Math W4045. Algebraic Curves **Spring 2011**

Assignment 9

Due Monday, April 18, 2011

(1) Let X be a Riemann surface. Prove the following statements:
  (a) If $F : X \to \mathbb{P}^1$ is a nonconstant holomorphic map, then $F^*(q_1) \sim F^*(q_2)$ for any $q_1, q_2 \in \mathbb{P}^1$.
  (b) Suppose that E_1, E_2 are divisors on X such that (i) $E_1 \geq 0$, $E_2 \geq 0$, (ii) the supports of E_1 and E_2 are nonempty and disjoint, (iii) $E_1 \sim E_2$. Then for any two distinct points $q_1, q_2 \in \mathbb{P}^1$, there exists a holomorphic map $F : X \to \mathbb{P}^1$ such that $F^*(q_1) = E_1$ and $F^*(q_2) = E_2$.

(2) Let $X = \{[x : y : z] \in \mathbb{P}^2 \mid F(x, y, z) = 0\}$ be a smooth projective curve, where $F \in \mathbb{C}[x, y, z]$ is a homogeneous polynomial of degree $d \geq 2$. We assume that $p_0 = [0 : 1 : 0] \in X$.
  (a) Show that $\frac{\partial F}{\partial x}(0, 1, 0)$ and $\frac{\partial F}{\partial z}(0, 1, 0)$ are not both zero.
  (b) Define $\pi : X \to \mathbb{P}^1$ by

$$\pi([x : y : z]) = \begin{cases} [x : z], & \text{if } [x : y : z] \in X - \{p_0\}, \\ [\frac{\partial F}{\partial z}(0, 1, 0) : -\frac{\partial F}{\partial x}(0, 1, 0)], & \text{if } [x : y : z] = [0 : 1 : 0]. \end{cases}$$

  Show that π is a nonconstant holomorphic map.
  (c) Show that for any $[a : c] \in \mathbb{P}^1$,

$$\pi^*([a : c]) = \operatorname{div}(cx - az) - p_0.$$

  What is the degree of π?
  (d) Show that $R_\pi = \operatorname{div}(\frac{\partial F}{\partial y}) - 2p_0$.
  (e) Compute the genus $g(X)$ of X by applying the Hurwitz's formula to π.

(3) Let X be a compact Riemann surface of genus 4. Show that X is not isomorphic to any smooth projective plane curve.