R}},u}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{Q}^{\mathrm{ac}}$. Indeed, we have a commutative diagram (in the context of the proof of Proposition 4.20)

$$\begin{array}{ccccc}
\mathrm{CH}^r(\mathcal{X}_m) & & \longrightarrow & & \mathrm{H}^{2r}(\mathcal{X}_m, \mathbb{Q}_\ell(r)) \\
\downarrow & & & & \downarrow \\
\mathrm{CH}^r(X_{L_{\mathtt{R}}L^{\mathtt{R}},u}) & \longrightarrow & \mathrm{H}^{2r}_{\dagger}(X_{L_{\mathtt{R}}L^{\mathtt{R}},u}, \mathbb{Q}_\ell(r)) & \hookrightarrow & \mathrm{H}^{2r}(X_{L_{\mathtt{R}}L^{\mathtt{R}},u}, \mathbb{Q}_\ell(r))
\end{array}$$

in which the left vertical arrow is surjective, which implies that $\mathrm{H}^{2r}_{\dagger}(X_{L_{\mathtt{R}}L^{\mathtt{R}},u}, \mathbb{Q}_\ell(r))$ is a quotient of $\mathrm{H}^{2r}(\mathcal{X}_m, \mathbb{Q}_\ell(r))$.

It follows that with this new definition of $(\mathbb{S}^{\mathtt{R}}_{\mathbb{L}})^{\langle\ell\rangle}_{L_{\mathtt{R}}}$, [LL21, Proposition 6.9(2)] holds when $\mathtt{R} \subseteq \mathtt{V}^{\mathrm{spl}}_F \cap \mathtt{V}^{\heartsuit}_F$ without assuming [LL21, Hypothesis 6.6].

Remark 4.33. Finally, we explain the main difficulty on lifting the restriction $F \neq \mathbb{Q}$ (when $r \geqslant 2$). Suppose that $F = \mathbb{Q}$ and $r \geqslant 2$. Then the Shimura variety X_L from Subsection 4.2 is never proper over the base field. Nevertheless, it is well-known that X_L admits a canonical toroidal compactification, which is smooth. However, to run our argument, we need suitable compactification of their integral models at every finite place u of E as well. As far as we can see, the main obstacle is the compactification of integral models using Drinfeld level structures when u splits over F, together with a vanishing result like Theorem 4.21.

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