**Math W4045. Algebraic Curves** **Spring 2011**

# Assignment 4

*Due Monday, February 28, 2011*

(1) Suppose that $f : X \to Y \subset \mathbb{C}^2$ is a map from a Riemann surface $X$ to an irreducible nonsingular affine plane curve $Y$. Prove that $f$ is a holomorphic map to $Y$ if and only if $f$ is a holomorphic map to $\mathbb{C}^2$.

(2) Suppose that $f : X \to Y \subset \mathbb{P}^2$ is a map from a Riemann surface $X$ to a nonsingular projective plane curve $Y$. Prove that $f$ is a holomorphic map to $Y$ if and only if $f$ is a holomoprhic map to $\mathbb{P}^2$.

(3) Let $f_0(z,w), f_1(z,w), f_2(z,w) \in \mathbb{C}[z,w]$ be homogeneous polynomials of the same degree, such that

$$\{(z,w) \in \mathbb{C}^2 \mid f_0(z,w) = f_1(z,w) = f_2(z,w) = 0\} = \{(0,0)\}.$$

Show that the map $F : \mathbb{P}^1 \to \mathbb{P}^2$ defined by

$$F([z : w]) = [f_0(z,w) : f_1(z,w) : f_2(z,w)]$$

is well-defined and holomoprhic.

(4) Recall that a lattice $L \subset \mathbb{C}$ is an additive subgroup generated (over $\mathbb{Z}$) by two complex numbers $\omega_1$ and $\omega_2$ which are linearly independent over $\mathbb{R}$. Thus $L = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}$.

(a) Suppose that $L \subset L'$ are two lattices in $\mathbb{C}$. Show that the natural map $\mathbb{C}/L \to \mathbb{C}/L'$ is holomorphic, and is biholomorphic if and only if $L = L'$.

(b) Let $L$ be a lattice in $\mathbb{C}$ and let $\alpha$ be a nonzero complex number. Show that $\alpha L$ is a lattice in $\mathbb{C}$ and that the map $\phi : \mathbb{C}/L \to \mathbb{C}/(\alpha L)$ sending $z + L$ to $(\alpha z) + (\alpha L)$ is a well-defined biholomorphic map.

(c) Show that every torus $\mathbb{C}/L$ is isomorphic to a torus which has the form $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, where $\tau$ is a complex number with strictly positive imaginary part.