Mathematics W4045. Algebraic Curves, Spring 2011

Lecture 1 (Wednesday, January 19)

1. Warmup: real algebraic curves in $\mathbb{R}^2$. Examples: (1) $y - x^2 = 0$, (2a) $x^2 + y^2 - 1 = 0$, (2b) $x^2 - y^2 - 1 = 0$, (2c) $x^2 - y^2 = 0$, (2d) $x^2 + y^2 = 0$, (2e) $x^2 + y^2 + 1 = 0$.

2. Complex algebraic curves in $\mathbb{C}^2$ (i.e. affine plane curves. Examples: (1) $w - z^2 = 0$, (2a) $z^2 + w^2 - 1 = 0$, (2b) $z^2 - w^2 - 1 = 0$, (2c) $z^2 - w^2 = 0$, (2d) $z^2 + w^2 = 0$, (2e) $z^2 + w^2 + 1 = 0$. (2a), (2b), (2e) are equivalent; (2c), (2d) are equivalent. Bijections between $\mathbb{C}$ and (1), between $\mathbb{C} - \{0\}$ and (2b).

3. Irreducibility, irreducible components. Check examples in 2. above.

4. Smooth points, singular points, smooth (nonsingular) curves. Check examples in 2. above.

5. Implicit Function Theorem.

Lecture 2 (Monday, January 24)

1. Definition of topological spaces. Examples: $\mathbb{C}$ with the standard topology, $\mathbb{C}$ with the Zariski topology.

2. Defintion of a Hausdorff topological space. Examples: the standard topology on $\mathbb{C}$ is Hausdorff; the Zarisky topology on $\mathbb{C}$ is not.

3. Definition of a metric space. Examples: $\mathbb{R}^n$, $\mathbb{C}^n$. A metric space is a Hausdorff topological space.

4. Subspace of a topological space or a metric space. Example: a complex algebraic curve in $\mathbb{C}^2$ (i.e. affine plane curve) with the subspace topology is a Hausdorff topological space.

5. Continuity, homeomorphism. Examples: (1) The affine plane curve $w - z^2 = 0$ is homeomorphic to $\mathbb{C}$, (2) The affine plane curve $z^2 - w^2 - 1 = 0$ (or $z^2 + w^2 - 1 = 0$ or $z^2 + w^2 + 1 = 0$) is homeomorphic to $\mathbb{C} - \{0\}$.

6. Open basis, second countability. Examples: $\mathbb{R}^n$, $\mathbb{C}^n$.

7. An affine plane curve is a second countable Hausdorff topological space.

8. Definition of a complex chart. Examples: $w - z^2 = 0$, $z^2 - w^2 - 1 = 0$.

Lecture 3 (Wednesday, January 26)

1. A complex chart near a smooth point of a affine plane curve. Compatibility of two complex charts.

2. Definition of a complex atlas. Equivalence of two complex atlases. Definition of a complex structure. Definition of a Riemann surface.

3. State: an irreducible affine plane curve is connected. Prove: an irreducible smooth affine plane curve is a Riemann surface.

4. Compactness. Affine plane curves are not compact. First example of a compact Riemann surface: $\mathbb{P}^1$; quotient topology.

Lecture 4 (Monday, January 31)

1. Second example of a compact Riemann surface: complex tori $\mathbb{C}/L$.

2. Definition of an n-dimensional complex chart, compatibility of two n-dimensional complex charts. Definition of an n-dimensional complex atlas, equivalence of two n-dimensional complex atlases.

3. Definition of an n-dimensional complex manifold. Example: $\mathbb{P}^n$.

Lecture 5 (Wednesday, February 2)

1. Definition of a projective plane curve, i.e., a complex algebraic curves in $\mathbb{P}^2$. A projective plane curve is a compact Hausdorff topological space.

2. Irreducibility, irreducible components. Examples: $x^2 + y^2 + z^2 = 0$, $x^2 + y^2 = 0$. A line is a projective plane curve of the form $ax + by + cz = 0$, where $(a, b, c) \in \mathbb{C}^3 - \{(0, 0, 0)\}$.

3. Smooth points and singular points of a projective plane curve. Euler's formula. Example: singular points of $x^2y^3 + x^2z^3 + y^2z^3 = 0$.

4. The three affine plane curves $X_i = X \cap U_i$ $(i = 0, 1, 2)$ associated to a projective plane curve X. $[1, a, b]/[a, 1, b]/[a, b, 1]$ is a singular point of X iff (a, b) is a singular point of $X_0/X_1/X_2$. Revisit the example $x^2y^3 + x^2z^3 + y^2z^3 = 0$.

5. State: any nonsingular projective plane curve is irreducible. Prove: any nonsingular projective plane curve is a compact Riemann surface.

Lecture 6 (Monday, February 7)

1. More preliminaries on topology: let X, Y be topological spaces.

   (1) $f : X \to Y$ is continuous, $A \subset X$, $B \subset Y$, $f(A) \subset B$
   $\Rightarrow f|_A : A \to B$ is continuous.

(2) X is compact, A is a closed subset of $X \Rightarrow A$ is compact.

(3) $f : X \to Y$ is continuous, X is compact $\Rightarrow f(X)$ is compact.

(4) X is Hausdorff, A is a compact subset of X, $p \in X \setminus A \Rightarrow$ There exists open set U, V in X such that $p \in U$, $A \subset V$, $U \cap V = \emptyset$.

(5) X is Hausdorff, A is a compact subset of $X \Rightarrow A$ is a closed set in X.

(6) $f : X \to Y$ is a bijective continuous map, X is compact, Y is Hausdorff $\Rightarrow f$ is a homeomorphism.

2. Definitions of holomorphic functions on an open subset W of a Riemann surface X. The set $\mathcal{O}_X(W)$ of holomorphic functions on W is a $\mathbb{C}$-algebra. Examples:

   (1) $X = \mathbb{C}$: agrees with the old definition.

   (2) If $\phi : U \to V$ a complex chart on X then $f : U \to \mathbb{C}$ is holomophic $\Leftrightarrow f \circ \phi^{-1} : V \to \mathbb{C}$ is holomorphic.

   (3) Let $g, h \in \mathbb{C}[z, w]$ be homogenenous of the same degree d. Then $\frac{g}{h}$ is a holomorphic function on the open subset $W = \{[z : w] \in \mathbb{P}^1 \mid h(z, w) \neq 0\}$ of $\mathbb{P}^1$.

   (4) Let $X = \{(z, w) \in \mathbb{C}^2 \mid f(z, w) = 0\}$ be a smooth irreducible affine plane curve, so that it is a Riemann surface. Let $g, h \in \mathbb{C}[z, w]$, where h is not divisble by f. Then $\frac{g}{h}$ is a holomoprhic function on the open subset $W = \{(z, w) \in X \mid h(z, w) \neq 0\}$ of X. In particular, g is a holomorphic function on X.

Lecture 7 (Wednesday, February 9)

1. Singularities of a function f defined and holomorphic on a punctured neighborhood of a point p in a Riemann surface X. Meromorphic functions. The set $\mathcal{M}_X(W)$ of meromorphic functions on an open subset W in X is a field.

2. Definition of $\mathrm{ord}_p(f)$ for a meromorphic function f at $p \in X$. f is holomorphic or has a removable singularity at p iff $\mathrm{ord}_p(f) \geq 0$; f has a zero (resp. pole) at p iff $\mathrm{ord}_p(f) > 0$ (resp. $\mathrm{ord}_p(f) < 0$).

3. Examples of meromorphic functions:

   (1) $X = \mathbb{C}$: usual definition.

   (2) $\phi : U \to V$ a complex chart on X, f is a meromorphic function on U iff $f \circ \phi^{-1}$ is a meromorphic function on V. $\mathcal{M}_X(U) \cong \mathcal{M}_{\mathbb{C}}(V)$.

   (3) $g, h \in \mathbb{C}[z, w]$ homogeneous of the same degree, $h \neq 0$. Then $Z = \{[z : w] \in \mathbb{P}^1 \mid h(z, w) = 0\}$ is a finite subset of $\mathbb{P}^1$ and $\#Z \leq \deg h$. $W = \mathbb{P}^1 - Z$ is open in the standard and Zariski topology. (Aside: let X be a compact Riemann surface. $A \subset X$ is a closed set in

the Zariski topology on X iff A is empty or $A = X$ or A is a finite nonempty set.)

$$\frac{g}{h} : \mathbb{P}^1 \to \mathbb{C} \cup \infty, \quad [z : w] \mapsto \frac{g(z,w)}{h(z,w)}$$

is holomorphic on W and meromorphic on $\mathbb{P}^1$.

(4) $X = \{(z,w) \in \mathbb{C}^2 \mid f(z,w) = 0\}$ irreducible nonsingular affine plane curve. $g, h \in \mathbb{C}[z,w]$, $h \notin (f)$. We will see later that

$$Z = \{(z : w] \in \mathbb{P}^1 \mid f(z,w) = h(z,w) = 0\}$$

is a finite subset of X. $W = X - Z$ is open in X.

$$\frac{g}{h} : X \to \mathbb{C} \cup \infty, \quad (z,w) \mapsto \frac{g(z,w)}{h(z,w)}$$

is holomorphic on W and meromorphic on X.

(5) $X = \{[x : y : z] \in \mathbb{P}^2 \mid F(x,y,z) = 0\}$ nonsingular projective plane curve. $G, H \in \mathbb{C}[x,y,z]$ homogeneous of the same degree, $H \notin (F)$. We will see later that

$$Z = \{[x : y : z] \in \mathbb{P}^2 \mid F(x,y,z) = H(x,y,z) = 0\}$$

is a finite subset of X and $\#Z \leq \deg F \deg H$. $W = X - Z$ is open in the standard and Zariski topology on X.

$$\frac{G}{H} : X \to \mathbb{C} \cup \infty, \quad [x : y : z] \mapsto \frac{G(x,y,z)}{H(x,y,z)}$$

is holomorphic on W and meromorphic on X.

Lecture 8 (Monday, February 14)

1. Generalization of the following theorems from $\mathbb{C}$ to a general Riemann surface X: (1) The zeros and poles of a nonzero meromorphic functions form a discrete set. (2) The identity theorem. (3) The maximum modulus theorem for holomorphic functions.

2. If X is a compact Riemann surface then (1) any nonzero meromorphic function on X has a finite number of zeros of poles, and (3) any holomophic function on X is constant. Any bounded entire function on $\mathbb{C}$ can be extended to a holomorphic function on $\mathbb{P}^1$, so it must be constant.

3. Any meromorphic function on $\mathbb{P}^1$ is of the form $\frac{g}{h}$, where $g, h \in \mathbb{C}[z,w]$ are homogeneous polynomials of the same degree.

Lecture 9 (Wednesday, February 16)

1. The theta function $\theta(z) = \sum_{n=-\infty}^{\infty} e^{\pi i(n^2\tau+2nz)}$ is holomorphic on $\mathbb{C}$. It satisfies $\theta(z+1) = \theta(z)$ and $\theta(z+\tau) = e^{-\pi i(\tau+2z)}\theta(z)$. It has simple zeros at $\frac{1+\tau}{2} + m + n\tau$, $m, n \in \mathbb{Z}$.

2. For any $x \in \mathbb{C}$, the translated theta function $\theta^{(x)} = \theta(z - \frac{1+\tau}{2} - x)$ is holomorophic on $\mathbb{C}$. It satisfies $\theta^{(x)}(z+1) = \theta^{(x)}(z)$ and $\theta(z+\tau) = -e^{-2\pi i(z-x)}\theta(z)$. It has simple zeros at $x + m + n\tau$, $m, n \in \mathbb{Z}$.

Lecture 10 (Monday, February 21)

1. Given complex numbers $x_1, \ldots, x_k$ and nonzero integers $e_1, \ldots, e_k$ satisfying (i) $x_i - x_j \notin L = \mathbb{Z} \oplus \mathbb{Z}\tau$ if $i \neq j$, (ii) $\displaystyle\sum_{i=1}^{k} e_i = 0$, and (iii) $\displaystyle\sum_{i=1}^{k} e_i x_i \in \mathbb{Z}$, there exists a meromorphic function f on $\mathbb{C}/L$ such that

$$\mathrm{ord}_p f = \begin{cases} e_i, & \text{if } p = x_i + L \text{ for some } i \in \{1, \ldots, k\}, \\ 0, & \text{otherwise.} \end{cases}$$

   Moreover, such f is unique up to multiplication by a nonzero constant $c \in \mathbb{C}^*$. Indeed,

$$f \circ \pi(z) = c \prod_{i=1}^{k} \theta^{(x_i)}(z)^{e_i}$$

   where $\pi : \mathbb{C} \to \mathbb{C}/L$ is the natrual projection.

2. We will see later that any nonzero meromorphic function on $\mathbb{C}/L$ is of the form described above. This implies that if f is any meromorphic function on $\mathbb{C}/L$ then

$$\sum_{p \in \mathbb{C}/L} \mathrm{ord}_p f = 0 \in \mathbb{Z}, \qquad \sum_{p \in \mathbb{C}/L} (\mathrm{ord}_p f)p = L \in \mathbb{C}/L.$$

3. Definition of holomorphic maps between complex manifolds. Example: $F : X \to \mathbb{C}$ is a holomorphic map iff F is a holomorphic function on X.

4. Let $\{\phi_\alpha : U_\alpha \to V_\alpha\}$ be an n-dimensional complex atlas on X. Let $\{\phi'_\beta : U'_\beta \to V'_\beta\}$ be an m-dimensional complx atlas on Y. A map $F : X \to Y$ is holomorphic if and only if

$$\phi'_\beta \circ F \circ \phi_\alpha^{-1} : \phi_\alpha(F^{-1}(U'_\beta) \cap U_\alpha) \subset \mathbb{C}^n \to \mathbb{C}^m$$

   is holomorphic whever $F(U_\alpha) \cap U'_\beta$ is nonempty. Example: $g, h \in \mathbb{C}[z, w]$ homogeneous polynomial of the same degree, with no common factors, not both identically zero. Then $F : \mathbb{P}^1 \to \mathbb{P}^1$, $[z : w] \mapsto [g(z, w) : h(z, w)]$, is a well-defined holomorphic map.

5. Properties fo holomorphic maps:

   (a) If $F : X \to Y$ is a holomorphic maps between Riemann surfaces, then $F : X \to Y$ is a continuous map between topological spaces.

   (b) If $F : X \to Y$, $G : Y \to Z$ are holomorphic maps between Riemann surfaces then $G \circ F : X \to Z$ is a holomorphic map.

   (c) If $F : X \to Y$ is a holomorphic map, f is a holomorphic function on an open subset W of Y, then $f \circ F$ is a holomorphic function on $F^{-1}(W)$.

   (d) Let $F : X \to Y$ is a holomorphic map. Let f be a meromorphic function on an open subset W of Y, such that $F(X)$ is not contained in the set of poles of f. Then $f \circ F$ is a meromorphic function on $F^{-1}(W)$.

   From (c), if $F : X \to Y$ is a holomorphic map then for every open subset $W \subset Y$, there is a $\mathbb{C}$-algebra homomorphism

   $$F^* : \mathcal{O}_Y(W) \to \mathcal{O}_X(F^{-1}(W)), \quad f \mapsto f \circ F.$$

   From (d), if $F : X \to Y$ is a *nonconstant* holomorphic map then for every open subset $W \subset Y$, there is a $\mathbb{C}$-algebra homomorphism

   $$F^* : \mathcal{M}_Y(W) \to \mathcal{M}_X(F^{-1}(W)), \quad f \mapsto f \circ F.$$

6. Isomorphism between Riemann surfaces. Automorphism of a Riemann surface. Example: given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$, the map $F_A : \mathbb{P}^1 \to \mathbb{P}^1$, $[z : w] \mapsto [az + bw : cz + dw]$ is an automorphism of $\mathbb{P}^1$. $F_{A^{-1}} = F_A^{-1}$.

Lecture 11 (Wednesday, February 23)

1. Theorems on holomorphic maps between Riemann surfaces: (1) Discrete Preimage Theorem, (2) Identity Theorem, (3) Open Mapping Theorem. If $F : X \to Y$ is an injective holomorphic map then $F^{-1} : F(X) \to X$ is a holomorphic map.

2. If $F : X \to Y$ is a *nonconstant* holomorphic map from a *compact* Riemann surface X to a Riemann surface Y, then F is sujective, Y is compact, and $f^{-1}(y)$ is a nonempty finite set for any $y \in Y$. Example: $F : \mathbb{P}^1 \to X = \{[x, y, z] \in \mathbb{P}^2 \mid x^2 + y^2 - z^2 = 0\}$, $[z_0 : z_1] \mapsto [2z_0z_1 : z_0^2 - z_1^2 : z_0^2 + z_1^2]$, is an isomorphism.

3. Correspondence between nonconstant meromorphic functions and nonconstant holomorphic maps to $\mathbb{P}^1$.

Lecture 12 (Monday, February 28)

1. Correspondence between nonconstant meromorphic functions and nonconstant holomorphic maps to $\mathbb{P}^1$ (continued). Any nonconstant holomorphic map $F : \mathbb{P}^1 \to \mathbb{P}^1$ is of the form $[z : w] \mapsto [g(z, w) : h(z, w)]$, where $g, h \in \mathbb{C}[z, w]$ are homogeneous of the same degree $d > 0$, with no common factors. Any automorphism of $\mathbb{P}^1$ is of the form described in item 6 of Lecture 10.

2. Let $F : X \to Y$ be a nonconstant holomorphic map between Riemann surfaces. Definition of $\mathrm{mult}_p F$, the multiplicity of F at a point $p \in X$. Ramification points and branch points. The Local Normal Form.

3. Let $X = \{(x, y) \in \mathbb{C}^2 \mid f(x, y) = 0\}$ be an irreducible, smooth affine plane curve, and define $\pi : X \to \mathbb{C}$, $(x, y) \mapsto x$. Then π is a holomorphic map. We assume that $f(x, y)$ is not of the form $ax+b$, so that π is not a constant map. Then $(a, b) \in X$ is a ramification point of π iff $\frac{\partial f}{\partial y}(a, b) = 0$.

Lecture 13 (Wednesday, March 2)

1. Let $X = \{[x : y : z] \in \mathbb{P}^2 \mid F(x, y, z) = 0\}$ be a smooth projective plane curve. Suppose that $[0 : 1 : 0] \notin X$. Define $\pi : X \to \mathbb{P}^1$, $[x : y : z] \mapsto [x : z]$. Then π is a nonconstant holomorphic map. $[a : b : c] \in X$ is a ramification point of π iff $\frac{\partial F}{\partial y}(a, b, c) = 0$.

2. Example: $X = \{(x, y) \mid y^2 = h(x)\}$, where $h(x) \in \mathbb{C}[x]$ is a polynomial of degree $d > 0$, with distinct roots $\alpha_1, \ldots, \alpha_d$. Then the ramification points of $\pi : X \to \mathbb{C}$, $(x, y) \mapsto x$, are $(\alpha_1, 0), \ldots, (\alpha_d, 0) \in X$. The branch points of π are $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$. For $k = 1, \ldots, d$, $\mathrm{mult}_{(\alpha_k,0)}\pi = 2$.

3. Let F be the nonconstant holomorphic map from X to $\mathbb{P}^1$ associated to a nonconstant meromorphic function on X. Then for any $p \in X$,
$$\mathrm{mult}_p F = \begin{cases} \mathrm{ord}_p f, & \mathrm{ord}_p f > 0, \\ \mathrm{ord}_p(f - f(p)), & \mathrm{ord}_p f = 0, \\ -\mathrm{ord}_p f, & \mathrm{ord}_p f < 0. \end{cases}$$

4. The degree of a holomorphic map between compact Riemann surfaces. If f is a nonconstant meromorphic function on a compact Riemann surface X then $\sum_{p \in X} \mathrm{ord}_p f = 0$.

Lecture 14 (Monday, March 7)

1. Proof of Proposition 4.8 on page 47 of Miranda's book.

2. Definition of C^∞ 2-manifold. Orientation. Any Riemann surface is an oriented C^∞ 2-manifold.

3. The genus and Euler characteristic of compact Riemann surfaces.

Lecture 15 (Monday, March 21)

1. Connected sum of compact orientable surfaces.

2. Outline of the proof of the Hurwitz's formula.

Lecture 16 (Wednesday, March 23)

1. Holomorphic/meromophic 1-forms on an open set V of $\mathbb{C}$. Example: the differential df of a holomorphic/meromorphic function f on V.

2. The order $\mathrm{ord}_{z_0}(\omega)$ of a nonzero meromorphic 1-form ω on a connected open set V at $z_0 \in V$. Zeros and poles of ω.

3. The pullback $T^*\omega$ of a meromorphic 1-form ω on an open set V_2 of $\mathbb{C}$ under a holomorphic map $T : V_1 \to V_2$, where V_1 is an open set of $\mathbb{C}$ and the image of T is not contained in the set of poles of ω. Example: $z = T(w) = w^m$ (m is positive integer), $T^*dz = mw^{m-1}dw$. If $T : V_1 \to V_2$ and $S : V_2 \to V_3$ are nonconstant holomorphic maps between open sets in $\mathbb{C}$, and ω is a meromorphic 1-form on V_3, then $(S \circ T)^*\omega = T^*S^*\omega$.

4. Let $T : V_1 \to V_2$ and ω be as in 3. above. For any $z_0 \in V_1$, we have

$$\mathrm{ord}_{z_0}(T^*\omega) = \mathrm{ord}_{T(z_0)}(\omega) \cdot \mathrm{mult}_{z_0}(T) + \mathrm{mult}_{z_0}(T) - 1.$$

   In particular, $\mathrm{ord}_{z_0}(T^*\omega) = \mathrm{ord}_{T(z_0)}(\omega)$ if T is a biholomorphic map.

5. Holomorphic/meromorphic 1-form ω on a Riemann surface X. Example: X is an open set in $\mathbb{C}$. If $\mathcal{A} = \{\phi_\alpha : U_\alpha \to V_\alpha \mid \alpha \in I\}$ is a complex atlas on X then a holomorphic/meromorphic 1-form ω is equivalent to a collection $\{\omega_\alpha \mid \alpha \in I\}$, where ω_α is a holomorphic/meromorphic 1-form on V_α, such that if $U_\alpha \cap U_\beta \neq \emptyset$, then $\omega_\alpha = (\phi_\beta \circ \phi_\alpha^{-1})^*\omega_\beta$ on $\phi_\alpha(U_\alpha \cap U_\beta) \subset V_\alpha$.

6. The order $\mathrm{ord}_p(\omega)$ of a nonzero meromorphic 1-form ω on a Riemann surface X at $p \in X$. Zeros and poles of ω. If f is a nonzero meromorphic function on X, then $f\omega$ is a nonzero meromorphic 1-form on X, and $\mathrm{ord}_p(f\omega) = \mathrm{ord}_p(f) + \mathrm{ord}_p(\omega)$ for any $p \in X$.

References for this lecture are the following sections in Chapter IV of Miranda's book: Holomorphic 1-Forms, Meromorphic 1-Forms, Multiplication of 1-Forms by Functions, Differential of Functions, Pulling Back Differential Forms.

Lecture 17 (Monday, March 28)

1. The differential df of a meromorphic function f on a Riemann surface X. Example 1: $X = \mathbb{P}^1$, $f([z : w]) = \frac{z}{w}$,

$$\mathrm{ord}_p(df) = \begin{cases} 0, & p \in U_1 = \{[z : w] \in \mathbb{P}^1 \mid w \neq 0\}, \\ -2, & p = [1 : 0] \end{cases}.$$

2. Example 2: $X = \mathbb{C}/L$, dz is a holomorphic 1-form on $\mathbb{C}$ which descends to a holomorphic 1-form ω_0 on $\mathbb{C}/L$. ω_0 has no zeros. (Reference: Exercise B on page 111 of Miranda's book.)

3. If ω is a nonzero meromorphic 1-form on a Riemann surface X, then any meromorphic 1-form on X is of the form $h\omega$, where h is a meromorphic function on X. If ω_1, ω_2 are two nonzero meromoprhic 1-forms on a *compact* Riemann surface X, then

$$\sum_{p \in X} \mathrm{ord}_p(\omega_1) = \sum_{p \in X} \mathrm{ord}_p(\omega_2).$$

   - Example 1 (continued): Any meromorphic 1-form on $\mathbb{P}^1$ is of the form $\omega = \dfrac{g(z,w)}{h(z,w)} df$, where $g, h \in \mathbb{C}[z,w]$ are homogeneuous of the same degree, and f is defined as in 1. above. If ω is nonzero than $\sum_{p \in \mathbb{P}^1} \mathrm{ord}_p(\omega) = -2$. In particular, there is no nonzero holomorphic 1-form on $\mathbb{P}^1$.
   - Example 2 (continued): Any meromorphic 1-form on $\mathbb{C}/L$ is of the form $\omega = h\omega_0$, where h is a meromoprhic function on $\mathbb{C}/L$. If ω is nonzero then $\mathrm{ord}_p(\omega) = \mathrm{ord}_p(h)$ for all $p \in X$, and $\sum_{p \in \mathbb{C}/L} \mathrm{ord}_p(\omega) = 0$. ω is holomorphic iff h is holomorphic iff $\omega = c\omega_0$ for some constant $c \in \mathbb{C}$.

4. Fact: There is a nonconstant meromorphic function on any compact Riemann surface. Corollary: There is a nonzero meromorphic 1-form on any compact Riemann surface.

5. Let $X = \{(u,v) \in \mathbb{C}^2 \mid f(u,v) = 0\}$ be an irreducible smooth affine plane curve, so that it is a Riemann surface. $u, v, p(u,v) \in \mathbb{C}[u,v]$ restrict to holomorphic functions on X. So $du, dv, p(u,v)du, p(u,v)dv$ are holomorphic 1-forms on X. If $q(u,v) \in \mathbb{C}[u,v]$ is not in the ideal (f) in $\mathbb{C}[u,v]$ generated by $f(u,v)$, then $\dfrac{p(u,v)}{q(u,v)}du, \dfrac{p(u,v)}{q(u,v)}dv$ are meromorphic 1-forms on X. (Reference: Exercise C on page 111 of Miranda's book.)

6. Geometry of the hyperelliptic Riemann surface X defined in Assignment 6 (3). The mermorphic 1-form $y^{-1}dx$ on X_1 is indeed a holomorphic 1-form on X_1. (Reference: Exercise G on page 112 of Miranda's book.)

Lecture 18 (Wednesday, March 30)

1. Geometry of genus 0 hyperelliptic Riemann surfaces $y^2 = x$ and $y^2 = x^2 - 1$.

2. We use the notation in Assignment 6 (3). We have seen that $y^{-1}dx$ is a holomorphic 1-form on X_1. If $p \in X_1$ is not a ramification point of π, then $\mathrm{ord}_p(dx) = 0$ and $\mathrm{ord}_p(y^{-1}) = 0$, so $\mathrm{ord}_p(y^{-1}dx) = 0$. If $p \in X_1$ is a ramification point of π, then $\mathrm{ord}_p(dx) = 1$ and $\mathrm{ord}_p(y^{-1}) = -1$, so $\mathrm{ord}_p(y^{-1}dx) = 0$. Therefore $y^{-1}dx$ has no zeros in X_1. $(\phi^{-1})^*(y^{-1}dx) = -z^{g-1}w^{-1}dz$. By similar argument, $w^{-1}dz$ is a holomorphic 1-form on X_2 with nonzeros on X_2.

   - If $g = 0$ then $y^{-1}dx$ extends to a meromorphic 1-form on X which has no zeros and has poles at the points in $X \setminus X_1 = \pi^{-1}([1:0])$.
   - If $g = 1$ then $y^{-1}dx$ extends to a holomorphic 1-form on X which has no zeros. Any holomorphic 1-form on X is a constant multiple of this holomorphic 1-form.
   - Suppose that $g \geq 2$. For any $p(x) \in \mathbb{C}[x]$, $p(x)y^{-1}dx$ is a holomorphic 1-form on X_1.

     $$(\phi^{-1})^*(y^{-1}dx) = -z^{g-1}p(\frac{1}{z})w^{-1}dz$$

     which is holomorphic on X_2 if $\deg p \geq g-1$. Therefore,

     $$y^{-1}dx, xy^{-1}dx, \ldots, x^{g-1}y^{-1}dx$$

     are holomorphic 1-forms on X. Indeed (to be proved later), they form a basis of $\Omega^1_X(X)$, the space of holomorphic 1-forms on X:

     $$\Omega^1_X(X) = \bigoplus_{i=0}^{g-1} \mathbb{C}x^i y^{-1}dx.$$

3. Fact (to be proved later): If X be a compact Riemann surface of genus g then $\dim_{\mathbb{C}} \Omega^1_X(X) = g$.

4. Let $F : X \to Y$ be a holomorphic map between Riemannn surfaces, let ω be a meromorphic 1-form on Y. Definition of the pull back $F^*\omega$ of ω (when F is not a constant map to a pole of ω). Composition: $(G \circ F)^*\omega = F^*G^*\omega$.

5. Lemma: Let $F : X \to Y$ be a nonconstant holomorphic map, and let ω be a nonzero meromorphic 1-form on Y. Then for any $p \in X$.

   $$\mathrm{ord}_p(F^*\omega) = \mathrm{mult}_p(F)\mathrm{ord}_{F(p)}(\omega) + \mathrm{mult}_p(F) - 1.$$

6. Theorem: Let ω be a nonzero meromorphic 1-form on a compact Riemann surface X. Then

   $$\sum_{p \in X} \mathrm{ord}_p(\omega) = 2g(X) - 2.$$

7. Definition of a divisor D on a Riemann surface X. Divisors on X form an additive group $\mathrm{Div}(X)$. Examples: (i) principal divisors, (ii) canonical divisors, (iii) on $\mathbb{P}^1$,

$$\mathrm{div}(\frac{z(z-1)}{w^2}) = [0:1] + [1:1] - 2[1:0], \quad \mathrm{div}(d(\frac{z}{w})) = -2[1:0].$$

References for this lecture are the following sections in Miranda's book: Chapter III Section 1, "Hyperelliptic Riemann Surfaces"; Chapter IV Section 2, "Pulling Back Differential Forms"; Chapter V Section 1, "The Definition of a Divisor", "The Divisor of a Meromorphic Function: Principal Divisors", "The Divisor of a Meromorphic 1-Form: Canonical Divisors", "The Degree of a Canonical Divisor on a Compact Riemann Surface".

Lecture 19 (Monday, April 4)

1. Let X be a Riemann surface. The set of principal divisors on X, $\mathrm{PDiv}(X)$, is a subgroup of $\mathrm{Div}(X)$. The set of canonincal divisors on X, $\mathrm{KDiv}(X)$, is a coset in $\mathrm{Div}(X)/\mathrm{PDiv}(X)$. We say two divisors $D_1, D_2 \in \mathrm{Div}(X)$ are linearly equivalent, written $D_1 \sim D_2$, if $D_1 - D_2 \in \mathrm{PDiv}(X)$.

2. If X is a *compact* Riemann surface then there is a surjective group homomorphism $\deg : \mathrm{Div}(X) \to \mathbb{Z}$. Let $\mathrm{Div}_d(X) = \deg^{-1}(d)$ be the set of degree d divisors on X. In particular, $\mathrm{Div}_0(X) = \mathrm{Ker}(\deg)$ is a subgroup of $\mathrm{Div}(X)$. Then $\mathrm{PDiv}(X) \subset \mathrm{Div}_0(X)$ and $\mathrm{KDiv}(X) \subset \mathrm{Div}_{2g-2}(X)$, where g is the genus of X.

3. State the Fact (proved in Chapter VIII of Miranda's book): If X is a compact Riemann surface of genus g, then there is a group isomorphism $\mathrm{Div}_0(X)/\mathrm{PDiv}(X) \cong \mathbb{C}^g/\Lambda$, where Λ is a rank $2g$ lattice in $\mathbb{C}^g$. $\mathbb{C}^g/\Lambda$ is homeomorphic to $(S^1)^{2g}$. Prove the Fact when $X = \mathbb{P}^1$ (the projective line) or $X = \mathbb{C}/L$ (a compact torus).

4. Let $F : X \to Y$ be a nonconstant holomorphic map between Riemann surfaces. Definitions of the inverse image divisor $F^*(q) \in \mathrm{Div}(X)$, where $q \in Y$, and the pullback divisor $F^*(D) \in \mathrm{Div}(X)$, where $D \in \mathrm{Div}(Y)$.

References for this lecture are the following sections in Miranda's book: Chapter V Section 1, "The Definition of a Divisor", "The Divisor of a Meromorphic Function: Principal Divisors", "The Degree of a Divisor on a Compact Riemann Surface", "The Divisor of a Meromorphic 1-Form: Canonical Divisors", "The Inverse Image Divisor of a Holomorphic Map"; Chapter V Section 2, "The Definition of Linear Equivalence", "Principal Divisors on a Complex Torus"; Chapter II Section 4, "Meromorphic Functions on a Complex Torus, Yet Again"

Lecture 20 (Wednesday, April 6)

1. Let $F : X \to Y$ be a nonconstant holomorphic map between Riemann surfaces. Definitions of the ramification divisor $R_F \in \mathrm{Div}(X)$ and the branch divisor $B_F \in \mathrm{Div}(Y)$ of F.

2. Lemma: Let $F : X \to Y$ be a nonconstant holomorphic map between Riemann surfaces.

   (1) $F^* : \mathrm{Div}(X) \to \mathrm{Div}(Y)$ is a group holomorphism.

   (2) If f is a nonzero meromorphic function on Y, then
   $$\mathrm{div}(F^*f) = F^*\mathrm{div}(f) \in \mathrm{PDiv}(X).$$

   (3) If ω is a nonzero meromorphic 1-form on Y , then
   $$\mathrm{div}(F^*\omega) = F^*\mathrm{div}(\omega) + R_F \in \mathrm{KDiv}(X).$$

   (4) If X, Y are compact, then $\deg(F^*D) = \deg F \deg D$.

3. Remarks on the above Lemma:

   - By (2), $F^*(\mathrm{PDiv}(Y)) \subset \mathrm{PDiv}(X)$.
   - When X, Y are compact, taking the degree of the equality in (3), we recover the Hurwitz's formula
   $$2g(X) - 2 = \deg F(2g(Y) - 2) + \sum_{x \in X}(\mathrm{mult}_p(F) - 1).$$
   - By (4), when X, Y are compact, $F^*(\mathrm{Div}_d(Y)) \subset \mathrm{Div}_{d \deg F}(X)$.

4. The divisor of zeros, $\mathrm{div}_0(f)$, and the divisor of poles, $\mathrm{div}_\infty(f)$, of a nonzero meromorphic function f on a Riemann surface. $\mathrm{div}(f) = \mathrm{div}_0(f) - \mathrm{div}_\infty(f)$.

5. The partial ordering on divisor: given two divisors D_1, D_2 on a Riemann surface X, $D_1 \geq D_2 \Leftrightarrow D_1(p) \geq D_2(p)$ for all $p \in X$. We say a divisor is effective if $D \geq 0$. Examples: (1) Given a nonconstant holomorphic map $F : X \to Y$ between Riemann surfaces, $R_F \geq 0$, $B_F \geq 0$, and $F^*(q) \geq 0$ for any $q \in Y$. (2) Given a nonzero meromoprhic function f on a Riemann surfaces, $\mathrm{div}_0(f) \geq 0$, $\mathrm{div}_\infty(f) \geq 0$.

   Let D be a divisor on a Riemann surface X. Then $D = D_1 - D_2$, where
   $$D_1 = \sum_{p \in X, D(p) > 0} D(p) \cdot p \geq 0, \quad D_2 = \sum_{p \in X, D(p) < 0} (-D(p)) \cdot p \geq 0.$$

6. The intersection divisor $\mathrm{div}(G)$ of a homogeneous polynomial $G \in \mathbb{C}[x, y, z]$ on a smooth projective plane curve $X = \{[x : y : z] \in \mathbb{P}^2 \mid F(x, y, z) = 0\}$, where G is not in the (prime) ideal generated by F. Lemma: (a) $\mathrm{div}(G_1G_2) = \mathrm{div}(G_1) + \mathrm{div}(G_2)$, (b) $\deg G_1 = \deg G_2 \Rightarrow \mathrm{div}(G_1) - \mathrm{div}(G_2) = \mathrm{div}(\frac{G_1}{G_2}) \Rightarrow \mathrm{div}(G_1) \sim \mathrm{div}(G_2)$.

References for this lecture are the following sections in Miranda's book: Chapter V Section 1, "The Divisor of a Meromorphic Function: Principal Divisors", "The Inverse Image Divisor of a Holomorphic Map", "The Ramification and Branch Divisor of a Holomorphic Map", "Intersection Divisors on a Smooth Projecitve Curves", "The Partial Ordering on Divisors".

Lecture 21 (Monday, April 11)

In this lecture, $X = \{[x : y : z] \in \mathbb{P}^2 \mid F(x, y, z) = 0\}$ is a smooth projective plane curve, where $F \in \mathbb{C}[x, y, z]$ is a homogeneous polynomial of degree $d > 0$.

1. Assume that $[0 : 1 : 0] \notin X$. Define

   $$\pi : X \to \mathbb{P}^1, \quad [x : y : z] \mapsto [x : z].$$

   By item 1 of Lecture 13,

   - π is a nonconstant holomorphic map,
   - $[a : b : c] \in X$ is a ramification point of π iff $\frac{\partial F}{\partial y}(a, b, c) = 0$.

   Proposition: (1) For any $[a : c] \in \mathbb{P}^1$, $\pi^*([a : c]) = \mathrm{div}(cx - az) \in \mathrm{Div}(X)$. (2) $R_\pi = \mathrm{div}(\frac{\partial F}{\partial y}) \in \mathrm{Div}(X)$. (3) $\deg(\pi) = \deg(\mathrm{div}(cx - az)) = d$.

2. Bezout's Theorem: Let $G \in \mathbb{C}[x, y, z]$ be a homogeneous polynomial of degree e which is not in the ideal generated by F. Then $\deg(\mathrm{div}G) = de$.

3. Plücker's formula: $g(X) = \dfrac{(d-1)(d-2)}{2}$.

References for this lecture are the following sections in Miranda's book: Chapter V Section 2, "The Degree of a Smooth Projective Curve", "Bezout's Theorem for Smooth Projective Plane Curves", "Plücker's Formula".

Lecture 22 (Wednesday, April 13)

1. Let X be a Riemann surface. Let 0 be the zero function on X, and let $d0$ be the zero holomorphic 1-form on X. We define $\mathrm{ord}_p(0) = \mathrm{ord}_p(d0) = +\infty$ for any $p \in X$.

2. Let X be a Riemann surface. For any open set U in X, let $\mathcal{M}_X(U)$ (resp. $\mathcal{M}_X^{(1)}(U)$) denote the space of meromorphic functions (resp. 1-forms) on U. Then $\mathcal{M}_X(U)$ and $\mathcal{M}_X^{(1)}(U)$ are complex vector spaces. Given $D \in \mathrm{Div}(X)$, define

   $$\begin{aligned} L(D) &= \{f \in \mathcal{M}_X(X) \mid f = 0 \text{ or } \mathrm{div}(f) \geq -D\} \\ &= \{f \in \mathcal{M}_X(X) \mid \mathrm{ord}_p(f) \geq -D(p) \text{ for all } p \in X\}, \\ L^{(1)}(D) &= \{\omega \in \mathcal{M}_X^{(1)}(X) \mid f = 0 \text{ or } \mathrm{div}(\omega) \geq -D\} \\ &= \{\omega \in \mathcal{M}_X^{(1)}(X) \mid \mathrm{ord}_p(\omega) \geq -D(p) \text{ for all } p \in X\}. \end{aligned}$$

   $L(D)$ (resp. $L^{(1)}(D)$) is called the space of meromorphic functions (resp. 1-forms) with poles bounded by D.

3. Lemma: Let X be a Riemannn surface.

   (1) $L(D)$ is a complex linear subspace of $\mathcal{M}_X(X)$; $L^{(1)}(D)$ is a complex linear subspace of $\mathcal{M}_X^{(1)}(X)$.
   (2) $L(0) = \mathcal{O}_X(X)$, $L^{(1)}(0) = \Omega_X(X)$.
   (3) If $D_1 \leq D_2$ then $L(D_1) \subset L(D_2)$ and $L^{(1)}(D_1) \subset L^{(1)}(D_2)$.
   (4) If $D_1 \sim D_2$ then there are linear isomorphisms $L(D_1) \cong L(D_2)$ and $L^{(1)}(D_1) \cong L^{(1)}(D_2)$.
   (5) Let $K = \mathrm{div}(\omega)$ be a canoncial divisor on X. Then there is a linear isomorphism $L(D + K) \cong L^{(1)}(D)$.

   Outline of proofs of (4) and (5):

   (4) If h is a nonzero meromorphic function on X then there is a linear isomorphism $\mu_h : \mathcal{M}_X(X) \to \mathcal{M}_X(X)$, $f \mapsto fh$.
   $\mu_h(L(\mathrm{div}(h) + D)) = L(D)$.
   (5) If ω is a nonzero meromorphic 1-form on X then there is a linear isomoprhism $\mu_\omega : \mathcal{M}_X(X) \to \mathcal{M}_X^{(1)}(X)$, $f \mapsto f\omega$.
   $\mu_h(L(\mathrm{div}(\omega) + D)) = L^{(1)}(D)$.

4. Lemma: Let X be a *compact* Riemann surface. Then
   (a) $L(0) \cong \mathbb{C}$. (b) $\deg D < 0 \Rightarrow L(D) = \{0\}$.

5. Let D be a divisor on a Riemann surface X. The *complete linear system* of D, $|D|$, is the set of all effective divisors $E \geq 0$ on X which are linearly equivalent to X:

$$|D| = \{E \in \mathrm{Div}(X) \mid E \sim D \text{ and } E \geq 0\}.$$

   There is a surjective map $\pi : L(D) - \{0\} \to |D|$ given by $f \mapsto \mathrm{div}(f) + D$. $\pi(f_1) = \pi(f_2)$ iff $\frac{f_1}{f_2}$ is a meromorphic functions on X with no zero and pole.

   We now assume that X is *compact*. Then $\pi_1(f_1) = \pi_2(f_2)$ iff $\frac{f_1}{f_2}$ is a nonzero constant, and we have a bijection

$$\mathbb{P}(L(D)) := (L(D) - \{0\})/\mathbb{C}^* \to |D|.$$

6. Let $D \in \mathrm{Div}(\mathbb{P}^1)$. If $\deg D < 0$ then $L(D) = \{0\}$. We want to compute $L(D)$ when $d = \deg D \geq 0$. We have $D \sim dp_0$, where $p_0 = [1 : 0]$.

$$\begin{aligned} L(dp_0) &= \Big\{\frac{p(z,w)}{w^d} \mid p(z,w) \in \mathbb{C}[z,w] \text{is either the zero polynomial} \\ &\qquad\qquad \text{or a homogeneous polynomial of degree } d\Big\} \\ &\cong \mathbb{C}^{d+1}. \end{aligned}$$

References for this lecture are the following sections in Miranda's book: Chapter V Section 3, "The Definition of the Space $L(D)$", "Isomorphisms between $L(D)$'s under Linear Equivalence", "The Definition of the Space $L^{(1)}(D)$", "The Isomorphism between $L^{(1)}(D)$ and $L(D+K)$", "Computation of $L(D)$ for the Riemann Sphere".

Lecture 23 (Monday, April 18)

1. Let $D = \sum_{i=1}^{k} e_i[a_i : b_i]$ be a divisor of degree $d \geq 0$ on $\mathbb{P}^1$, where $[a_1 : b_1]$, $\;$, $[a_k : b_k]$ are distinct points in $\mathbb{P}^1$, $e_1, \ldots, e_k$ are nonzero integers, and $e_1 + \cdots + e_k = 0\}$. Then
$$\begin{aligned} L(D) &= \Big\{\frac{p(z,w)}{\prod_{i=1}^{k}(b_i z - a_i w)^{e_i}} \mid p(z,w) \in \mathbb{C}[z,w] \text{ is the either zero} \\ &\qquad\qquad \text{polynomial or a homogeneous polynomial of degree } d\Big\} \\ &\cong \mathbb{C}^{d+1}. \end{aligned}$$

2. Let X be a Riemann surface, $D \in \mathrm{Div}(X)$, $p \in X$. Then the codimension of $L(D-p)$ in $L(D)$ is either 0 or 1.

3. Let X be a *compact* Riemann surface. Then $L(D)$ is finite dimensional for any $D \in \mathrm{Div}(X)$. Indeed, if we write $D = P - N$, where $P, N \geq 0$, then $\ell(D) \leq \deg P + 1$, where $\ell(D) = \dim_{\mathbb{C}} L(D)$.

4. Let $X = \mathbb{C}/L$ be a complex torus. Define the Abel-Jacobi map $A : \mathrm{Div}(X) \to X = \mathbb{C}/L$ by $A(D) = \sum_{p \in X} D(p)p$ (sum in the additive group $\mathbb{C}/L$). Then $D_1 \sim D_1$ iff $\deg D_1 = \deg D_2$ and $A(D_1) = A(D_2)$. Lemma: If $D \in \mathrm{Div}(X)$ and $\deg D > 0$, then D is linearly equivalent to a positive divisior (i.e. a nonzero effective divisor). Moreover, if $\deg D = 1$ then $D \sim q$ for a unique point $q \in X$; if $\deg D > 1$ then for any $x \in X$ there exists a positive divisor $E > 0$ on X such that $E \sim D$ and $E(x) = 0$.

5. Let $D \in \mathrm{Div}(\mathbb{C}/L)$. Then (a) $\deg D < 0 \Rightarrow \ell(D) = 0$, (b) $\deg D = 0$ and $D \sim 0 \Rightarrow \ell(D) = 1$, (c) $\deg D = 0$ and $D \not\sim 0 \Rightarrow \ell(D) = 0$, (d) $\deg D > 0 \Rightarrow \ell(D) = \deg D$.

References for this lecture are the following sections in Miranda's book: Chapter V Section 2, "Principal Divisors on a Complex Torus"; Chapter V Section 3, "Computation of $L(D)$ for the Riemann Sphere". "Computaiont of $L(D)$ for a Complex Torus", "A Bound on the Dimension of $L(D)$".

Lecture 24 (Wednesdayday, April 20)

1. Definition of an algebraic curve. Examples: $\mathbb{P}^1$, $\mathbb{C}/L$. Fact: Any compact Riemann surface is an algebraic curve.

2. Statement of the Riemann-Roch Theorem (Second Form): Let X be an algebraic curve of genus g. Let D be any divisor on X, and let K by any canonical divisor on X. Then

$$\ell(D) - \ell(K - D) = \deg D + 1 - g.$$

   Proof of the Riemann-Roch Theorem for $\mathbb{P}^1$ and $\mathbb{C}/\Lambda$.

3. A consequence of the Riemann-Roch Theorem: if X is an algebraic curve of genus g, then $\dim_{\mathbb{C}} \Omega_X(X) = \ell(K) = g$. (This is item 3 of Lecture 18.)

4. Let $X = \{[x : y : z] \in \mathbb{P}^2 \mid F(x, y, z) = 0\}$ be a smooth projective curve, where $F \in \mathbb{C}[x, y, z]$ is a homogeneous polynomial of degree $d \geq 3$. We have seen that if $p(u, v) \in \mathbb{C}[u, v]$ is a polynomial of degree at most $d - 3$, then

$$p(u, v) \frac{du}{\frac{\partial f}{\partial v}(u, v)}$$

   is a holomorphic 1-form on $X_2 = \{(u, v) \in \mathbb{C}^2 \mid F(u, v, 1) = 0\}$ which extends to a holomorphic 1-form on X. There is an injective linear map $i : V = \{p(u, v)\mathbb{C}[u, v] \mid \deg p \leq d - 3\} \to \Omega_X(X)$ sending $p(u, v)$ to (the extension of) $p(u, v) \frac{du}{\frac{\partial f}{\partial v}(u, v)}$. Moreover, $\dim_{\mathbb{C}} V = \frac{(d-1)(d-2)}{2} = g(X) = \dim_{\mathbb{C}} \Omega_X(X)$. So i is a linear isomorphism.

5. Let X be a hyperelliptic Riemann surface defined as in Assignment 6 (3). Assume that $g \geq 1$. Then there is an injective linear map $i : V = \{p(x) \in \mathbb{C}[x] \mid \deg p \leq g - 1\} \to \Omega_X(X)$ sending $p(x)$ to (the extension of) $p(x)y^{-1}dx$. Moreover, $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} \Omega_X(X) = g$. So i is a linear isomorphism.

References for this lecture are the following sections in Miranda's book: Chapter VI Section 1, "Separating Points and Tangents", Chapter VI Section 3, "The Riemann-Roch Theorem II", Problem VI.3H, I.

Lecture 25 (Monday, April 26)

1. Definition of presheaves/sheaves of abelian groups on a topological space. Examples: X is a Riemann surface, $\mathcal{O}_X, \mathcal{M}_X, \mathcal{O}_X[D], \Omega^1_X, \mathcal{M}^{(1)}_X, \Omega^1_X[D]$ are sheaves of abelian groups on X.

2. Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X. Definition of the stalk $\mathcal{F}_p$ of $\mathcal{F}$ at at a point $p \in X$. For any open neighborhood U of p there is a group homorphism $\mathcal{F}(U) \to \mathcal{F}_p$ sending $f \in \mathcal{F}(U)$ to its equivalence class in $\mathcal{F}_p$.

3. Let D be a divisor on a *compact* Riemann surface X, and let $p \in X$. Define $\mathcal{T}[D]_p := (\mathcal{M}_X)_p/\mathcal{O}_X[D]_p$. There is a short exact sequence of abelian groups (indeed complex vector spaces):

$$0 \to \mathcal{O}_X[D]_p \to (\mathcal{M}_X)_P \stackrel{\alpha_{D,p}}{\to} \mathcal{T}[D]_p \to 0.$$

4. Lemma: Let f be a meromorphic function on a compact Riemann surface X. Let $D \in \mathrm{Div}(X)$. If $\alpha_{D,p}(f_p) \neq 0$ then p is a pole of f or p is in the support of D.

   Therefore $\{p \in X \mid \alpha_{D,p}(f_p) \neq 0\}$ is a finite subset of X. This allows us to define a complex linear map.

$$\alpha_D : \mathcal{M}_X(X) \to \mathcal{T}[D] := \bigoplus_{p\in X} \mathcal{T}[D]_p, \quad f \mapsto \sum_{p\in X} \alpha_{D,p}(f_p)p.$$

   Then $\mathrm{Ker}(\alpha_D) = L(D)$. We define $H^1(D)$ to be the cokernel of α_D. Theorem: $H^1(D)$ is finite dimensional.

5. The Riemann-Roch Theorem (First Form): Let D be a divisor on an algebraic curve X. Then

$$\dim L(D) - \dim H^1(D) = \deg D + 1 - \dim H^1(0).$$

   Note that the above equation can be rewritten as

$$\dim L(D) - \dim H^1(D) - \deg D = \dim L(0) - \dim H^1(0) - \deg(0).$$

   It suffices to show that the left hand side is a constant independent of $D \in \mathrm{Div}(X)$.

References for this lecture are the following sections in Miranda's book: Chapter IX Section 1, "Presheaves", "Examples of Presheaves", "The Sheaf Axiom", Problem IX.1I; Chapter VI Section 2, "Definition of Laurent Tail Divisors", "Mittag-Leffler Problems adn $H^1(D)$" ; Chapter VI Section 3, "The Riemann-Roch Theorem I".

Lecture 26 (Wednesday, April 27)

1. Proof of Riemann-Roch Theorem (First Form).

2. Residue map: Let X be a compact Riemann surface, and let D be a divisor on X.

   (1) Given $p \in X$, define $\mathrm{Res}_p : (\mathcal{M}_X^{(1)})_p \to \mathbb{C}$.

   (2) Given $p \in X$ and $\omega_p \in (\mathcal{M}_X^{(1)})_p$, define $\mathrm{Res}_{\omega_p} : (\mathcal{M}_X)_p \to \mathbb{C}$ by $\mathrm{Res}_{\omega_p}(f_p) = \mathrm{Res}_p(f_p\omega_p)$. If

$$\omega_p \in \Omega^1_X[-D]_p = \{\omega_p \in (\mathcal{M}_X^{(1)})_p \mid \mathrm{ord}_p(\omega) \geq D(p)$$

   and

$$f_p \in \mathcal{O}_X[D]_p = \{f_p \in (\mathcal{M}_X)_p \mid \mathrm{ord}_p(f) \geq -D(p)\},$$

   then $f_p\omega_p \in (\Omega^1_X)_p \subset \mathrm{Ker}(\mathrm{Res}_p)$. So if $\omega_p \in \Omega^1_X[-D]_p$ then Res_{ω_p} descends to a complex linear functional

$$\mathrm{Res}_{\omega_p} : \mathcal{T}[D]_p = (\mathcal{M}_X)_p/\mathcal{O}_X[D]_p \to \mathbb{C}.$$

(3) Given $\omega \in \Omega^1_X[-D](X) = L^{(1)}(-D)$, define $\mathrm{Res}_\omega : \mathcal{T}[D] \to \mathbb{C}$ by

$$\mathrm{Res}_\omega(\sum_{p\in X} r_p p) = \sum_{p\in X} \mathrm{Res}_{\omega_p} r_p = \sum_{p\in X} \mathrm{Res}_p(r_p\omega_p),$$

where $r_p \in \mathcal{T}[D]_p$.

Lemma: If ω is a meromorphic 1-form on a compact Riemann surface X then $\sum_{p\in X} \mathrm{Res}_p(\omega_p) = 0$. Therefore if $f \in \mathcal{M}_X(X)$ and $\omega \in L^{(1)}(-D)$ then $\mathrm{Res}_\omega \circ \alpha_D(f) = \sum_{p\in X} \mathrm{Res}_p(f_p\omega_p) = 0$. Therefore Res_ω descends to a linear functional

$$\mathrm{Res}_\omega : H^1(D) = \mathcal{T}[D]/\mathrm{Im}(\alpha_D) \to \mathbb{C}.$$

(4) There is a complex linear map $\mathrm{Res} : L^{(1)}(D) \to H^1(D)^*$ given by $\omega \mapsto \mathrm{Res}_\omega$.

3. Serre Duality: Let D be a divisor on an algebraic curve X. Then $\mathrm{Res} : L^{(1)}(D) \to H^1(D)^*$ is a linear isomorphism. Derivation of Riemann-Roch Theorem (Second Form) from Riemann-Roch Theorem (First Form) and Serre Duality.

The reference for this lecture is Chapter VI Section 3 of Miranda's book.

Lecture 27 (Monday, May 2)
Proof of the Serre Duality.

The reference for this lecture is the following section of Miranda's book: Chapter VI, Section 3, "Serre duality".