Math W4045. Algebraic Curves **Spring 2011**

Assignment 5

Due Monday, March 7, 2011

(1) Let $F : X \to Y$ be a nonconstant holomorphic map between Riemann surfaces. Given $p \in X$, choose complex charts $\phi_1 : U_1 \to V_1$ on X and $\phi_2 : U_2 \to V_2$ on Y such that $p \in U_1$, $F(U_1) \subset U_2$, $\phi_1(p) = \phi_2(F(p)) = 0$, so that $h := \phi_2 \circ F \circ \phi_1^{-1}$ is a holomorphic function on V_1, $h(0) = 0$. Show that

$$\mathrm{mult}_p(F) := \mathrm{ord}_0(h)$$

is a well-defined postive integer, i.e., $\mathrm{ord}_0(h)$ is independent of choices of $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$.

(2) Let $F : X \to Y$ and $G : Y \to Z$ be two nonconstant holomorphic maps between Riemann surfaces. Use Local Normal Form to prove the following statements.

(a) If $p \in X$ then $\mathrm{mult}_p(G \circ F) = \mathrm{mult}_{F(p)}(G)\mathrm{mult}_p(F)$.

(b) If f is a meromorphic function on Y then $\mathrm{ord}_p(f \circ F) = \mathrm{mult}_p(F)\mathrm{ord}_{F(p)}(f)$.

(c) The set of ramification points of F is a discrete subset of X.

(3) Let $F : \mathbb{P}^1 \to \mathbb{P}^1$ be the holomorphic map defined by $[z : w] \mapsto [z^3 : w^3 - wz^2]$.

(a) Find all the ramification points and branch points of F.

(b) Find the multiplicities of F at all its ramification points.

(4) Let $X = \{[x : y : z] \in \mathbb{P}^2 \mid x^d + y^d + z^d = 0\}$, where d is a positive integer.

(a) Show that X is a smooth projective plane curve, so that it is a compact Riemannn surface.

(b) Note that $[0 : 1 : 0] \notin X$, so that the map $\pi : X \to \mathbb{P}^1$ defined by $[x : y : z] \mapsto [x : z]$ is holomorphic. Show that $\deg \pi = d$.

(c) Find all the ramification and branch points of π.

(d) Find the multiplicities of F at all its ramification points.