REMARKS ON THE ARITHMETIC FUNDAMENTAL LEMMA

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ABSTRACT. W. Zhang's arithmetic fundamental lemma (AFL) is a conjectural identity between the derivative of an orbital integral on a symmetric space with an arithmetic intersection number on a unitary Rapoport–Zink space. In the minuscule case, Rapoport–Terstiege–Zhang have verified the AFL conjecture via explicit evaluation of both sides of the identity. We present a simpler way for evaluating the arithmetic intersection number, thereby providing a new proof of the AFL conjecture in the minuscule case.

1. INTRODUCTION

1.1. **Zhang's arithmetic fundamental lemma.** The *arithmetic Gan–Gross–Prasad conjectures* (arithmetic GGP) generalize the celebrated Gross–Zagier formula to higher dimensional Shimura varieties ([GGP12, §27], [Zha12, §3.2]). The arithmetic fundamental lemma (AFL) conjecture arises from Zhang's relative trace formula approach for establishing the arithmetic GGP for the group $U(1, n-2) \times U(1, n-1)$. It relates a derivative of orbital integrals on symmetric spaces to an arithmetic intersection number of cycles on unitary Rapoport–Zink spaces,

$$(1.1.0.1) \qquad O'(\gamma, \mathbf{1}_{S_n(\mathbb{Z}_p)}) = -\omega(\gamma) \langle \Delta(\mathcal{N}_{n-1}), (\mathrm{id} \times g) \Delta(\mathcal{N}_{n-1}) \rangle.$$

For the precise definitions of quantities appearing in the identity, see [RTZ13, Conjecture 1.2]. The left-hand side of (1.1.0.1) is known as the *analytic side* and the right-hand side is known as the *arithmetic-geometric* side. The AFL conjecture has been verified for $n = 2, 3$ ([Zha12]), and for general n in the minuscule case (in the sense that g satisfies a certain minuscule condition) by Rapoport–Terstiege–Zhang [RTZ13]. In all these cases, the identity (1.1.0.1) is proved via explicit evaluation of both sides. When g satisfies a certain inductive condition, Mihatsch [Mih16] has recently developed a recursive algorithm which reduces the identity (1.1.0.1) to smaller n, thus establishing some new cases of the AFL conjecture.

In the minuscule case, the evaluation of the analytic side is relatively straightforward. The bulk of [RTZ13] is devoted to a highly nontrivial evaluation of the arithmetic-geometric side, which is truly a tour de force. Our main goal in this short note is to present a new (and arguably simpler) way to evaluate the arithmetic-geometric side in [RTZ13].

1.2. **The main results.** Let p be an odd prime. Let $F = \mathbb{Q}_p$, $k = \overline{\mathbb{F}}_p$, $W = W(k)$ and $K = W[1/p]$. Let σ be the p-Frobenius acting on $\overline{\mathbb{F}}_p$, W and K. Let $E = \mathbb{Q}_{p^2}$ be the unramified quadratic extension of F. The *unitary Rapoport–Zink space* $\mathcal{N}_n$ is the formal scheme over $\mathrm{Spf}\, W$ parameterizing deformations up to quasi-isogeny of height 0 of unitary p-divisible groups of signature $(1, n-1)$ (definitions recalled in §2.1). Fix $n \geq 2$ and write $\mathcal{N} = \mathcal{N}_n$, $\mathcal{M} = \mathcal{N}_{n-1}$ for short. There is a natural closed immersion $\delta : \mathcal{M} \to \mathcal{N}$. Denote by $\Delta \subseteq \mathcal{M} \times_W \mathcal{N}$ the image of $(\mathrm{id}, \delta) : \mathcal{M} \to \mathcal{M} \times_W \mathcal{N}$, known as the (local) *diagonal cycle* or *GGP cycle* on $\mathcal{M} \times_W \mathcal{N}$.

Let C_{n-1} be a non-split σ-Hermitian E-space of dimension $n-1$. Let $C_n = C_{n-1} \oplus Eu$ (where the direct sum is orthogonal and u has norm 1) be a non-split σ-Hermitian E-space of dimension n. The unitary group $J = \mathrm{U}(C_n)$ acts on $\mathcal{N}$ in a natural way (see §2.2). Let $g \in J(\mathbb{Q}_p)$. The arithmetic-geometric side of the AFL conjecture (1.1.0.1) concerns the arithmetic intersection number of the diagonal cycle Δ and its translate by $\mathrm{id} \times g$, defined as

$$\langle \Delta, (\mathrm{id} \times g)\Delta \rangle := \log p \cdot \chi(\mathcal{M} \times_W \mathcal{N}, \mathcal{O}_\Delta \otimes^{\mathbb{L}} \mathcal{O}_{(\mathrm{id} \times g)\Delta}).$$

When Δ and $(\mathrm{id} \times g)\Delta$ intersect properly, namely when the formal scheme

$$(1.2.0.2) \qquad\qquad\qquad \Delta \cap (\mathrm{id} \times g)\Delta \cong \delta(\mathcal{M}) \cap \mathcal{N}^g$$

is an Artinian scheme (where $\mathcal{N}^g$ denotes the fixed points of g), the intersection number is simply $\log p$ times the W-length of the Artinian scheme (1.2.0.2).

Recall that $g \in J(\mathbb{Q}_p)$ is called *regular semisimple* if

$$L(g) := \mathcal{O}_E \cdot u + \mathcal{O}_E \cdot gu + \cdots + \mathcal{O}_E \cdot g^{n-1}u$$

is an $\mathcal{O}_E$-lattice in C_n. In this case, the *invariant* of g is the unique sequence of integers

$$\mathrm{inv}(g) := (r_1 \geq r_2 \geq \ldots \geq r_n)$$

characterized by the condition that there exists a basis $\{e_i\}$ of the lattice $L(g)$ such that $\{p^{-r_i}e_i\}$ is a basis of the dual lattice $L(g)^\vee$. It turns out that the "bigger" $\mathrm{inv}(g)$ is, the more difficult it is to compute the intersection. With this in mind, recall that a regular semisimple element g is called *minuscule* if $r_1 = 1$ and $r_n \geq 0$ (equivalently, $pL(g)^\vee \subseteq L(g) \subseteq L(g)^\vee$). In this minuscule case, the intersection turns out to be proper, and one of the main results of [RTZ13] is an explicit formula for the W-length of (1.2.0.2) at each of its k-point.

To state the formula, assume g is regular semisimple and minuscule, and suppose $\mathcal{N}^g$ is nonempty. Then g stabilizes both $L(g)^\vee$ and $L(g)$ and thus acts on the $\mathbb{F}_{p^2}$-vector space $L(g)^\vee/L(g)$. Let $P(T) \in \mathbb{F}_{p^2}[T]$ be the characteristic polynomial of g acting on $L(g)^\vee/L(g)$. For any irreducible polynomial $R(T) \in \mathbb{F}_{p^2}[T]$, we denote its multiplicity in $P(T)$ by $m(R(T))$ and define its *reciprocal* by

$$R^*(T) := T^{\deg R(T)} \cdot \sigma(R(1/T)).$$

We say $R(T)$ is *self-reciprocal* if $R(T) = R^*(T)$. By [RTZ13, 8.1], if $(\delta(M) \cap \mathcal{N}^g)(k)$ is nonempty, then $P(T)$ has a unique self-reciprocal monic irreducible factor $Q(T) | P(T)$ such that $m(Q(T))$ is odd. We denote $c := \frac{m(Q(T))+1}{2}$. Then $1 \leq c \leq \frac{n+1}{2}$. Now we are ready to state the intersection length formula.

Theorem A ([RTZ13, Theorem 9.5]). *Assume g is regular semisimple and minuscule. Assume $p > c$. Then for any $x \in (\delta(M) \cap \mathcal{N}^g)(k)$, the complete local ring of $\delta(M) \cap \mathcal{N}^g$ at x is isomorphic to $k[X]/X^c$, and hence has W-length equal to c.*

We will present a simpler proof of Theorem A in Theorem 4.3.5. Along the way, we will also give a simpler proof of the following Theorem B in Corollary 3.2.3, which concerns minuscule special cycles (recalled in §2.10) on unitary Rapoport–Zink spaces and may be of independent interest.

Theorem B ([RTZ13, Theorems 9.4, 10.1]). *Let $\mathbf{v} = (v_1, \ldots, v_n)$ be an n-tuple of vectors in C_n. Assume it is minuscule in the sense that $L(\mathbf{v}) := \mathrm{span}_{\mathcal{O}_E} \mathbf{v}$ is an $\mathcal{O}_E$-lattice in C_n satisfying $pL(\mathbf{v})^\vee \subseteq L(\mathbf{v}) \subseteq L(\mathbf{v})^\vee$. Let $\mathcal{Z}(\mathbf{v}) \subseteq \mathcal{N}$ be the associated special cycle. Then $\mathcal{Z}(\mathbf{v})$ is a reduced k-scheme.*

1.3. **Novelty of the proof.** The original proofs of Theorems A and B form the technical heart of [RTZ13] and occupies its two sections §10-§11. As explained below, our new proofs presented here have the merit of being much shorter and more conceptual.

1.3.1. *Theorem A.* The original proof of Theorem A uses Zink's theory of windows to compute the local equations of (1.2.0.2). It requires explicitly writing down the window of the universal deformation of p-divisible groups and solving quite involved linear algebra problems. Theorem B ensures that the intersection is entirely concentrated in the special fiber so that each local ring has the form $k[X]/X^\ell$. The assumption $p > c$ ensures $\ell < p$ so that the ideal of local equations is *admissible* (see the last paragraph of [RTZ13, p. 1661]), which is crucial in order to construct the frames for the relevant windows needed in Zink's theory.

Our new proof of Theorem A does not use Zink's theory and involves little explicit computation. Our key observation is that Theorem B indeed allows us to identify the intersection (1.2.0.2) as the fixed point scheme $\mathcal{V}(\Lambda)^{\bar g}$ of a finite order automorphism $\bar g$ on a generalized Deligne–Lusztig variety $\mathcal{V}(\Lambda)$ (§4.1), which becomes purely an algebraic geometry problem over the *residue field k*. When $p > c$, it further simplifies to a more elementary problem of determining the fixed point scheme of a finite order automorphism $\bar g \in \mathrm{GL}_{d+1}(k)$ on a projective space $\mathbb{P}^d$ over k (§4.2). This elementary problem has an answer in terms of the sizes of the Jordan blocks of $\bar g$ (Lemma 4.3.4), which explains conceptually why the intersection multiplicity should be equal to c. Notice that our method completely avoids computation within Zink's theory, and it would be interesting to explore the possibility to remove the assumption $p > c$ using this method.

1.3.2. *Theorem B.* The original proof of Theorem B relies on showing two things (by [RTZ13, Lemma 10.2]): (1) the minuscule special cycle $\mathcal{Z}(\mathbf{v})$ has no W/p^2-points and (2) its special fiber $\mathcal{Z}(\mathbf{v})_k$ is regular. Step (1) is relatively easy using Grothendieck–Messing theory. Step (2) is more difficult: for super-general points x on $\mathcal{Z}(\mathbf{v})_k$, the regularity is shown by explicitly computing the local equation of $\mathcal{Z}(\mathbf{v})_k$ at x using Zink's theory; for non-super-general points, the regularity is shown using induction and reduces to the regularity of certain special divisors, whose the local equations can again be explicitly computed using Zink's theory.

Our new proof of Theorem B does not use Zink's theory either and involves little explicit computation. Our key observation is that to show both (1) and (2), it suffices to consider the thickenings $\mathcal{O}$ of k which are objects of the crystalline site of k. These $\mathcal{O}$-points of $\mathcal{Z}(\mathbf{v})$ then can be understood using only Grothendieck–Messing theory (Theorem 3.1.3). We prove a slight generalization of (1) which applies to possibly non-minuscule special cycles (Corollary 3.2.1). We then prove the tangent space of the minuscule special cycle $\mathcal{Z}(\mathbf{v})_k$ has the expected dimension (Corollary 3.2.2). The desired regularity (2) follows immediately.

1.3.3. Our new proofs are largely inspired by our previous work on arithmetic intersections on GSpin Rapoport–Zink spaces [LZ17]. The GSpin Rapoport–Zink spaces considered in [LZ17] are not of PEL type, which makes them technically more complicated. So the unitary case treated here can serve as a guide to [LZ17]. We have tried to indicate similarities between certain statements and proofs, for both clarity and the convenience of the readers.

1.4. **Structure of the paper.** In §2, we recall necessary backgrounds on unitary Rapoport–Zink spaces and the formulation of the arithmetic intersection problem. In §3, we study the local structure of the minuscule special cycles and prove Theorem B. In §4, we provide an alternative moduli interpretation of the generalized Deligne–Lusztig variety $\mathcal{V}(\Lambda)$ and prove Theorem A.

1.5. **Acknowledgments.** We are very grateful to M. Rapoport and W. Zhang for helpful conversations and comments. Our debt to the paper [RTZ13] should be clear to the readers.

2. Unitary Rapoport–Zink spaces

In this section we review the structure of unitary Rapoport-Zink spaces. We refer to [Vol10], [VW11] and [KR11] for the proofs of these facts.

2.1. Unitary Rapoport–Zink spaces. Let p be an odd prime. Let $F = \mathbb{Q}_p$, $k = \bar{\mathbb{F}}_p$, $W = W(k)$ and $K = W[1/p]$. Let σ be the p-Frobenius acting on $\bar{\mathbb{F}}_p$, and we also denote by σ the canonical lift of the p-Frobenius to W and K. For any $\mathbb{F}_p$-algebra R, we also denote by σ the Frobenius $x \mapsto x^p$ on R.

Let $E = \mathbb{Q}_{p^2}$ be the unramified quadratic extension of F. Fix a $\mathbb{Q}_p$-algebra embedding $\phi_0 : \mathcal{O}_E \hookrightarrow W$ and denote by ϕ_1 the embedding $\sigma \circ \phi_0 : \mathcal{O}_E \hookrightarrow W$. The embedding ϕ_0 induces an embedding between the residue fields $\mathbb{F}_{p^2} \hookrightarrow k$, which we shall think of as the natural embedding. For any $\mathcal{O}_E$-module Λ we shall write Λ_W for $\Lambda \otimes_{\mathcal{O}_E, \phi_0} W$.

Let r, s be positive integers and let $n = r + s$. We denote by $\mathcal{N}_{r,s}$ the unitary Rapoport–Zink spaces of signature (r, s), a formally smooth formal W-scheme, parameterizing deformations up to quasi-isogeny of height 0 of unitary p-divisible groups of signature (r, s). More precisely, for a W-scheme S, a *unitary p-divisible groups* of signature (r, s) over S is a triple (X, ι, λ), where

(1) X is a p-divisible group of dimension n and height $2n$ over S,
(2) $\iota : \mathcal{O}_E \to \mathrm{End}(X)$ is an action satisfying the signature (r, s) condition, i.e., for $\alpha \in \mathcal{O}_E$,

$$\mathrm{char}(\iota(\alpha) : \mathrm{Lie}\, X)(T) = (T - \phi_0(\alpha))^r (T - \phi_1(\sigma))^s \in \mathcal{O}_S[T],$$

(3) $\lambda : X \to X^\vee$ is a principal polarization such that the associated Rosati involution induces $\alpha \mapsto \sigma(\alpha)$ on $\mathcal{O}_E$ via ι.

Over k, there is a unique such triple (X, ι, λ) such that X is supersingular, up to $\mathcal{O}_E$-linear isogeny preserving the polarization up to scalar. Fix such a framing triple and denote it by $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$.

Let Nilp_W be the category of W-schemes on which p is locally nilpotent. Then the *unitary Rapoport–Zink space $\mathcal{N}_{r,s}$* represents the functor $\mathrm{Nilp}_W \to \mathbf{Sets}$ which sends $S \in \mathrm{Nilp}_W$ to the set of isomorphism classes of quadruples $(X, \iota, \lambda, \rho)$, where (X, ι, λ) is a unitary p-divisible group over S of signature (r, s) and $\rho : X \times_S S_k \to \mathbb{X} \times_k S_k$ is an $\mathcal{O}_E$-linear quasi-isogeny of height zero which respects λ and $\lambda_{\mathbb{X}}$ up to a scalar $c(\rho) \in \mathcal{O}_F^\times = \mathbb{Z}_p^\times$ (i.e., $\rho^\vee \circ \lambda_{\mathbb{X}} \circ \rho = c(\rho) \cdot \lambda$).

In the following we denote $\mathcal{N} := \mathcal{N}_{1,n-1}$, $\mathcal{M} := \mathcal{N}_{1,n-2}$ and $\bar{\mathcal{N}}_0 := \mathcal{N}_{0,1} \cong \mathrm{Spf}\, W$. They have relative dimension $n - 1$, n and 0 over $\mathrm{Spf}\, W$ respectively. We denote by $\bar{\mathbb{Y}} = (\bar{\mathbb{Y}}, \iota_{\bar{\mathbb{Y}}}, \lambda_{\bar{\mathbb{Y}}})$ the framing object for $\bar{\mathcal{N}}_0$ and denote by $\bar{Y} = (\bar{Y}, \iota_{\bar{Y}}, \lambda_{\bar{Y}})$ the universal p-divisible group over $\bar{\mathcal{N}}_0$. We may and shall choose framing objects $\mathbb{X} = (\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ and $\mathbb{X}^\flat = (\mathbb{X}^\flat, \iota_{\mathbb{X}^\flat}, \lambda_{\mathbb{X}^\flat})$ for $\mathcal{N}$ and $\mathcal{M}$ respectively such that

$$\mathbb{X} = \mathbb{X}^\flat \times \bar{\mathbb{Y}}$$

as unitary p-divisible groups.

2.2. The group J. The covariant Dieudonné module $M = \mathbb{D}(\mathbb{X})$ of the framing unitary p-divisible group is a free W-module of rank $2n$ together with an $\mathcal{O}_E$-action (induced by ι) and a perfect symplectic W-bilinear form $\langle \cdot, \cdot \rangle : M \times M \to W$ (induced by λ), cf. [VW11, §2.3]. Let $N = M \otimes_W K$ be the associated isocrystal and extend $\langle \cdot, \cdot \rangle$ to N bilinearly. Let F, V be the usual operators on N. We have

(2.2.0.1) $$\langle Fx, Fy \rangle = p\sigma(\langle x, y \rangle), \quad \forall x, y \in N.$$

The E-action decomposes N into a direct sum of two K-vector spaces of dimension n,

(2.2.0.2) $$N = N_0 \oplus N_1,$$

where the action of E on N_i is induced by the embedding ϕ_i. Both N_0 and N_1 are totally isotropic under the symplectic form. The operator F is of degree one and induces a σ-linear bijection $N_0 \xrightarrow{\sim} N_1$. Since the isocrystal N is supersingular, the degree 0 and σ^2-linear operator

$$\Phi = V^{-1}F = p^{-1}F^2$$

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has all slopes zero ([KR11, §2.1]). We have a K-vector space N_0 together with a σ^2-linear automorphism Φ.[1] The fixed points

$$C = N_0^{\Phi}$$

is an E-vector space of dimension n and $N_0 = C \otimes_{E,\phi_0} K$. Fix $\delta \in \mathcal{O}_E^{\times}$ such that $\sigma(\delta) = -\delta$. Define a non-degenerate σ-sesquilinear form on N_0 by

$$(2.2.0.3) \qquad\qquad\qquad \{x, y\} := (p\delta)^{-1} \langle x, Fy \rangle.$$

Using (2.2.0.1) it is easy to see that

$$(2.2.0.4) \qquad\qquad\qquad \sigma(\{x, y\}) = \{\Phi y, x\}, \quad \forall x, y \in N_0.$$

In particular, when restricted to C, the form $\{\cdot, \cdot\}$ is σ-Hermitian, namely

$$(2.2.0.5) \qquad\qquad\qquad \sigma(\{x, y\}) = \{y, x\}, \quad \forall x, y \in C.$$

In fact, $(C, \{\cdot, \cdot\})$ is the unique (up to isomorphism) non-degenerate non-split σ-Hermitian E-space of dimension n. Let $J = \mathrm{U}(C)$ be the unitary group of $(C, \{\cdot, \cdot\})$. It is an algebraic group over $F = \mathbb{Q}_p$. By Dieudonné theory, the group $J(\mathbb{Q}_p)$ can be identified with the automorphism group of the framing unitary p-divisible group $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ and hence acts on the Rapoport–Zink space $\mathcal{N}$.

2.3. **Special homomorphisms.** By definition, the *space of special homomorphisms* is the $\mathcal{O}_E$-module $\mathrm{Hom}_{\mathcal{O}_E}(\overline{\mathbb{Y}}, \mathbb{X})$. There is a natural $\mathcal{O}_E$-valued σ-Hermitian form on $\mathrm{Hom}_{\mathcal{O}_E}(\overline{\mathbb{Y}}, \mathbb{X})$ given by

$$(x, y) \mapsto \lambda_{\overline{\mathbb{Y}}}^{-1} \circ \hat{y} \circ \lambda_{\mathbb{X}} \circ x \in \mathrm{End}_{\mathcal{O}_E}(\overline{\mathbb{Y}}) \xrightarrow{\sim} \mathcal{O}_E.$$

By [KR11, Lemma 3.9], there is an isomorphism of σ-Hermitian E-spaces

$$(2.3.0.6) \qquad\qquad\qquad \mathrm{Hom}_{\mathcal{O}_E}(\overline{\mathbb{Y}}, \mathbb{X}) \otimes_{\mathcal{O}_E} E \xrightarrow{\sim} C.$$

Therefore we may view elements of C as special quasi-homomorphisms.

2.4. **Vertex lattices.** For any $\mathcal{O}_E$-lattice $\Lambda \subset C$, we define the dual lattice $\Lambda^{\vee} := \{x \in C : \{x, \Lambda\} \subseteq \mathcal{O}_E\}$. It follows from the σ-Hermitian property (2.2.0.5) that we have $(\Lambda^{\vee})^{\vee} = \Lambda$.

A *vertex lattice* is an $\mathcal{O}_E$-lattice $\Lambda \subseteq C$ such that $p\Lambda \subseteq \Lambda^{\vee} \subseteq \Lambda$. Such lattices correspond to the vertices of the Bruhat–Tits building of the unitary group $\mathrm{U}(C)$. Fix a vertex lattice Λ. The *type* of Λ is defined to be $t_{\Lambda} := \dim_{\mathbb{F}_{p^2}} \Lambda/\Lambda^{\vee}$, which is always an *odd* integer such that $1 \leq t_{\Lambda} \leq n$ (cf. [Vol10, Remark 2.3]).

We define $\Omega_0(\Lambda) := \Lambda/\Lambda^{\vee}$ and equip it with the perfect σ-Hermitian form

$$(\cdot, \cdot) : \Omega_0(\Lambda) \times \Omega_0(\Lambda) \to \mathbb{F}_{p^2}, \quad (x, y) := p \{\tilde{x}, \tilde{y}\} \bmod p,$$

where $\{\cdot, \cdot\}$ is the Hermitian form on C defined in (2.2.0.3), and $\tilde{x}, \tilde{y} \in \Lambda$ are lifts of x, y.

We define

$$\Omega(\Lambda) := \Omega_0(\Lambda) \otimes_{\mathbb{F}_{p^2}} k.$$

Remark 2.4.1. To compare with the definitions in [Vol10], our $\Omega_0(\Lambda)$ is the space V in [Vol10, (2.11)], and our pairing $(\cdot, \cdot)$ differs from the pairing $(\cdot, \cdot)$ defined loc. cit. by a factor of the reduction $\bar{\delta} \in \mathbb{F}_{p^2}^{\times}$ of δ.

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[1]Such a pair (N_0, Φ) is sometimes called a relative isocrystal.

2.5. The variety $\mathcal{V}(\Lambda)$. Let Λ be a vertex lattice and let $\Omega_0 = \Omega_0(\Lambda)$. Recall from §2.4 that Ω_0 is an $\mathbb{F}_{p^2}$-vector space whose dimension is equal to the type $t = t_\Lambda$ of Λ, an odd number. Let $d := (t-1)/2$. We define $\mathcal{V}(\Omega_0)$ to be the closed $\mathbb{F}_{p^2}$-subscheme of the Grassmannian $\mathrm{Gr}_{d+1}(\Omega_0)$ (viewed as a scheme over $\mathbb{F}_{p^2}$) such that for any $\mathbb{F}_{p^2}$-algebra R,

(2.5.0.1)
$$\mathcal{V}(\Omega_0)(R) = \{U \subseteq \Omega_0 \otimes_{\mathbb{F}_{p^2}} R : \ U \text{ is an } R\text{-module local direct summand of rank } d+1 \text{ such that } U^\perp \subseteq U\}.$$

Here $U^\perp$ is by definition $\{v \in \Omega_0 \otimes R : (v,u)_R = 0, \ \forall u \in U\}$, where $(\cdot, \cdot)_R$ is the R-sesquilinear form on $\Omega_0 \otimes R$ obtained from $(\cdot, \cdot)$ by extension of scalars (linearly in the first variable and σ-linearly in the second variable). Then $\mathcal{V}(\Omega_0)$ is a smooth projective $\mathbb{F}_{p^2}$-scheme of dimension d by [Vol10, Proposition 2.13] and Remark 2.4.1. In fact, $\mathcal{V}(\Omega_0)$ can be identified as a (generalized) Deligne–Lusztig variety, by [VW11, §4.5] (though we will not use this identification in the following).

We write $\mathcal{V}(\Lambda)$ for the base change of $\mathcal{V}(\Omega_0)$ from $\mathbb{F}_{p^2}$ to k.

2.6. Structure of the reduced scheme $\mathcal{N}^{\mathrm{red}}$. For each vertex lattice $\Lambda \subseteq C$, we define $\mathcal{N}_\Lambda \subseteq \mathcal{N}$ to be the locus where $\rho_X^{-1} \circ \Lambda^\vee \subseteq \mathrm{Hom}(\bar{Y}, X)$, i.e., the quasi-homomorphisms $\rho^{-1} \circ v$ lift to actual homomorphisms for any $v \in \Lambda^\vee$. Then $\mathcal{N}_\Lambda$ is a closed formal subscheme by [RZ96, Proposition 2.9]. By [VW11, §4] we have an isomorphism of k-varieties

(2.6.0.2)
$$\mathcal{N}_\Lambda^{\mathrm{red}} \xrightarrow{\sim} \mathcal{V}(\Lambda).$$

2.7. Some invariants associated to a k-point of $\mathcal{N}$. We follow [KR11, §2.1].

Let x be a point in $\mathcal{N}(k)$. Then x represents a tuple $(X, \iota, \lambda, \rho)$ over k as recalled in §2.1. Via ρ, we view the Dieudonné module of X as a W-lattice M_x in N, which is stable under the operators F and V. The endomorphism structure ι induces an action of $\mathcal{O}_E \otimes_{\mathbb{Z}_p} W \cong W \oplus W$ on M_x, which is equivalent to the structure of a $\mathbb{Z}/2\mathbb{Z}$-grading on M_x (into W-modules). We denote this grading by

$$M_x = \mathrm{gr}_0 M_x \oplus \mathrm{gr}_1 M_x.$$

This grading is compatible with (2.2.0.2) in the sense that

$$\mathrm{gr}_i M_x = M \cap N_i, \ i = 0, 1.$$

Moreover both $\mathrm{gr}_0 M_x$ and $\mathrm{gr}_1 M_x$ are free W-modules of rank n.

Consider the k-vector space $M_{x,k} := M_x \otimes_W k$. It has an induced $\mathbb{Z}/2\mathbb{Z}$-grading, as well as a canonical filtration $\mathrm{Fil}^1(M_{x,k}) \subset M_{x,k}$. Explicitly, $\mathrm{Fil}^1(M_{x,k})$ is the image of $V(M_x) \subseteq M_x$ under the reduction map $M_x \to M_{x,k}$. Define

$$\mathrm{Fil}^1(\mathrm{gr}_i M_{x,k}) := \mathrm{Fil}^1(M_{x,k}) \cap \mathrm{gr}_i M_{x,k}.$$

Then

$$\mathrm{Fil}^1(M_{x,k}) = \mathrm{Fil}^1(\mathrm{gr}_0 M_{x,k}) \oplus \mathrm{Fil}^1(\mathrm{gr}_1 M_{x,k}),$$

and by the signature $(1, n-1)$ condition we know that $\mathrm{Fil}^1(\mathrm{gr}_0 M_{x,k})$ (resp. $\mathrm{Fil}^1(\mathrm{gr}_1 M_{x,k})$) is a hyperplane (resp. line) in $\mathrm{gr}_0 M_{x,k}$ (resp. $\mathrm{gr}_1 M_{x,k}$).

The symplectic form $\langle \cdot, \cdot \rangle$ on N takes values in W on M_x, and hence induces a symplectic form on $M_{x,k}$ by reduction. The latter restricts to a k-bilinear non-degenerate pairing

$$\mathrm{gr}_0 M_{x,k} \times \mathrm{gr}_1 M_{x,k} \to k.$$

Under the above pairing, the spaces $\mathrm{Fil}^1(\mathrm{gr}_0 M_{x,k})$ and $\mathrm{Fil}^1(\mathrm{gr}_1 M_{x,k})$ are annihilators of each other. Equivalently, $\mathrm{Fil}^1(M_{x,k})$ is a totally isotropic subspace of $M_{x,k}$.

2.8. **Description of k-points by special lattices.** For a W-lattice A in N_0, we define its dual lattice to be $A^\vee := \{x \in N_0 : \{x, A\} \subseteq W\}$. If Λ is an $\mathcal{O}_E$-lattice in C, then we have $(\Lambda_W)^\vee = (\Lambda^\vee)_W$. In the following we denote both of them by $\Lambda_W^\vee$.

Definition 2.8.1. A *special lattice* is a W-lattice A in N_0 such that

$$A^\vee \subseteq A \subseteq p^{-1} A^\vee$$

and such that $A/A^\vee$ is a one-dimensional k-vector space.

Remark 2.8.2. The apparent difference between the above definition and the condition in [Vol10, Proposition 1.10] (for $i = 0$) is caused by the fact that we have normalized the pairing $\{\cdot, \cdot\}$ on N_0 differently from loc. cit., using an extra factor $(p\delta)^{-1}$ (cf. (2.2.0.3)). Our normalization is the same as that in [RTZ13].

Recall the following result from [Vol10].

Proposition 2.8.3 ([Vol10, Proposition 1.10]). *There is a bijection from $\mathcal{N}(k)$ to the set of special lattices, sending a point x to $\mathrm{gr}_0 M_x$ considered in §2.7.* $\qquad\square$

Remark 2.8.4. Let $x \in \mathcal{N}(k)$ and let A be the special lattice associated to it by Proposition 2.8.3. Let Λ be a vertex lattice. Then $x \in \mathcal{N}_\Lambda(k)$ if and only if $A \subseteq \Lambda_W$, if and only if $\Lambda_W^\vee \subseteq A^\vee$. (See also Remark 3.1.5 below.)

2.9. **Filtrations.** We introduce the following notation:

Definition 2.9.1. Let A be a special lattice. Write $A_k := A \otimes_W k$. Let $x \in \mathcal{N}(k)$ correspond to A under Proposition 2.8.3. Thus $A_k = \mathrm{gr}_0 M_{x,k}$. Define $\mathrm{Fil}^1(A_k) := \mathrm{Fil}^1(\mathrm{gr}_0 M_{x,k})$ (cf. §2.7). It is a hyperplane in A_k.

Lemma 2.9.2. *Let A be a special lattice. Then $\Phi^{-1}(A^\vee)$ is contained inside A, and its image in A_k is equal to $\mathrm{Fil}^1(A_k)$.*

Proof. Let A correspond to $x \in \mathcal{N}(k)$ under Proposition 2.8.3. Then F, V both preserve the W-lattice M_x in N (cf. §2.7). By definition, $\mathrm{Fil}^1(M_{x,k})$ is the image of $V(M_x) \subseteq M_x$ under the reduction map $M_x \to M_{x,k}$. Since the operator V is of degree 1 with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading, we see that $\mathrm{Fil}^1(A_k)$ is the image of $V(\mathrm{gr}_1 M_x) \subseteq A$ under $A \to A_k$. It suffices to prove that

$$(2.9.2.1) \qquad\qquad \Phi^{-1}(A^\vee) = V(\mathrm{gr}_1 M_x).$$

By the proof of [Vol10, Proposition 1.10], we have $\mathrm{gr}_1 M_x = F^{-1} A^\vee$. (Note that because of the difference of normalizations as discussed in Remark 2.8.2, what is denoted by $A^\vee$ here is denoted by $pA^\vee$ in [Vol10]. Also note that the integer i appearing loc. cit. is 0 in our case.) Therefore $V(\mathrm{gr}_1 M_x) = V(F^{-1} A^\vee)$. But $VF^{-1} = (V^{-1} F)^{-1} = \Phi^{-1}$ because $VF = FV = p$. Thus (2.9.2.1) holds as desired. $\qquad\square$

2.10. **Special cycles.** Let $\mathbf{v}$ be an arbitrary subset of C. We define the *special cycle* $\mathcal{Z}(\mathbf{v}) \subseteq \mathcal{N}$ to be the locus where $\rho^{-1} \circ v \in \mathrm{Hom}(\bar{Y}, X)$ for all $v \in \mathbf{v}$, i.e., all the quasi-homomorphisms $\rho^{-1} \circ v$ lift to actual homomorphisms. Note that $\mathcal{Z}(\mathbf{v})$ only depends on the $\mathcal{O}_E$-submodule $L(\mathbf{v})$ spanned by $\mathbf{v}$ in C, and we have $\mathcal{Z}(\mathbf{v}) = \mathcal{Z}(L(\mathbf{v}))$.

We say $\mathbf{v}$ is *minuscule* if $L(\mathbf{v})$ is an $\mathcal{O}_E$-lattice in C satisfying $pL(\mathbf{v})^\vee \subseteq L(\mathbf{v}) \subseteq L(\mathbf{v})^\vee$, or equivalently, if $L(\mathbf{v})$ is the dual of a vertex lattice. When this is the case we have $\mathcal{Z}(\mathbf{v}) = \mathcal{N}_{L(\mathbf{v})^\vee}$ by definition.

2.11. The intersection problem. Let $C^\flat$ be the analogue for $\mathcal{M}$ of the Hermitian space C. Then $C \cong C^\flat \oplus Eu$ for some vector denoted by u which is of norm 1 and orthogonal to $C^\flat$. We have a closed immersion

$$\delta : \mathcal{M} \to \mathcal{N},$$

sending $(X, \iota, \lambda, \rho)$ to $(X \times \bar{Y}, \iota \times \iota_{\bar{Y}}, \lambda \times \lambda_{\bar{Y}}, \rho \times \mathrm{id})$. We have $\delta(\mathcal{M}) = \mathcal{Z}(u)$. The closed immersion δ induces a closed immersion of formal schemes

$$(\mathrm{id}, \delta) : \mathcal{M} \to \mathcal{M} \times_W \mathcal{N}.$$

Denote by Δ the image of (id, δ), which we call the (local) *GGP cycle*. For any $g \in J(\mathbb{Q}_p)$, we obtain a formal subscheme

$$(\mathrm{id} \times g)\Delta \subseteq \mathcal{M} \times_W \mathcal{N},$$

via the action of g on $\mathcal{N}$. Let $g \in J(\mathbb{Q}_p)$ and let $\mathcal{N}^g \subseteq \mathcal{N}$ be the fixed locus of g. Then by definition we have

$$\Delta \cap (\mathrm{id} \times g)\Delta \cong \delta(\mathcal{M}) \cap \mathcal{N}^g.$$

Our goal is to compute the arithmetic intersection number

$$\langle \Delta, (\mathrm{id} \times g)\Delta \rangle,$$

when g is *regular semisimple* and *minuscule* (as defined in the introduction). Notice that $g \in J(\mathbb{Q}_p)$ is regular semisimple if and only if $\mathbf{v}(g) := (u, gu, \ldots, g^{n-1}u)$ is an E-basis of C. Also notice that a regular semisimple element g is minuscule if and only if $\mathbf{v}(g)$ is minuscule in the sense of §2.10.

3. Reducedness of minuscule special cycles

3.1. **Local structure of special cycles.**

Definition 3.1.1. Let $\mathscr{C}$ be the following category:

- Objects in $\mathscr{C}$ are triples $(\mathcal{O}, \mathcal{O} \to k, \delta)$, where $\mathcal{O}$ is a local Artinian W-algebra, $\mathcal{O} \to k$ is a W-algebra map, and δ is a nilpotent divided power structure on $\ker(\mathcal{O} \to k)$ (cf. [BO78, Definitions 3.1, 3.27]).
- Morphisms in $\mathscr{C}$ are W-algebra maps that are compatible with the structure maps to k and the divided power structures.

3.1.2. Let $x \in \mathcal{N}(k)$ correspond to a special lattice A under Proposition 2.8.3. Let $\mathcal{O} \in \mathscr{C}$. By a *hyperplane* in $A_\mathcal{O} := A \otimes_W \mathcal{O}$ we mean a free direct summand of $A_\mathcal{O}$ of rank $n - 1$. We define the $\mathbb{Z}/2\mathbb{Z}$-grading on $M_{x,\mathcal{O}} := M_x \otimes_W \mathcal{O}$ by linearly extending that on M_x (cf. §2.7). Denote by $\widehat{\mathcal{N}}_x$ the completion of $\mathcal{N}$ at x. For any $\tilde{x} \in \widehat{\mathcal{N}}_x(\mathcal{O})$, we have a unitary p-divisible group of signature $(1, n-1)$ over $\mathcal{O}$ deforming that over k defined by x. By Grothendieck-Messing theory, we obtain the Hodge filtration $\mathrm{Fil}^1_{\tilde{x}} M_{x,\mathcal{O}} \subseteq M_{x,\mathcal{O}}$. Define $f_\mathcal{O}(\tilde{x})$ to be the intersection

$$\mathrm{Fil}^1_{\tilde{x}} M_{x,\mathcal{O}} \cap \mathrm{gr}_0 M_{x,\mathcal{O}}$$

inside $M_{x,\mathcal{O}}$. By the signature $(1, n-1)$ condition, $f_\mathcal{O}(\tilde{x})$ is a hyperplane in $A_\mathcal{O}$. It also lifts $\mathrm{Fil}^1 A_k$ (cf. Definition 2.9.1) by construction. Thus we have defined a map

(3.1.2.1)
$$f_\mathcal{O} : \widehat{\mathcal{N}}_x(\mathcal{O}) \xrightarrow{\sim} \{\text{hyperplanes in } A_\mathcal{O} \text{ lifting } \mathrm{Fil}^1 A_k\}.$$

By construction, $f_\mathcal{O}$ is functorial in $\mathcal{O}$ in the sense that the collection $(f_\mathcal{O})_{\mathcal{O} \in \mathscr{C}}$ is a natural transformation between two set-valued functors on $\mathscr{C}$. Here we are viewing the right hand side of (3.1.2.1) as a functor in $\mathcal{O}$ using the base change maps.

The following result is the analogue of [LZ17, Theorem 4.1.7]. As a direct consequence of the PEL moduli problem, it should be well known to the experts and is essentially proved in [KR11, Proposition 3.5].

Theorem 3.1.3. *Keep the notations in §3.1.2.*

(1) The natural transformation $(f_\mathcal{O})_{\mathcal{O} \in \mathscr{C}}$ is an isomorphism.

(2) Let $\mathbf{v}$ be a subset of C. If $x \in \mathcal{Z}(\mathbf{v})(k)$, then $\mathbf{v} \subseteq A$. Suppose $x \in \mathcal{Z}(\mathbf{v})(k)$. Then for any $\mathcal{O} \in \mathscr{C}$ the map $f_\mathcal{O}$ induces a bijection

$$\widehat{\mathcal{Z}(\mathbf{v})}_x(\mathcal{O}) \xrightarrow{\sim}$$

$\{$hyperplanes in $A_\mathcal{O}$ lifting $\mathrm{Fil}^1 A_k$ and containing the image of $\mathbf{v}$ in $A_\mathcal{O}$ $\}$.

Proof. **(1)** We need to check that for all $\mathcal{O} \in \mathscr{C}$ the map $f_\mathcal{O}$ is a bijection. Let $\tilde{x} \in \widehat{\mathcal{N}}_x(\mathcal{O})$. This represents a deformation over $\mathcal{O}$ of the p-divisible group at x. Similarly to the situation in §2.7, the compatibility with the endomorphism structure implies that

$$\mathrm{Fil}^1_{\tilde{x}} M_{x,\mathcal{O}} = \bigoplus_{i=0}^1 \mathrm{Fil}^1_{\tilde{x}} M_{x,\mathcal{O}} \cap \mathrm{gr}_i M_{x,\mathcal{O}}.$$

By the compatibility with the polarization, we know that $\mathrm{Fil}^1_{\tilde{x}} M_{x,\mathcal{O}}$ is totally isotropic under the symplectic form on $M_{x,\mathcal{O}}$. It follows that the two modules $\mathrm{Fil}^1_{\tilde{x}} M_{x,\mathcal{O}} \cap \mathrm{gr}_1 M_{x,\mathcal{O}}$ and $\mathrm{Fil}^1_{\tilde{x}} M_{x,\mathcal{O}} \cap \mathrm{gr}_0 M_{x,\mathcal{O}}$ are annihilators of each other if we identify $\mathrm{gr}_1 M_{x,\mathcal{O}}$ as the $\mathcal{O}$-linear dual of $\mathrm{gr}_0 M_{x,\mathcal{O}}$ using the symplectic form on $M_{x,\mathcal{O}}$. Therefore, $\mathrm{Fil}^1_{\tilde{x}} M_{x,\mathcal{O}}$ can be recovered from $f_\mathcal{O}(\tilde{x})$. This together with Grothendieck-Messing theory proves the injectivity of $f_\mathcal{O}$. The surjectivity of $f_\mathcal{O}$ also follows from Grothendieck-Messing theory and the above way of reconstructing $\mathrm{Fil}^1_{\tilde{x}} M_{x,\mathcal{O}}$ from its intersection with $\mathrm{gr}_0 M_{x,\mathcal{O}}$. Note that the unitary p-divisible groups reconstructed in this way do satisfy the signature condition because we have started with *hyperplanes* in $A_\mathcal{O}$.

(2) The statements follow from the proof of [KR11, Proposition 3.5] and the definition of (2.3.0.6) in [KR11, Lemma 3.9]. We briefly recall the arguments here. If $\phi \in \mathrm{Hom}_{\mathcal{O}_E}(\overline{\mathbb{Y}}, \mathbb{X}) \otimes_{\mathcal{O}_E} E$ is a special quasi-homomorphism, the element $v \in C$ corresponding to ϕ under (2.3.0.6) is by definition the projection to N_0 of $\phi_*(\bar{1}_0) \in N$, where ϕ_* is the map $\mathbb{D}(\overline{\mathbb{Y}}) \otimes_W K \to \mathbb{D}(\mathbb{X}) \otimes_W K = N$ induced by ϕ, and $\bar{1}_0$ is a certain fixed element in $\mathbb{D}(\overline{\mathbb{Y}})$. In fact, $\bar{1}_0$ is chosen such that

- $W\bar{1}_0 = \mathrm{gr}_0 \mathbb{D}(\overline{\mathbb{Y}})$, where the grading is with respect to the $\mathcal{O}_E$-action on $\overline{\mathbb{Y}}$,
- $W\bar{1}_0 = \mathrm{Fil}^1_{\bar{Y}} \mathbb{D}(\overline{\mathbb{Y}})$, the Hodge filtration for the deformation $\bar{Y}$ of $\overline{\mathbb{Y}}$ over W.

In particular v and ϕ are related by the formula $v = \phi_*(\bar{1}_0)$, as the projection to N_0 is not needed.

From now on we assume without loss of generality that $\mathbf{v} = \{v\}$, with v corresponding to ϕ as in the above paragraph. If $x \in \mathcal{Z}(v)(k)$, then ϕ_* has to map $\mathbb{D}(\overline{\mathbb{Y}})$ into M_x, so $v \in M_x$. Since ϕ_* is compatible with the $\mathbb{Z}/2\mathbb{Z}$-gradings, we further have $v \in A$. We have shown that if $x \in \mathcal{Z}(v)(k)$, then $v \in A$.

Now suppose $x \in \mathcal{Z}(\mathbf{v})(k)$. Let $\mathcal{O} \in \mathscr{C}$. Write $v_\mathcal{O} := v \otimes 1 \in A_\mathcal{O} \subset M_{x,\mathcal{O}}$. For all $\tilde{x} \in \widehat{\mathcal{N}}_x(\mathcal{O})$, by Grothendieck-Messing theory we know that $\tilde{x} \in \widehat{\mathcal{Z}(v)}_x(\mathcal{O})$ if and only if the base change of ϕ_* to $\mathcal{O}$ (still denoted by ϕ_*) preserves the Hodge filtrations, i.e.

$$\phi_*(\mathrm{Fil}^1_{\bar{Y}} \mathbb{D}(\overline{\mathbb{Y}})) \subseteq \mathrm{Fil}^1_{\tilde{x}} M_{x,\mathcal{O}}.$$

Since $W\bar{1}_0 = \mathrm{Fil}^1_{\bar{Y}} \mathbb{D}(\overline{\mathbb{Y}})$, this last condition is equivalent to $v_\mathcal{O} \in \mathrm{Fil}^1_{\tilde{x}} M_{x,\mathcal{O}}$. Again, because ϕ_* is compatible with the $\mathbb{Z}/2\mathbb{Z}$-gradings, the last condition is equivalent to $v_\mathcal{O} \in f_\mathcal{O}(\tilde{x})$. In conclusion, we have shown that $\tilde{x} \in \widehat{\mathcal{N}}_x(\mathcal{O})$ is in $\widehat{\mathcal{Z}(v)}_x(\mathcal{O})$ if and only if $v_\mathcal{O} \in f_\mathcal{O}(\tilde{x})$, as desired. $\square$

Corollary 3.1.4. *Let $x \in \mathcal{N}(k)$ correspond to the special lattice A. Let $\mathbf{v}$ be a subset of C. Then $x \in \mathcal{Z}(\mathbf{v})(k)$ if and only if $\mathbf{v} \subseteq A^\vee$.*

Proof. By part (2) of Theorem 3.1.3 applied to $\mathcal{O} = k$, we see that $x \in \mathcal{Z}(\mathbf{v})(k)$ if and only if $\mathbf{v} \subseteq A$ and the image of $\mathbf{v}$ in A_k is contained in $\mathrm{Fil}^1(A_k)$. The corollary follows from Lemma 2.9.2 and the Φ-invariance of $\mathbf{v}$. $\square$

Remark 3.1.5. Note that Remark 2.8.4 is a special case of Corollary 3.1.4.

3.2. **Proof of the reducedness.**

Corollary 3.2.1. *Let* Λ *be an* $\mathcal{O}_E$*-lattice in* C *with* $p^i\Lambda \subseteq \Lambda^\vee \subseteq \Lambda$ *for some* $i \in \mathbb{Z}_{\geq 1}$. *Then the special cycle* $\mathcal{Z}(\Lambda^\vee)$ *defined by* $\Lambda^\vee$ *has no* (W/p^{i+1})*-points. In particular, taking* $i = 1$ *we see that* $\mathcal{N}_\Lambda(W/p^2) = \emptyset$ *for any vertex lattice* Λ.

Proof. Let $\mathcal{O} = W/p^{i+1}$, equipped with the reduction map $W/p^{i+1} \to k$ and the natural divided power structure on the kernel $p\mathcal{O}$. Then $\mathcal{O} \in \mathscr{C}$. Assume $\mathcal{Z}(\Lambda^\vee)$ has an $\mathcal{O}$-point $\tilde{x}$ reducing to a k-point x. Let A be the special lattice corresponding to x (cf. §2.8). By Theorem 3.1.3, there exists a hyperplane P in $A_\mathcal{O}$ lifting $\mathrm{Fil}^1(A_k)$, such that $P \supseteq \Lambda^\vee \otimes_{\mathcal{O}_E} \mathcal{O}$. Since P is a hyperplane in $A_\mathcal{O}$, there exists an element $l \in \mathrm{Hom}_\mathcal{O}(A_\mathcal{O}, \mathcal{O})$ such that

(3.2.1.1) $$l(P) = 0 \quad \text{and} \quad l(A_\mathcal{O}) = \mathcal{O}.$$

We may find an element $\tilde{l} \in A^\vee \subseteq N_0$ to represent l, in the sense that for all $a \otimes 1 \in A_\mathcal{O}$ with $a \in A$, we have

$$l(a \otimes 1) = \text{image of } \{a, \tilde{l}\} \text{ under } W \to \mathcal{O}.$$

Since $l(\Lambda^\vee \otimes \mathcal{O}) \subseteq l(P) = 0$, we know that $\{v, \tilde{l}\} \in p^{i+1}W$ for all $v \in \Lambda^\vee$. Since $\Lambda^\vee \subseteq C = N_0^\Phi$, applying (2.2.0.4) we see that $\{\tilde{l}, v\} \in p^{i+1}W$ for all $v \in \Lambda^\vee$. Therefore

$$p^{-i-1}\tilde{l} \in (\Lambda_W^\vee)^\vee = \Lambda_W,$$

and thus $\tilde{l} \in p^{i+1}\Lambda_W$, which is contained in $p\Lambda_W^\vee$ by hypothesis. Since Λ is Φ-invariant, we also have $\Phi(\tilde{l}) \in p\Lambda_W^\vee$. But $\Lambda_W^\vee \subseteq A^\vee$ by Corollary 3.1.4, so $\Phi(\tilde{l}) \in pA^\vee$. It follows that for all $a \in A$, we have $\{\Phi(\tilde{l}), a\} \in pW$, and therefore

$$\{a, \tilde{l}\} \xrightarrow{(2.2.0.4)} \sigma^{-1}(\{\Phi(\tilde{l}), a\}) \in pW$$

contradicting with the second condition in (3.2.1.1). $\square$

Corollary 3.2.2. *Let* Λ *be a vertex lattice of type* t *and let* $x \in \mathcal{N}_\Lambda(k)$. *Then the tangent space* $\mathcal{T}_x\mathcal{N}_{\Lambda,k}$ *to* $\mathcal{N}_{\Lambda,k}$ *at* x, *where* $\mathcal{N}_{\Lambda,k}$ *is the special fiber (i.e. base change to* k*) of* $\mathcal{N}_\Lambda$, *is of* k*-dimension* $(t-1)/2$.

Proof. This can be deduced from Theorem 3.1.3 elementarily, in the same way as in [LZ17, §4.2]. Here we provide a shorter proof. Firstly we make an easy observation. Denote by $\mathscr{C}_k$ the full subcategory of $\mathscr{C}$ consisting of characteristic p objects. Let $\mathcal{W}_1, \mathcal{W}_2$ be two formal schemes over k. Fix $y_i \in \mathcal{W}_i(k), i = 1, 2$. For $i = 1, 2$, define the set-valued functor $\mathcal{F}_i$ on $\mathscr{C}_k$ sending $\mathcal{O}$ to the set of $\mathcal{O}$-points of $\mathcal{W}_i$ which induce y_i under the structure map $\mathcal{O} \to k$. Assume $\mathcal{F}_1 \cong \mathcal{F}_2$. Then the tangent spaces $\mathcal{T}_{x_i}\mathcal{W}_i$ are isomorphic. In fact, this observation is a direct consequence of the definition of the vector space structure on the tangent spaces from the point of view of functor of points, as recalled in the proof of [LZ17, Lemma 4.2.6] for instance.

Denote by B the k-subspace of A_k spanned by the image of $\Lambda^\vee$ in A_k. Consider the Grassmannian $\mathrm{Gr}_{n-1}(A_k)$ parametrizing hyperplanes in the n-dimensional k-vector space A_k. Let $\mathcal{W}_1$ be the sub-variety of $\mathrm{Gr}_{n-1}(A_k)$ defined by the condition that the hyperplane should contain B, and let $y_1 \in \mathcal{W}_1(k)$ corresponding to $\mathrm{Fil}^1(A_k) \subseteq A_k$. Let $\mathcal{W}_2 := \mathcal{N}_{\Lambda,k}$ and $y_2 := x$. By Theorem 3.1.3, the assumption on $(\mathcal{W}_i, y_i), i = 1, 2$ in the previous paragraph is satisfied. Hence it suffices to compute the dimension of $\mathcal{T}_{y_1}\mathcal{W}_1$. Note that $\mathcal{W}_1$ is itself a Grassmannian, parametrizing hyperplanes in A_k/B. The proof is finished once we know that A_k/B has k-dimension $(t+1)/2$. But this is true by the (σ-linear) duality between the k-vector spaces $A_k/B = A/\Lambda_W^\vee$ and $\Lambda_W/A^\vee$ under the σ-sesquilinear form on $\Omega(\Lambda)$ obtained by extension of scalars from the σ-Hermitian form $(\cdot, \cdot)$ on $\Omega_0(\Lambda)$ (cf. §2.4) and the fact that $A/A^\vee$ is a 1-dimensional k-vector space (cf. Definition 2.8.1). $\square$

In the following corollary we re-prove [RTZ13, Theorems 9.4, 10.1].

Corollary 3.2.3. *Let Λ be a vertex lattice. Then $\mathcal{N}_\Lambda = \mathcal{N}_{\Lambda,k} = \mathcal{N}_\Lambda^{\mathrm{red}}$ and it is regular.*

Proof. Let t be the type of Λ. Recall from §2.6 that $\mathcal{N}_\Lambda^{\mathrm{red}}$ is a smooth k-scheme of dimension $(t-1)/2$. By Corollary 3.2.2, all the tangent spaces of $\mathcal{N}_{\Lambda,k}$ have k-dimension $(t-1)/2$, and so $\mathcal{N}_{\Lambda,k}$ is regular. In particular $\mathcal{N}_{\Lambda,k}$ is reduced, namely $\mathcal{N}_{\Lambda,k} = \mathcal{N}_\Lambda^{\mathrm{red}}$. Knowing that $\mathcal{N}_{\Lambda,k}$ is regular, and that $\mathcal{N}_\Lambda$ has no (W/p^2)-points (Corollary 3.2.1), it follows that $\mathcal{N}_\Lambda = \mathcal{N}_{\Lambda,k}$ by the general criterion [RTZ13, Lemma 10.3]. $\qquad\square$

4. The intersection length formula

4.1. **The arithmetic intersection as a fixed point scheme.** Fix a regular semisimple and minuscule element $g \in J(\mathbb{Q}_p)$. Let $L := L(\mathbf{v}(g))$ and $\Lambda := L^\vee$. They are both $\mathcal{O}_E$-lattices in C. Recall from the end of §2.10 that Λ is a vertex lattice and $\mathcal{Z}(L) = \mathcal{N}_\Lambda$. From now on we assume $\mathcal{N}^g(k) \neq \emptyset$. As shown in [RTZ13, §5], this assumption implies that both L and Λ are g-cyclic and stable under g. In particular, the natural action of g on $\mathcal{N}$ stabilizes $\mathcal{N}_\Lambda$.

Let $\Omega_0 = \Omega_0(\Lambda)$ and $\Omega = \Omega(\Lambda)$. Let $t = t_\Lambda$ and $d = (t-1)/2$ as in §2.5. Let $\bar{g} \in \mathrm{GL}(\Omega_0)(\mathbb{F}_{p^2})$ be the induced action of g on Ω_0. Then $\bar{g}$ preserves the Hermitian form $(\cdot,\cdot)$ on Ω_0 and hence acts on $\mathcal{V}(\Lambda)$. It is clear from the definition of the isomorphism (2.6.0.2) given in [VW11, §4] that it is equivariant for the actions of g and $\bar{g}$ on the two sides.

Remark 4.1.1. Since both Λ and $\Lambda^\vee$ are g-cyclic, the linear operator $\bar{g} \in \mathrm{GL}(\Omega_0)(\mathbb{F}_{p^2})$ has equal minimal polynomial and characteristic polynomial. Equivalently, in the Jordan normal form of $\bar{g}$ (over k) there is a unique Jordan block associated to any eigenvalue.

Proposition 4.1.2. $\delta(\mathcal{M}) \cap \mathcal{N}^g$ *is a scheme of characteristic p (i.e. a k-scheme) isomorphic to $\mathcal{V}(\Lambda)^{\bar{g}}$.*

Proof. Recall from §2.11 that $\delta(\mathcal{M}) = \mathcal{Z}(u)$. Since the $\mathcal{O}_E$-module L is generated by $u, gu, \cdots, g^{n-1}u$ and stable under g, we have $\delta(\mathcal{M}) \cap \mathcal{N}^g = \mathcal{Z}(L)^g = \mathcal{N}_\Lambda^g$. By Corollary 3.2.3, we know that $\mathcal{N}_\Lambda^g = (\mathcal{N}_\Lambda^{\mathrm{red}})^g$. But the latter is isomorphic to the characteristic p scheme $\mathcal{V}(\Lambda)^{\bar{g}}$ under (2.6.0.2). $\qquad\square$

4.2. **Study of $\mathcal{V}(\Lambda)^{\bar{g}}$.** We start with an alternative moduli interpretation of $\mathcal{V}(\Omega_0)$. The idea is to rewrite (in Lemma 4.2.2) the procedure of taking the complement $U \mapsto U^\perp$ with respect to the Hermitian form, in terms of taking the complement with respect to some quadratic form and taking a Frobenius. The alternative moduli interpretation is given in Corollary 4.2.3 below.

Let Θ_0 be a t-dimensional non-degenerate quadratic space over $\mathbb{F}_p$. Let $[\cdot,\cdot] : \Theta_0 \times \Theta_0 \to \mathbb{F}_p$ be the associated bilinear form. Since there is a unique isomorphism class of non-degenerate σ-Hermitian spaces over $\mathbb{F}_{p^2}$, we may assume that $\Omega_0 = \Theta_0 \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}$ and that the σ-Hermitian form $(\cdot,\cdot)$ (cf. §2.5) is obtained by extension of scalars (linearly in the first variable and σ-linearly in the second variable) from $[\cdot,\cdot]$.

Definition 4.2.1. Let R be an $\mathbb{F}_p$-algebra. We define $[\cdot,\cdot]_R$ to be the R-bilinear form on $\Theta_0 \otimes_{\mathbb{F}_p} R$ obtained from $[\cdot,\cdot]$ by extension of scalars. For any R-submodule $\mathcal{L} \subset \Theta_0 \otimes_{\mathbb{F}_p} R$, define

$$\mathcal{L}^{\mathrm{lin}\perp} := \left\{ v \in \Theta_0 \otimes_{\mathbb{F}_p} R : [v,l]_R = 0, \ \forall l \in \mathcal{L} \right\}.$$

Define $\sigma_*(\mathcal{L})$ to be the R-module generated by the image of $\mathcal{L}$ under the map

$$\sigma : \Theta_0 \otimes_{\mathbb{F}_p} R \to \Theta_0 \otimes_{\mathbb{F}_p} R, \quad v \otimes r \mapsto v \otimes r^p.$$

Let R be an $\mathbb{F}_{p^2}$-algebra. Let U be an R-submodule of $\Omega_0 \otimes_{\mathbb{F}_{p^2}} R$. Since $\Omega_0 \otimes_{\mathbb{F}_{p^2}} R = \Theta_0 \otimes_{\mathbb{F}_p} R$, we may view U as an R-submodule of the latter and define $\sigma_*(U)$ as in Definition 4.2.1.

Lemma 4.2.2. *We have $U^\perp = (\sigma_*(U))^{\mathrm{lin}\perp}$.*

Proof. Consider two arbitrary elements

$$x = \sum_j u_j \otimes r_j, \quad y = \sum_k v_k \otimes s_k$$

of $\Theta_0 \otimes_{\mathbb{F}_p} R$. We have

$$(y, x)_R = \sum_{j,k} s_k r_j^p \cdot [v_k, u_j]_R = \sum_{j,k} s_k r_j^p \cdot [u_j, v_k]_R = [\sigma(x), y]_R.$$

Hence for $y \in \Theta_0 \otimes_{\mathbb{F}_p} R$, we have $y \in U^\perp$ if and only if $(y, x)_R = 0$ for all $x \in U$, if and only if $[\sigma(x), y]_R = 0$ for all $x \in U$, if and only if $y \in (\sigma_*(U))^{\mathrm{lin}\perp}$. $\qquad\square$

Corollary 4.2.3. *For any $\mathbb{F}_{p^2}$-algebra R, the set $\mathcal{V}(\Omega_0)(R)$ is equal to the set of R-submodules U of*

$$\Omega_0 \otimes_{\mathbb{F}_{p^2}} R = \Theta_0 \otimes_{\mathbb{F}_p} R,$$

such that U is an R-module local direct summand of rank $d + 1$, satisfying

$$(\sigma_*(U))^{\mathrm{lin}\perp} \subseteq U.$$

Proof. This is a direct consequence of (2.5.0.1) and Lemma 4.2.2. $\qquad\square$

In the following we denote $\mathcal{V}(\Lambda)$ by $\mathcal{V}$ for simplicity, where Λ is always fixed as in the beginning of §4.1. Denote $\Theta := \Theta_0 \otimes_{\mathbb{F}_p} k$. Fix a point $x_0 \in \mathcal{V}^{\bar{g}}(k)$. Let U_0 correspond to x_0 under (2.5.0.1) or Corollary 4.2.3. Define

$$\mathcal{L}_{d+1} := U_0, \quad \mathcal{L}_d := (\sigma_*(U_0))^{\mathrm{lin}\perp} \xepsilon{\underline{\text{Lemma 4.2.2}}} U_0^\perp.$$

They are subspaces of Θ stable under $\bar{g}$, of k-dimensions $d + 1$ and d respectively.

Definition 4.2.4. Define $\mathcal{I} := \mathbb{P}(\Theta/\mathcal{L}_d)$, a projective space of dimension d over k.

Then $\mathcal{L}_{d+1}$ defines an element in $\mathcal{I}(k)$, which we still denote by x_0 by abuse of notation. We have a natural action of $\bar{g}$ on $\mathcal{I}$ that fixes x_0. Let $\mathcal{R}_p$ (resp. $\mathcal{S}_p$) be the quotient of the local ring of $\mathcal{I}^{\bar{g}}$ (resp. of $\mathcal{V}^{\bar{g}}$) at x_0 divided by the p-th power of its maximal ideal.

Lemma 4.2.5. *There is a k-algebra isomorphism $\mathcal{R}_p \cong \mathcal{S}_p$.*

Proof. The proof is based on exactly the same idea as [LZ17, Lemma 5.2.9]. Let $\tilde{\mathcal{R}}_p$ (resp. $\tilde{\mathcal{S}}_p$) be the quotient of the local ring of $\mathcal{I}$ (resp. of $\mathcal{V}$) at x_0 divided by the p-th power of its maximal ideal. Let R be an arbitrary local k-algebra with residue field k such that the p-th power of its maximal ideal is zero. Then by Lemma 4.2.2, the R-points of $\mathcal{V}$ lifting x_0 classify R-module local direct summands U of $\Theta \otimes_k R$ of rank $d + 1$ that lift $\mathcal{L}_{d+1}$, and such that

$$U \supseteq (\sigma_*(U))^{\mathrm{lin}\perp}.$$

But by the assumption that the p-th power of the maximal ideal of R is zero, we have

$$\sigma_{R,*}(U) = (\sigma_{k,*}(\mathcal{L}_{d+1})) \otimes_k R,$$

where we have written σ_R and σ_k to distinguish between the Frobenius on R and on k. Therefore

$$(\sigma_{R,*}(U))^{\mathrm{lin}\perp} = \left((\sigma_{k,*}(\mathcal{L}_{d+1})) \otimes_k R\right)^{\mathrm{lin}\perp} = (\sigma_{k,*}(\mathcal{L}_{d+1}))^{\mathrm{lin}\perp} \otimes_k R = \mathcal{L}_d \otimes_k R.$$

Thus we see that the set of R-points of $\mathcal{V}$ lifting x_0 is in canonical bijection with the set of R-points of $\mathcal{I}$ lifting x_0. We thus obtain a canonical $\tilde{\mathcal{R}}_p$-point of $\mathcal{V}$ lifting $x_0 \in \mathcal{V}(k)$, and a canonical $\tilde{\mathcal{S}}_p$-point of $\mathcal{I}$ lifting $x_0 \in \mathcal{I}(k)$. These two points induce maps $\tilde{\mathcal{S}}_p \to \tilde{\mathcal{R}}_p$ and $\tilde{\mathcal{R}}_p \to \tilde{\mathcal{S}}_p$ respectively. From the moduli

interpretation of these two maps we see that they are k-algebra homomorphisms inverse to each other and equivariant with respect to the actions of $\bar{g}$ on both sides. Note that $\mathcal{S}_p$ (resp. $\mathcal{R}_p$) is the quotient of $\tilde{\mathcal{S}}_p$ (resp. $\tilde{\mathcal{R}}_p$) by the augmentation ideal for the $\bar{g}$-action. It follows that $\mathcal{R}_p \cong \mathcal{S}_p$. $\qquad\square$

4.3. Study of $\mathcal{I}^{\bar{g}}$.

Definition 4.3.1. Let λ be the eigenvalue of $\bar{g}$ on the 1-dimensional k-vector space $\mathcal{L}_{d+1}/\mathcal{L}_d = U_0/U_0^{\perp}$, and let c be the size of the unique (cf. Remark 4.1.1) Jordan block of $\bar{g}|_{\mathcal{L}_{d+1}}$ associated to λ. Notice our c is denoted by $c+1$ in [RTZ13, §9].

Remark 4.3.2. By the discussion before [RTZ13, Proposition 9.1], c is the size of the unique Jordan block associated to λ of $\bar{g}$ on $\Theta/\mathcal{L}_d = \Omega/U_0^{\perp}$, and is also equal to the quantity $\frac{m(Q(T))+1}{2}$ introduced in the introduction.

Proposition 4.3.3. *The local ring $\mathcal{O}_{\mathcal{I}^{\bar{g}},x_0}$ of $\mathcal{I}^{\bar{g}}$ at x_0 is isomorphic to $k[X]/X^c$ as a k-algebra.*

Proof. By Remark 4.3.2 and Definition 4.2.4, the proposition is a consequence of the following general lemma applied to $\mathcal{L} = \Theta/\mathcal{L}_d$ and $h = \bar{g}$. $\qquad\square$

Lemma 4.3.4. *Let $\mathcal{L}$ be a k-vector space of dimension $d+1$. Let $\mathbb{P}(\mathcal{L}) = \mathbb{P}^d$ be the associated projective space. Let $x_0 \in \mathbb{P}(\mathcal{L})(k)$, represented by a vector $\ell \in \mathcal{L}$. Let $h \in \mathrm{GL}(\mathcal{L})(k) = \mathrm{GL}_{d+1}(k)$. Assume that*

(1) the natural action of h on $\mathbb{P}(\mathcal{L})$ fixes x_0. Denote the eigenvalue of h on ℓ by λ.
(2) there is a unique Jordan block of h associated to the eigenvalue λ. Denote its size by c.

Let $R := \mathcal{O}_{\mathbb{P}(\mathcal{L})^h,x_0}$ be the local ring of the fixed point scheme $\mathbb{P}(\mathcal{L})^h$ at x_0. Then

$$R \cong k[X]/X^c.$$

Proof. Extend ℓ to a basis $\{\ell_0 = \ell, \ell_1, \ldots, \ell_d\}$ of $\mathcal{L}$ such that the matrix $(h_{ij})_{0 \leq i,j \leq d}$ of h under this basis is in the Jordan normal form. Under this basis, the point x_0 has projective coordinates $[X_0 : \cdots : X_d] = [1 : 0 : \cdots : 0] \in \mathbb{P}^d$. Let $Z_i = X_i/X_0$ $(1 \leq i \leq d)$ and let $\mathbb{A}^d$ be the affine space with coordinates $(Z_1, \ldots, Z_d)$. Then we can identify the local ring of $\mathbb{P}^d$ at x_0 with the local ring of $\mathbb{A}^d$ at the origin. Since h fixes x_0, we know that h acts on the local ring of $\mathbb{A}^d$ at the origin (although h does not stabilize $\mathbb{A}^d$ in general). Since (h_{ij}) is in the Jordan normal form, we know that the action of h on the latter is given explicitly by

$$hZ_i = \frac{h_{i,i}X_i + h_{i,i+1}X_{i+1}}{h_{0,0}X_0 + h_{0,1}X_1} = \frac{h_{i,i}Z_i + h_{i,i+1}Z_{i+1}}{h_{0,0} + h_{0,1}Z_1}, \ 1 \leq i \leq d,$$

where $h_{i,i+1}Z_{i+1}$ is understood as 0 when $i = d$. Hence the local equations at the origin of $\mathbb{A}^d$ which cut out the h-fixed point scheme are given by

$$(h_{0,0} - h_{i,i})Z_i + h_{0,1}Z_1 Z_i = h_{i,i+1}Z_{i+1}, \ 1 \leq i \leq d.$$

By hypothesis (2), we have $h_{0,0} - h_{i,i} \neq 0$ if and only if $i \geq c$. Thus when $i \geq c$, we know that $(h_{0,0} - h_{i,i}) + h_{0,1}Z_1$ is a unit in the local ring of $\mathbb{A}^d$ at the origin, and so Z_i can be solved as a multiple of $h_{i,i+1}Z_{i+1}$ when $i \geq c$. It follows that

$$Z_i = 0, \ i \geq c.$$

If $c = 1$, then $Z_1 = \cdots = Z_d = 0$ and the local ring R in question is isomorphic to k as desired. If $c > 1$, then $h_{0,1} = 1$ and we find the equations for $i = 1, \cdots, c-1$ simplify to

$$Z_1 Z_1 = Z_2, Z_1 Z_2 = Z_3, \cdots, Z_1 Z_{c-2} = Z_{c-1}, Z_1 Z_{c-1} = 0.$$

Hence the local ring R in question is isomorphic to (the localization at the ideal $(Z_1, Z_2, \cdots, Z_{c-1})$ or (Z_1) of)

$$k[Z_1, Z_2, \ldots, Z_{c-1}]/(Z_1^2 - Z_2, Z_1^3 - Z_3, \cdots, Z_1^{c-1} - Z_{c-1}, Z_1^c) \cong k[Z_1]/Z_1^c,$$

as desired. $\qquad\square$

Theorem 4.3.5. *Let* $g \in J(\mathbb{Q}_p)$ *be regular semisimple and minuscule. Let* $x_0 \in (\delta(\mathcal{M}) \cap \mathcal{N}^g)(k)$. *Also denote by* x_0 *the image of* x_0 *in* $\mathcal{V}(\Lambda)(k)$ *as in Proposition 4.1.2 and define* λ, c *as in Definition 4.3.1. Assume* $p > c$. *Then the complete local ring of* $\delta(\mathcal{M}) \cap \mathcal{N}^g$ *at* x_0 *is isomorphic to* $k[X]/X^c$.

Proof. Let $\hat{\mathcal{S}}$ be the complete local ring of $\delta(\mathcal{M}) \cap \mathcal{N}^g$ at x_0. By Proposition 4.1.2 and by the fact that $\mathcal{V}(\Lambda)$ is smooth of dimension d (§2.5), we know that $\hat{\mathcal{S}}$ is a quotient of the power series ring $k[[X_1, \cdots, X_d]]$. By Proposition 4.1.2, Lemma 4.2.5, Proposition 4.3.3, we know that $\hat{\mathcal{S}}/\mathfrak{m}_{\hat{\mathcal{S}}}^p$ is isomorphic to $k[X]/X^c$ as a k-algebra. In such a situation, it follows from the next abstract lemma that $\hat{\mathcal{S}} \cong k[X]/X^c$. $\qquad\square$

Lemma 4.3.6. *Let* I *be a proper ideal of* $k[[X_1, \cdots, X_d]]$ *and let* $\hat{\mathcal{S}} = k[[X_1, \cdots, X_d]]/I$. *Let* $\mathfrak{m}$ *be the maximal ideal of* $k[[X_1, \cdots, X_d]]$ *and let* $\mathfrak{m}_{\hat{\mathcal{S}}}$ *be the maximal ideal of* $\hat{\mathcal{S}}$. *Assume there is a* k-algebra isomorphism $\beta : \hat{\mathcal{S}}/\mathfrak{m}_{\hat{\mathcal{S}}}^p \xrightarrow{\sim} k[X]/X^c$ *for some integer* $1 \leq c < p$. *Then* $\hat{\mathcal{S}}$ *is isomorphic to* $k[X]/X^c$ *as a* k-algebra.

Proof. We first notice that if R_1 is any quotient ring of $k[[X_1, \cdots, X_d]]$ with its maximal ideal $\mathfrak{m}_1$ satisfying $\mathfrak{m}_1 = \mathfrak{m}_1^2$ (i.e. R_1 has zero cotangent space), then $R_1 = k$. In fact, R_1 is noetherian and we have $\mathfrak{m}_1^l = \mathfrak{m}_1$ for all $l \in \mathbb{Z}_{\geq 1}$, so by Krull's intersection theorem we conclude that $\mathfrak{m}_1 = 0$ and $R_1 = k$.

Suppose $c = 1$. Then $\hat{\mathcal{S}}/\mathfrak{m}_{\hat{\mathcal{S}}}^p \cong k$, so $\hat{\mathcal{S}}$ has zero cotangent space and thus $\hat{\mathcal{S}} = k$ as desired. Next we treat the case $c \geq 2$. Let α be the composite

$$\alpha : k[[X_1, \cdots, X_d]] \to \hat{\mathcal{S}}/\mathfrak{m}_{\hat{\mathcal{S}}}^p \xrightarrow{\beta} k[X]/X^c.$$

Let $J = \ker \alpha$. Since α is surjective, we reduce to prove that $I = J$. Note that because β is an isomorphism we have

(4.3.6.1) $$I + \mathfrak{m}^p = J.$$

In the following we prove $\mathfrak{m}^p \subset I$, which will imply $I = J$ and hence the lemma. The argument is a variant of [RTZ13, Lemma 11.1].

Let $Y \in k[[X_1, \cdots, X_d]]$ be such that $\alpha(Y) = X$. Since X generates the maximal ideal in $k[X]/X^c$, we have

(4.3.6.2) $$\mathfrak{m} = J + (Y).$$

Then by (4.3.6.1) and (4.3.6.2) we have $\mathfrak{m} = I + (Y) + \mathfrak{m}^p$, and so the local ring $k[[X_1, \cdots, X_d]]/(I + (Y))$ has zero cotangent space. We have observed that the cotangent space being zero implies that the ring has to be k, or equivalently

(4.3.6.3) $$\mathfrak{m} = I + (Y).$$

Now we start to show $\mathfrak{m}^p \subset I$. By (4.3.6.3) we have $\mathfrak{m}^p \subset I + (Y^p)$, so we only need to prove $Y^p \in I$. We will show the stronger statement that $Y^c \in I$. By Krull's intersection theorem, it suffices to show that $Y^c \in I + \mathfrak{m}^{pl}$ for all $l \geq 1$. In the following we show this by induction on l.

Assume $l = 1$. Note that $\alpha(Y^c) = 0$, so by (4.3.6.1) we have

$$Y^c \in J = I + \mathfrak{m}^p.$$

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Suppose $Y^c \in I + \mathfrak{m}^{pl}$ for an integer $l \geq 1$. Write

(4.3.6.4)
$$Y^c = i + m, \ i \in I, \ m \in \mathfrak{m}^{pl}.$$

By (4.3.6.2) we know

$$\mathfrak{m}^{pl} \subset (J + (Y))^{pl} \subset \sum_{s=0}^{pl} J^s (Y)^{pl-s}.$$

Thus we can decompose $m \in \mathfrak{m}^{pl}$ into a sum

(4.3.6.5)
$$m = \sum_{s=0}^{pl} j_s Y^{pl-s}, \ j_s \in J^s.$$

By (4.3.6.4) and (4.3.6.5), we have

$$Y^c = i + \sum_{s=0}^{pl} j_s Y^{pl-s},$$

and so

(4.3.6.6)
$$Y^c - \sum_{s=0}^{pl-c} j_s Y^{pl-s} = i + \sum_{s=pl-c+1}^{pl} j_s Y^{pl-s}.$$

Denote

$$A := \sum_{s=0}^{pl-c} j_s Y^{pl-s-c}.$$

Then the left hand side of (4.3.6.6) is equal to $(1-A)Y^c$. Hence we have

$$(1-A)Y^c = i + \sum_{s=pl-c+1}^{pl} j_s Y^{pl-s} \subset I + J^{pl-c+1} \overset{(4.3.6.1)}{=\!=\!=\!=\!=} I + (I + \mathfrak{m}^p)^{pl-c+1} = I + \mathfrak{m}^{p(pl-c+1)} \subset I + \mathfrak{m}^{p(l+1)},$$

where for the last inclusion we have used $c < p$. Since $1 - A$ is a unit in $k[[X_1, \cdots, X_d]]$ (because $c < p$), we have $Y^c \in I + \mathfrak{m}^{p(l+1)}$. By induction, $Y^c \in I + \mathfrak{m}^{pl}$ for all $l \in \mathbb{Z}_{\geq 1}$, as desired. $\square$

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