Math V1202. Calculus IV, Section 004, Spring 2007

Solutions to Extra Practice Problems for [F] Section 1, 2

[F] Section 1

1. Find all the possible values and write them in the form $a + bi$:

   (a) $\ln(\sqrt{3} - i)$

   Solution:

$$\begin{aligned} \sqrt{3} - i &= 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = 2\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right) = 2e^{-i\pi/6} \\ \ln(\sqrt{3} - i) &= \ln 2 + i\left(-\frac{\pi}{6} + 2k\pi\right) = \ln 2 + \left(2k - \frac{1}{6}\right)\pi i \end{aligned}$$

   where k is any integer.

   (b) $(-2)^{1+i}$

   Solution: $-2 = 2(\cos\pi + i\sin\pi) = 2e^{i\pi}$, so

$$\ln(-2) = \ln 2 + i(\pi + 2k\pi) = \ln 2 + i\pi(2k+1),$$

   where k is any integer.

$$\begin{aligned} (-2)^{1+i} &= e^{(1+i)\ln(-2)} = e^{(1+i)(\ln 2 + i\pi(2k+1))} = e^{(\ln 2 - \pi(2k+1)) + i(\ln 2 + \pi(2k+1))} \\ &= 2e^{-\pi(2k+1)}(\cos(\ln 2 + \pi(2k+1)) + i\sin(\ln 2 + \pi(2k+1))) \\ &= 2e^{-\pi(2k+1)}\cos(\ln 2 + \pi) + 2e^{-\pi(2k+1)}\sin(\ln 2 + \pi)i \end{aligned}$$

   where k is any integer.

2. Beginning with the formula

$$\sin t = \frac{e^{it} - e^{-it}}{2i},$$

   find a formula for $\sin^{-1} x$ in terms of ln and square roots.

   Solution: Let $t = \sin^{-1} x$. Then $x = \sin t = \frac{1}{2i}(z - 1/z)$, where $z = e^{it}$. We have $z - \frac{1}{z} - 2ix = 0$, so $z^2 - 2ixz - 1 = 0$,

$$\begin{aligned} z &= \frac{2ix \pm \sqrt{-4x^2 + 4}}{2} = \pm\sqrt{1 - x^2} + ix, \\ t &= \frac{1}{i}\ln z = -i\ln(\pm\sqrt{1 - x^2} + ix). \end{aligned}$$

   We obtain the formula $\sin^{-1} x = -i\ln(\pm\sqrt{1 - x^2} + ix)$.

[F] Section 2

1. Let $f(z) = z^2/\bar{z}$.

   (a) Can $f(z)$ be continuously extended to $z = 0$? Explain.

   Solution:

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{z^2}{\bar{z}} = \lim_{r\to 0^+} \frac{(re^{i\theta})^2}{re^{-i\theta}} = \lim_{r\to 0^+} re^{3i\theta} = 0$$

   If we define $f(0) = 0$, then $\lim_{z\to 0} f(z) = f(0)$ so that $f(z)$ will be continuous at $z = 0$. This shows that $f(z)$ can be continuously extended to $z = 0$.

   (b) Write $f(z)$ in the form $u + iv$.

   Solution:

$$\begin{aligned} f(z) &= \frac{z^2}{\bar{z}} = \frac{(x+iy)^2}{x-iy} = \frac{(x+iy)^3}{(x-iy)(x+iy)} \\ &= \frac{x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3}{x^2+y^2} \\ &= \frac{x^3 - 3xy^2}{x^2+y^2} + i\frac{3x^2y - y^3}{x^2+y^2} \end{aligned}$$

   (c) Is $f(z)$ analytic where it is defined?

   Solution: by part (b), $f(z) = u + iv$, where

$$u(x,y) = \frac{x^3 - 3xy^2}{x^2+y^2}, \quad v(x,y) = \frac{3x^2y - y^3}{x^2+y^2}.$$

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{(3x^2-3y^2)(x^2+y^2) - (x^3-3xy^2)2x}{(x^2+y^2)^2} = \frac{x^4 - 3y^4 + 6x^2y^2}{(x^2+y^2)^2} \\ \frac{\partial v}{\partial y} &= \frac{(3x^2-3y^2)(x^2+y^2)^2 - (3x^2y - y^3)2y}{(x^2+y^2)^2} = \frac{3x^4 - y^4 - 6x^2y^2}{(x^2+y^2)^2} \end{aligned}$$

   $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$, so $f(z)$ is not analytic.

2. Let $u(x,y) = 3x^2y - y^3$.

   (a) Show that u is harmonic.

   Solution:

$$\begin{aligned} \frac{\partial u}{\partial x} &= 6xy, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(6xy) = 6y \\ \frac{\partial u}{\partial y} &= 3x^2 - 3y^2, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(3x^2 - 3y^2) = -6y \end{aligned}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y - 6y = 0$$

So u is harmonicc.

(b) Find a harmonic function $v(x, y)$ such that $f(z) = u + iv$ is analytic. (Hint: Cauchy-Riemann equations)

Solution: We need to find $v(x, y)$ such that

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 6xy, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 3y^2 - 3x^2$$

We have

$$v(x.y) = \int (3y^2 - 3x^2)dx = 3xy^2 - x^3 + g(y)$$
$$\frac{\partial v}{\partial y} = 6xy + g'(y)$$

So $g'(y) = 0$ and g is a constant function. We may take $g = 0$ so that $v(x, y) = 3xy^2 - x^3$. u, v satisfy Cauchy-Riemann equations, so $f(z) = u + iv$ is analytic and v is harmonic. Indeed

$$f(z) = u + iv = (3x^2y - y^3) + i(3xy^2 - x^3) = -i(x + iy)^3 = -iz^3$$