**Math W4045. Algebraic Curves** **Spring 2011**

# Assignment 7

*Due Monday, April 4, 2011*

(1) Suppose that $\omega$ is a nonzero holomorphic 1-form on a compact Riemann surface $X$ of genus 1. Show that $\omega$ has no zeros, and that any holomorphic 1-form on $X$ is a constant multiple of $\omega$.

(2) Let $X = \{[x : y : z] \in \mathbb{P}^2 \mid F(x, y, z) = 0\}$ be a smooth projective curve, where $F(x, y, z)$ is a homogeneous polynomial. Let $f(u, v) = F(u, v, 1)$ define the associated smooth affine plane curve. (We assume that $f(u, v)$ is not a constant function of $u, v$.)

(a) Show that $du$ and $dv$ define meromorphic 1-forms on all of $X$, as do $r(u, v)du$ and $r(u, v)dv$ for any rational function $r(u, v) = \frac{g(u,v)}{h(u,v)}$, where $g, h \in \mathbb{C}[u, v]$ and $h$ is not in the ideal generated by $f(u, v)$.

(b) Show that $\frac{\partial f}{\partial u} du = -\frac{\partial f}{\partial v} dv$ as meromorphic 1-forms on $X$.

(c) Suppose that $F$ is of degree $d \geq 3$, and that $p(u, v) \in \mathbb{C}[u, v]$ is a polynomial of degree at most $d - 3$. Show that

$$\omega = \frac{p(u, v)du}{\frac{\partial f}{\partial v}(u, v)}$$

defines a holomorphic 1-form on the compact Riemann surface $X$.

(3) Let $g = 2$ and $X_1 = \{(x, y) \in \mathbb{C}^2 \mid y^2 = x^5 - x\}$ in Assignment 6 (3), so that $X$ is a compact Riemann surface of genus 2. Write your answers to (b) and (c) in the form

$$m_1(x_1, y_1) + \cdots m_n(x_n, y_n) + m\infty, \quad m_1, \ldots, m_n, m \in \mathbb{Z}.$$

(a) Show that $X \setminus X_1$ consists of a single point. We call this point $\infty$, so that $X = X_1 \cup \{\infty\}$.

(b) Note that $x$ and $y$ are meromorphic functions on $X$. Compute the principal divisors $\mathrm{div}(x)$, $\mathrm{div}(y)$.

(c) We have shown in class that $y^{-1}dx$ is a holomorphic 1-form on $X$ with no zeros in $X_1$. Compute the canonical divisor $\mathrm{div}(y^{-1}dx)$.

(4) Let $f$ be a nonzero meromorphic function on a Riemann surface $X$. Show that for any $p \in X$, $\mathrm{ord}_p(df) = \mathrm{ord}_p(f - f(p)) - 1$.