Example 1. (see 16.6 Exercise 42) Let S_1 be the part of the cylinder $x^2 + z^2 = R^2$ that lies inside $y^2 + z^2 = R^2$. Find the area of S_1.

Solution: We have Area(S_1)=8Area(S), where S is the portion of S_1 in the first octant (see Figure 1).

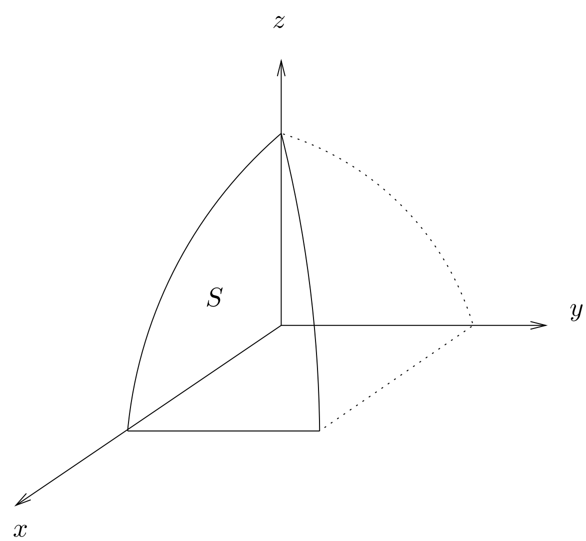

FIGURE 1

S is given by

$$\mathbf{r}(y,z) = \langle \sqrt{R^2 - z^2}, y, z \rangle, \quad (y,z) \in D$$

where $D = \{(y,z) \in \mathbb{R}^2 \mid y^2 + z^2 \leq R^2\}$.

$$\begin{aligned}
\mathbf{r}_y &= \langle 0,1,0 \rangle \\
\mathbf{r}_z &= \langle \frac{-z}{\sqrt{R^2 - z^2}}, 0, 1 \rangle \\
\mathbf{r}_y \times \mathbf{r}_z &= \langle 1, 0, \frac{z}{\sqrt{R^2 - z^2}} \rangle \\
|\mathbf{r}_y \times \mathbf{r}_z| &= \sqrt{1 + \frac{z^2}{R^2 - z^2}} = \frac{R}{\sqrt{R^2 - z^2}} \\
\text{Area}(S) &= \int\int_D |\mathbf{r}_y \times \mathbf{r}_z| dA = \int\int_D \frac{R}{\sqrt{R^2 - z^2}} dA \\
&= \int_0^R \int_0^{\sqrt{R^2 - z^2}} \frac{R}{\sqrt{R^2 - z^2}} dy dz = \int_0^R R dz = R^2.
\end{aligned}$$

So Area(S_1) $= 8R^2$.

Example 2. (see 15.6 Exercise 24) Let S_2 be the boundary of the solid that lies inside both cylinders $x^2 + z^2 = R^2$ and $y^2 + z^2 = R^2$. Find the area of S_2.

Solution: Let S_1 be the surface in Example 1. Then

$$\text{Area} S_2 = 2\text{Area}(S_1) = 16R^2.$$