Math W4045. Algebraic Curves **Spring 2011**

Assignment 8

Due Monday, April 11, 2011

(1) Fix $\tau \in \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$, and consider the lattice $L = \mathbb{Z} + \mathbb{Z}\tau$. Form the complex torus $X = \mathbb{C}/L$. Let $\pi : \mathbb{C} \to \mathbb{C}/L$ be the quotient map.

(a) (Meromorphic functions on a complex torus) Prove that if f is a nonzero meromorphic function then there exists a nonzero constant $c \in \mathbb{C}$ and two sets of d (not necessarily distinct) complex numbers $\{x_i\}$ and $\{y_j\}$ such that

$$\sum_{i=1}^{d} x_i = \sum_{j=1}^{d} y_j$$

and

$$f \circ \pi(z) = c \frac{\prod_{i=1}^{d} \theta^{(x_i)}(z)}{\prod_{j=1}^{d} \theta^{(y_j)}(z)}.$$

[Hint: see page 50 of Miranda's book.]

(b) (Abel's Theorem for a complex torus) Define a group homomorphism $A : \mathrm{Div}_0(X) \to X$ by sending a formal sum $\sum_i n_i p_i$ to the actual sum in the group X. Prove that A is surjective, and $\mathrm{Ker}(A) = \mathrm{PDiv}(X)$. Therefore, we have a group isomorphism $\mathrm{Div}_0(X)/\mathrm{PDiv}(X) \cong \mathbb{C}/L$. [Hint: Use (a) to prove $\mathrm{PDiv}(X) \subset \mathrm{Ker}(A)$. To prove $\mathrm{Ker}(A) \subset \mathrm{PDiv}(X)$, see Miranda page 141, line 17–24.]

(2) Let $f : X \to \mathbb{C}$ be a nonconstant holomorphic function on a Riemann surface X, which can also be viewed as a nonconstant holomophic map from the Riemann surface X to the Riemann surface $\mathbb{C}$. Show that the canonical divisor $\mathrm{div}(df)$ is equal to the ramification divisor R_f of f.

(3) Let f be a nonconstant meromorphic function on a Riemann surface X. Define $F : X \to \mathbb{P}^1$ by

$$F(p) = \begin{cases} [f(p) : 1], & p \text{ is not a pole of } f, \\ [1 : 0], & p \text{ is a pole of } f. \end{cases}$$

Then F is a nonconstant holomorphic map. Show that $\mathrm{div}_0(f) = F^*([0 : 1])$ and $\mathrm{div}_\infty(f) = F^*([1 : 0])$.

(4) Let $X = \{[x : y : z] \in \mathbb{P}^2 \mid y^2 z = x^3 - xz^2\}$. Then X is a projective plane curve.

(a) Check that X is smooth, so that it is a compact Riemann surface.

(b) Compute the intersection divisors of the lines defined by $x = 0$, $y = 0$, and $z = 0$ with X.

(c) Let $p_0 = [0 : 1 : 0]$, $p_1 = [0 : 0 : 1]$, $p_2 = [1 : 0 : 1]$, and $p_3 = [-1 : 0 : 1]$. Show that $2p_0 \sim 2p_i$ for each i, and that $p_1 + p_2 + p_3 \sim 3p_0$.