**Math V1202. Calculus IV, Section 004, Spring 2007**

# Solutions to Practice Midterm 2

**Problem 1** Let $C$ be the arc of the parabola $y = \frac{1}{2}x^2$ from $(1, \frac{1}{2})$ to $(2, 2)$. Evaluate the following line integrals.

(a) $\int_C x ds$

(b) $\int_C y dx + x^2 dy$

**Solution:** $x = t$, $y = t^2/2$, $1 \le t \le 2$, $dx = dt$, $dy = tdt$, $ds = \sqrt{1+t^2}dt$.

(a)

$$\int_C x ds = \int_1^2 t\sqrt{1+t^2}dt = \frac{1}{3}(1+t^2)^{3/2}\Big|_{t=1}^{t=2} = \frac{1}{3}(5\sqrt{5} - 2\sqrt{2}).$$

(b)

$$\int_C y dx + x^2 dy = \int_1^2 (\frac{t^2}{2} + t^3)dt = \left(\frac{t^3}{6} + \frac{t^4}{4}\right)\Big|_{t=1}^{t=2} = \frac{59}{12}.$$

**Problem 2** Consider the vector field

$$\mathbf{F} = (yz\cos(xz))\mathbf{i} + (\sin(xz) - z)\mathbf{j} + (xy\cos(xz) - y)\mathbf{k}.$$

(a) Find a function $f$ such that $\mathbf{F} = \nabla f$.

**Solution:**

$$\frac{\partial f}{\partial x} = yz\cos(xz), \quad \frac{\partial f}{\partial y} = \sin(xz) - z, \quad \frac{\partial f}{\partial z} = xy\cos(xz) - y$$

$$f(x,y,z) = \int yz\cos(xz)dx = y\sin(xz) + g(y,z)$$

$$\frac{\partial f}{\partial y} = \sin(xz) + \frac{\partial g}{\partial y} = \sin(xz) - z$$

$$\frac{\partial g}{\partial y} = -z, \qquad g(y,z) = \int (-z)dy = -yz + h(z)$$

$$f(x,y,z) = y\sin(xz) - yz + h(z)$$

$$\frac{\partial f}{\partial z} = xy\cos(xz) - y + \frac{dh}{dz} = xy\cos(xz) - y$$

$$\frac{dh}{dz} = 0, \qquad h(z) \text{ is a constant}$$

$$f(x, y, z) = y\sin(xz) - yz$$

(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $C$ is the curve given by

$$\mathbf{r}(t) = \langle (1-t)e^t, t^2, \sin(\frac{\pi}{2}t)\rangle, \quad 0 \le t \le 1.$$

**Solution:**

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) \\ &= f(0,1,1) - f(1,0,0) = -1 - 0 = -1 \end{aligned}$$

**Problem 3** Use Green's theorem to evaluate the line integral

$$\oint_C (x^3 - y^3)\ dx + (x^3 + y^3)\ dy$$

where $C$ is the oriented curve shown in Figure 1.

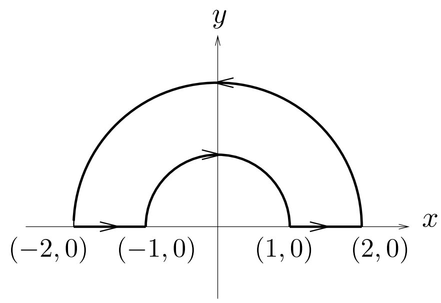


Figure 1: $C$ is the union of two semicircles and two line segments.

**Solution:** $C = \partial D$, where $D = \{(x, y) \mid 1 \le x^2 + y^2 \le 4, y \ge 0\}$. By Green's theorem,

$$\oint_C (x^3 - y^3)dx + (x^3 + y^3)dy = \iint_D (3x^2 + 3y^2)dxdy$$

$x = r\cos\theta, \quad y = r\sin\theta, \quad dxdy = rdrd\theta$

$$\iint_D (3x^2 + 3y^2)dxdy = \int_0^{\pi}\int_1^2 3r^3 drd\theta = \int_0^{\pi} \frac{3r^4}{4}\Bigg|_{r=1}^{r=2} d\theta = \int_0^{\pi} \frac{45}{4}d\theta = \frac{45\pi}{4}$$

**Problem 4** Consider the vector field $\mathbf{F} = xyz\,\mathbf{i} + yz\,\mathbf{j} - x^2\,\mathbf{k}$.

(a) Is $\mathbf{F}$ a conservative vector field? Explain.

**Solution:** No. The curl of a conservative vector field is $\mathbf{0}$, but

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & yz & -x^2 \end{vmatrix} = -y\mathbf{i} + (2x + xy)\mathbf{j} - xz\mathbf{k} \neq \mathbf{0}.$$

(b) Is there a vector field $\mathbf{G}$ such that $\operatorname{curl}\mathbf{G} = \mathbf{F}$? Explain.

**Solution:** No. We have $\operatorname{div}(\operatorname{curl}\mathbf{G}) = 0$ for any vector field $\mathbf{G}$, but

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(-x^2) = yz + z \neq 0.$$

**Problem 5** Let $T$ be the torus with vector equation

$\mathbf{r}(u,v) = (3{+}\cos u)\cos v\,\mathbf{i} + (3{+}\cos u)\sin v\,\mathbf{j} + \sin u\,\mathbf{k},\ 0 \le u \le 2\pi,\ 0 \le v \le 2\pi.$

(a) Calculate $\mathbf{r}_u$, $\mathbf{r}_v$, and $\mathbf{r}_u \times \mathbf{r}_v$.

**Solution:**

$$\begin{aligned} \mathbf{r}_u &= -\sin u\cos v\,\mathbf{i} - \sin u\sin v\,\mathbf{j} + \cos u\,\mathbf{k} \\ \mathbf{r}_v &= (3+\cos u)(-\sin v)\,\mathbf{i} + (3+\cos u)\cos v\,\mathbf{j} \\ &= (3+\cos u)(-\sin v\,\mathbf{i} + \cos v\,\mathbf{j}) \\ \mathbf{r}_u \times \mathbf{r}_v &= (3+\cos u)\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin u\cos v & -\sin u\sin v & \cos u \\ -\sin v & \cos v & \end{vmatrix} \\ &= (3+\cos u)(-\cos u\cos v\,\mathbf{i} - \cos u\sin v\,\mathbf{j} - \sin u\,\mathbf{k}) \end{aligned}$$

(b) Find an equation for the plane tangent to $T$ at $(\frac{5}{2}, 0, \frac{\sqrt{3}}{2})$.

**Solution:**

$$\mathbf{r}(\frac{2\pi}{3}, 0) = \langle \frac{5}{2}, 0, \frac{\sqrt{3}}{2}\rangle, \quad (\mathbf{r}_u \times \mathbf{r}_v)(\frac{2\pi}{3}, 0) = \langle \frac{5}{4}, 0, -\frac{5\sqrt{3}}{4}\rangle$$

So the tangent plane is given by

$$\frac{5}{4}(x - \frac{5}{2}) - \frac{\sqrt{3}}{4}(z - \frac{\sqrt{3}}{2}) = 0, \text{ i.e., } x - \sqrt{3}z = 1$$

(c) Find the surface area of $T$.

**Solution:** $|\mathbf{r}_u \times \mathbf{r}_v| = 3 + \cos u$.

$$\begin{aligned}\text{Area}(T) &= \int_0^{2\pi}\int_0^{2\pi}(3+\cos u)dudv = \int_0^{2\pi}(3u+\sin u)\Big|_{u=0}^{u=2\pi}dv \\ &= \int_0^{2\pi} 6\pi dv = 12\pi^2\end{aligned}$$

**Problem 6** Evaluate the surface integral $\iint_S z^4 dS$, where $S$ is the upper hemisphere $x^2+y^2+z^2=1,\ z\geq 0$.

**Solution:** $x=\sin\varphi\cos\theta$, $y=\sin\varphi\sin\theta$, $z=\cos\varphi$, $0\leq\theta\leq 2\pi$, $0\leq\varphi\leq\pi/2$, $dS=\sin\varphi d\theta d\varphi$.

$$\begin{aligned}\iint_S z^4 dS &= \int_0^{\pi/2}\int_0^{2\pi}\cos^4\varphi\sin\varphi d\theta d\varphi = \int_0^{\pi/2} 2\pi\cos^4\varphi\sin\varphi d\varphi \\ &= -\frac{2\pi}{5}\cos^5\varphi\Big|_{\varphi=0}^{\varphi=\pi/2} = \frac{2\pi}{5}.\end{aligned}$$

**Problem 7** Evaluate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$, and $S$ is the part of the cone $x^2+y^2=z^2$ between the planes $z=1$ and $z=4$, with upward orientation.

**Solution 1:** A vector equation of $S$ is given by $\mathbf{r}(x,y)=\langle x,y,g(x,y)\rangle$, where $g(x,y)=\sqrt{x^2+y^2}$ and $(x,y)$ is in $D=\{(x,y)\in\mathbb{R}^2 \mid 1\leq x^2+y^2\leq 16\}$. We have

$$\begin{aligned}\mathbf{F}(\mathbf{r}(x,y)) &= \langle -y, x, \sqrt{x^2+y^2}\rangle \\ \mathbf{r}_x\times\mathbf{r}_y &= \langle -g_x, -g_y, 1\rangle = \langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{x\sqrt{x^2+y^2}}, 1\rangle\end{aligned}$$

$\mathbf{r}_x\times\mathbf{r}_y$ is upward, so

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(x,y))\cdot\mathbf{r}_x\times\mathbf{r}_y\,dxdy = \iint_D \sqrt{x^2+y^2}dxdy$$

We use polar coordinates $x=r\cos\theta$, $y=r\sin\theta$, $dxdy=rdrd\theta$.

$$\iint_D \sqrt{x^2+y^2}dxdy = \int_0^{2\pi}\int_1^4 r^2 drd\theta = \int_0^{2\pi}\frac{r^3}{3}\Big|_{r=1}^{r=4}d\theta = \int_0^{2\pi} 21 d\theta = 42\pi.$$

**Solution 2:** A vector equation of $S$ is given by $\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r\rangle$, $1 \le r \le 4$, $0 \le \theta \le 2\pi$.

$$\mathbf{r}_r = \langle \cos\theta, \sin\theta, 1\rangle, \quad \mathbf{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$$

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = -r\cos\theta\mathbf{i} - r\sin\theta\mathbf{j} + r\mathbf{k}$$

$\mathbf{r}_r \times \mathbf{r}_\theta$ is upward, so

$$\begin{aligned} \iint_S \mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_1^4 \mathbf{F}(\mathbf{r}(r,\theta))\cdot \mathbf{r}_r \times \mathbf{r}_\theta \, dr d\theta \\ &= \int_0^{2\pi}\int_1^4 \langle -r\sin\theta, r\cos\theta, r\rangle \cdot \langle -r\cos\theta, -r\sin\theta, r\rangle dr d\theta \\ &= \int_0^{2}\int_1^4 r^2 dr d\theta = \int_0^{2\pi} \frac{r^3}{3}\bigg|_{r=1}^{r=4} d\theta = \int_0^{2\pi} 21 d\theta = 42\pi \end{aligned}$$