Math V1202. Calculus IV, Section 004, Spring 2007

Solutions to Midterm 2

Problem 1. Consider a vector field $\mathbf{F}(x,y,z) = \sin y\mathbf{i} + \cos x\mathbf{j} + z^2\mathbf{k}$.

(a) (8%) Is $\mathbf{F}$ conservative? Explain.

Solution: No. The curl of a conservative vector field is $\mathbf{0}$, but

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y & \cos x & z^2 \end{vmatrix} = (-\sin x - \cos y)\mathbf{k} \neq \mathbf{0}.$$

(b) (10%) Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the line segment from $(2,1,0)$ to $(0,1,3)$.

Solution: The line segment from $(2,1,0)$ to $(0,1,3)$ is given by

$$\mathbf{r}(t) = (1-t)\langle 2,1,0\rangle + t\langle 0,1,3\rangle = \langle 2-2t, 1, 3t\rangle, \quad 0 \le t \le 1.$$

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) &= \langle \sin 1, \cos(2-2t), 9t^2\rangle \\ \mathbf{r}'(t) &= \langle -2, 0, 3\rangle \\ \int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt = \int_0^1 (-2\sin 1 + 27t^2)dt \\ &= \left(-2\sin 1\cdot t + 9t^3\right)\Big|_{t=0}^{t=1} = 9 - 2\sin 1 \end{aligned}$$

Problem 2. Consider the vector field

$$\mathbf{F}(x,y,z) = (3x^2yz - 3y)\mathbf{i} + (x^3z - 3x)\mathbf{j} + (x^3y + 2z)\mathbf{k}.$$

(a) (12%) Find a function f such that $\mathbf{F} = \nabla f$.

Solution:

$$\frac{\partial f}{\partial x} = 3x^2yz - 3y, \quad \frac{\partial f}{\partial y} = x^3z - 3x, \quad \frac{\partial f}{\partial z} = x^3y + 2z$$

$$\begin{aligned} &f(x,y,z) = \int (3x^2yz - 3y)dx = x^3yz - 3xy + g(y,z) \\ &\frac{\partial f}{\partial y} = x^3z - 3x = x^3z - 3x + \frac{\partial g}{\partial y}, \quad g(y,z) = \int 0dy = h(z) \\ &f(x,y,z) = x^3yz - 3xy + h(z) \\ &\frac{\partial f}{\partial z} = x^3y + 2z = x^3y + \frac{dh}{dz}, \quad h(z) = \int 2zdz = z^2 + c \end{aligned}$$

where c is a constant. We may take $c = 0$ so that

$$f(x,y,z) = x^3yz - 3xy + z^2.$$

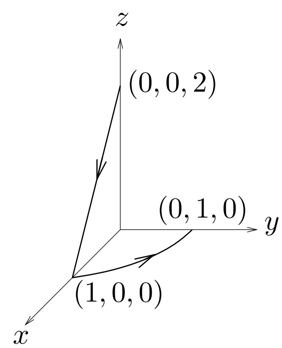

Figure 1: C is the union of the line segment from $(0,0,2)$ to $(1,0,0)$ and the arc $\{(x,y,z) \mid x^2+y^2=1,\ x\geq 0,\ y\geq 0,\ z=0\}$ from $(1,0,0)$ to $(0,1,0)$.

(b) (6%) Use part (a) to evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the oriented curve from $(0,0,2)$ to $(0,1,0)$ shown in Figure 1.

Solution:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f\cdot d\mathbf{r} = f(0,1,0) - f(0,0,2) = 0 - 4 = -4$$

where the second equality follows from the fundamental theorem for line integrals.

Problem 3. Use Green's theorem to evaluate the line integral

$$\int_C (\sin(x^3) - y)dx + (x - e^{y^2})dy$$

where C is the rectangle with vertices $(1,1)$, $(4,1)$, $(4,3)$, and $(1,3)$, oriented counterclockwise.
Solution: $C = \partial D$, where $D = \{(x,y) \mid 1\leq x\leq 4, 1\leq y\leq 3\}$.
By Green's theorem,

$$\begin{aligned}
&\oint_C (\sin(x^3) - y)dx + (x - e^{y^2})dy \\
=& \iint_D \frac{\partial}{\partial x}(x - e^{y^2}) - \frac{\partial}{\partial y}(\sin(x^3) - y)dxdy \\
=& \iint_D 2dxdy = \int_1^3\int_1^4 2dxdy = 12
\end{aligned}$$

Problem 4. Consider two vector fields

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}, \quad \mathbf{G}(x,y) = Q(x,y)\mathbf{i} - P(x,y)\mathbf{j}.$$

Show that $\operatorname{curl}\mathbf{F}\cdot\mathbf{k} = \operatorname{div}\mathbf{G}$.

Solution:

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + \left(\frac{\partial}{\partial x}Q(x,y) - \frac{\partial}{\partial y}P(x,y)\right)\mathbf{k}.$$

So

$$\operatorname{curl}\mathbf{F}\cdot\mathbf{k} = \frac{\partial}{\partial x}Q(x,y) - \frac{\partial}{\partial y}P(x,y).$$

We also have

$$\operatorname{div}\mathbf{G} = \frac{\partial}{\partial x}Q(x,y) + \frac{\partial}{\partial y}(-P(x,y)) = \frac{\partial}{\partial x}Q(x,y) - \frac{\partial}{\partial y}P(x,y).$$

So $\operatorname{curl}\mathbf{F}\cdot\mathbf{k} = \operatorname{div}\mathbf{G}$.

Problem 5. Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$, and S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the plane $z = 0$, with downward orientation.

Solution 1: A vector equation of S is given by $\mathbf{r}(x,y) = \langle x, y, g(x,y)\rangle$, where $g(x,y) = 4 - x^2 - y^2$ and $(x,y) \in D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 4\}$. We have

$$\begin{aligned} \mathbf{F}(\mathbf{r}(x,y)) &= \mathbf{F}(x,y,4-x^2-y^2) = \langle x, y, x^2+y^2-4\rangle \\ \mathbf{r}_x \times \mathbf{r}_y &= \langle -g_x, -g_y, 1\rangle = \langle 2x, 2y, 1\rangle \end{aligned}$$

$\mathbf{r}_x \times \mathbf{r}_y$ is upward, so

$$\begin{aligned} \iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}(\mathbf{r}(x,y))\cdot(-\mathbf{r}_x\times\mathbf{r}_y)\,dxdy \\ &= \iint_D \langle x, y, x^2+y^2-4\rangle\cdot\langle -2x, -2y, -1\rangle\,dxdy \\ &= \iint_D (4 - 3x^2 - 3y^2)\,dxdy \end{aligned}$$

We use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = rdrd\theta$.

$$\begin{aligned} \iint_D (4-3x^2-3y^2)\,dxdy &= \int_0^{2\pi}\int_0^2 (4-3r^2)\,rdrd\theta \\ = \int_0^{2\pi}\int_0^2 (4r - 3r^3)\,drd\theta &= \int_0^{2\pi}\left(2r^2 - \frac{3r^4}{4}\right)\Big|_{r=0}^{r=2} d\theta \\ = \int_0^{2\pi} -4d\theta &= -8\pi \end{aligned}$$

Solution 2: A vector equation of S is given by

$$\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 4-r^2\rangle, \quad 0 \le r \le 2, \quad 0 \le \theta \le 2\pi.$$

We have

$$\begin{aligned}
&\mathbf{F}(\mathbf{r}(r,\theta)) = \mathbf{F}(r\cos\theta, r\sin\theta, 4-r^2) = \langle r\cos\theta, r\sin\theta, r^2-4\rangle \\
&\mathbf{r}_r = \langle \cos\theta, \sin\theta, -2r\rangle, \quad \mathbf{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle \\
&\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = 2r^2\cos\theta\mathbf{i} + 2r^2\sin\theta\mathbf{j} + r\mathbf{k}
\end{aligned}$$

$\mathbf{r}_r \times \mathbf{r}_\theta$ is upward, so

$$\begin{aligned}
&\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^2 \mathbf{F}(\mathbf{r}(r,\theta))\cdot(-\mathbf{r}_r\times\mathbf{r}_\theta) dr dtheta \\
= &\int_0^{2\pi}\int_0^2 \langle r\cos\theta, r\sin\theta, r^2-4\rangle\cdot\langle -2r^2\cos\theta, -2r^2\sin\theta, -r\rangle dr d\theta \\
= &\int_0^{2\pi}\int_0^2 (-2r^3\cos^2\theta - 2r^3\sin^2\theta - r^3 + 4r) dr d\theta \\
= &\int_0^{2\pi}\int_0^2 (-3r^3+4r) dr d\theta = \int_0^{2\pi}\left(-\frac{3r^4}{4} + 2r^2\right)\Big|_{r=0}^{r=2} d\theta \\
= &\int_0^{2\pi} -4 d\theta = -8\pi
\end{aligned}$$

Problem 6. Evaluate $\iint_S yz dS$, where S is the part of the sphere $x^2+y^2+z^2=4$ in the first octant.
Solution 1: A vector equation of S is given by

$$\mathbf{r}(\theta,\varphi) = \langle 2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi\rangle, \quad 0 \le \theta \le \frac{\pi}{2}, 0 \le \varphi \le \frac{\pi}{2}$$

$$|\mathbf{r}_\theta \times \mathbf{r}_\varphi| = 4\sin\varphi$$

$$\begin{aligned}
&\iint_S yz dS = \int_0^{\pi/2}\int_0^{\pi/2} 2\sin\varphi\sin\theta\cdot 2\cos\varphi |\mathbf{r}_\theta\times\mathbf{r}_\varphi| d\theta d\varphi \\
= &\ 16\int_0^{\pi/2}\int_0^{\pi/2} \sin^2\varphi\cos\varphi\sin\theta d\theta d\varphi = 16\int_0^{\pi/2}\sin\theta d\theta \int_0^{\pi/2}\sin^2\varphi\cos\varphi d\varphi \\
= &\ 16\left(-\cos\theta\Big|_{\theta=0}^{\theta=\pi/2}\right)\cdot\left(\frac{\sin^3\varphi}{3}\Big|_{\theta=0}^{\theta=\pi/2}\right) = 16\cdot 1\cdot\frac{1}{3} = \frac{16}{3}
\end{aligned}$$

Solution 2: A vector equation of S is given by $\mathbf{r}(x,y) = \langle x, y, g(x,y)\rangle$, where $g(x,y) = \sqrt{4-x^2-y^2}$ and

$$(x,y) \in D = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \le 2, x \ge 0, y \ge 0\}.$$

$$\begin{aligned}\mathbf{r}_x \times \mathbf{r}_y &= \langle -g_x, -g_y, 1\rangle = \langle \frac{x}{\sqrt{4-x^2-y^2}}, \frac{y}{\sqrt{4-x^2-y^2}}, 1\rangle \\ |\mathbf{r}_x \times \mathbf{r}_y| &= \sqrt{\frac{x^2}{4-x^2-y^2} + \frac{y^2}{4-x^2-y^2} + 1} = \sqrt{\frac{4}{4-x^2-y^2}} = \frac{2}{\sqrt{4-x^2-y^2}}.\end{aligned}$$

$$\iint_S yz dS = \iint_D y\sqrt{4-x^2-y^2}|\mathbf{r}_x \times \mathbf{r}_y| dxdy = \iint_D 2y dxdy$$

We use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = rdrd\theta$.

$$\begin{aligned}\iint_D 2y dxdy &= \int_0^{\pi/2}\int_0^2 2r\sin\theta r dr d\theta = \int_0^{\pi/2} \sin\theta d\theta \int_0^2 2r^2 dr \\ &= \left(-\cos\theta\Big|_{\theta=0}^{\theta=\pi/2}\right)\cdot\left(\frac{2r^3}{3}\Big|_{r=0}^{r=2}\right) = 1\cdot\frac{16}{3} = \frac{16}{3}\end{aligned}$$

Solution 3: A vector equation of S is given by

$$\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, \sqrt{4-r^2}\rangle, \quad 0 \le r \le 2, 0 \le \theta \le \frac{\pi}{2}$$

$$\mathbf{r}_r = \langle \cos\theta, \sin\theta, \frac{-r}{\sqrt{4-r^2}}\rangle, \quad \mathbf{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$$

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -\frac{r}{\sqrt{4-r^2}} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \frac{r^2\cos\theta}{\sqrt{4-r^2}}\mathbf{i} + \frac{r^2\sin\theta}{\sqrt{4-r^2}}\mathbf{j} + r\mathbf{k}$$

$$|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\frac{r^4\cos^2\theta}{4-r^2} + \frac{r^4\sin^2\theta}{4-r^2} + r^2} = \sqrt{\frac{4r^2}{4-r^2}} = \frac{2r}{\sqrt{4-r^2}}$$

$$\begin{aligned}\iint_S yz dS &= \int_0^{\pi/2}\int_0^2 r\sin\theta\sqrt{4-r^2}|\mathbf{r}_r \times \mathbf{r}_\theta| dr d\theta = \int_0^{\pi/2}\int_0^2 2r^2\sin\theta dr d\theta \\ &= \int_0^{\pi/2}\sin\theta d\theta\int_0^2 2r^2 dr = \left(-\cos\theta\Big|_{\theta=0}^{\theta=\pi/2}\right)\cdot\left(\frac{2r^3}{3}\Big|_{r=0}^{r=2}\right) = 1\cdot\frac{16}{3} = \frac{16}{3}\end{aligned}$$

Problem 7. (12%) Let S be the part of the cylinder $x^2 + z^2 = R^2$ that lies inside the cylinder $y^2 + z^2 = R^2$. Find the area of S.

Solution 1: Let S_+ be the part of S in the first octant. A vector equation of S_+ is given by

$$\mathbf{r}(y,z) = \langle \sqrt{R^2-z^2}, y, z\rangle, \quad (y,z) \in D = \{(y,z) \in \mathbb{R}^2 \mid y^2+z^2 \le R^2, y, z \ge 0\}$$

$$\mathbf{r}_y = \langle 0,1,0\rangle, \quad \mathbf{r}_z = \langle \frac{-z}{\sqrt{R^2-z^2}}, 0, 1\rangle$$

$$\mathbf{r}_y \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ \frac{-z}{\sqrt{R^2-z^2}} & 0 & 1 \end{vmatrix} = \mathbf{i} + \frac{z}{\sqrt{R^2-z^2}}\mathbf{k}$$

$$|\mathbf{r}_y \times \mathbf{r}_z| = \sqrt{1+\frac{z^2}{R^2-z^2}} = \sqrt{\frac{R^2}{R^2-z^2}} = \frac{R}{\sqrt{R^2-z^2}}$$

$$\text{Area}(S_+) = \iint_D |\mathbf{r}_y \times \mathbf{r}_z| dydz = \int_0^R \int_0^{\sqrt{R^2-z^2}} \frac{R}{\sqrt{R^2-z^2}} dydz = \int_0^R Rdz = R^2$$

$$\text{Area}(S) = 8\text{Area}(S_+) = 8R^2$$

Solution 2: Let S_+ be the part of S in the first octant. A vector equation of S_+ is given by $\mathbf{r}(x,y) = \langle x, y, g(x,y)\rangle$, where $g(x,y) = \sqrt{R^2-x^2}$ and (x,y) is in the triangle D with vertices $(0,0)$, $(R,0)$, (R,R).

$$\begin{aligned} \mathbf{r}_x \times \mathbf{r}_y &= \langle -g_x. - g_y, 1\rangle = \langle \frac{x}{\sqrt{R^2-x^2}}, 0, 1\rangle \\ |\mathbf{r}_x \times \mathbf{r}_y| &= \sqrt{\frac{x^2}{R^2-x^2}+1} = \sqrt{\frac{R^2}{R^2-x^2}} = \frac{R}{\sqrt{R^2-x^2}} \end{aligned}$$

$$\begin{aligned} \text{Area}(S_+) &= \iint_D |\mathbf{r}_x \times \mathbf{r}_y| dxdy = \int_0^R \int_0^x \frac{R}{\sqrt{R^2-x^2}} dydx \\ &= \int_0^R \frac{Rx}{\sqrt{R^2-x^2}} dx = -R\sqrt{R^2-x^2}\Big|_{x=0}^{x=R} = R^2 \end{aligned}$$

$$\text{Area}(S) = 8\text{Area}(S_+) = 8R^2$$

Solution 3: A vector equation of S is given by $\mathbf{r}(\theta, y) = \langle R\sin\theta, y, R\cos\theta\rangle$, $(\theta, y) \in D$, where

$$\begin{aligned} D &= \{(\theta,y) \in \mathbb{R}^2 \mid 0 \le \theta \le 2\pi, y^2 \le R^2 - (R\cos\theta)^2\} \\ &= \{(\theta,y) \in \mathbb{R}^2 \mid 0 \le \theta \le 2\pi, -R|\sin\theta| \le y \le R|\sin\theta|\} \end{aligned}$$

$$\mathbf{r}_\theta = \langle R\cos\theta, 0, -R\sin\theta\rangle, \quad \mathbf{r}_y = \langle 0,1,0\rangle$$

$$\mathbf{r}_\theta \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ R\cos\theta & 0 & -R\sin\theta \\ 0 & 1 & 0 \end{vmatrix} = R\sin\theta\mathbf{i} + R\cos\theta\mathbf{k}$$

$$|\mathbf{r}_\theta \times \mathbf{r}_y| = R$$

$$\begin{aligned} \text{Area}(S) &= \iint_D |\mathbf{r}_\theta \times \mathbf{r}_y| d\theta dy = \int_0^{2\pi} \int_{-R|\sin\theta|}^{R|\sin\theta|} Rdyd\theta = \int_0^{2\pi} 2R^2|\sin\theta|d\theta \\ &= 4R^2 \int_0^\pi \sin\theta d\theta = 4R^2\left(-\cos\theta\Big|_{\theta=0}^{\theta=\pi}\right) = 8R^2 \end{aligned}$$