Math V1202. Calculus IV, Section 004, Spring 2007

Solutions to Practice Midterm 1

Problem 1. Evaluate $\iint_D y\, dA$, where D is the region in the first quadrant that is bounded by the lines $y = x$, $y = 2$ and the hyperbola $xy = 1$.

Solution:

$$\begin{aligned} \iint_D y dA &= \int_1^2 \int_{1/y}^{y} y dx dy = \int_1^2 y(y - 1/y) dy = \int_1^2 (y^2 - 1) \\ &= \left(\frac{y^3}{3} - y \right) \Bigg|_{y=1}^{y=2} = \frac{4}{3} \end{aligned}$$

Problem 2. Evaluate the integral

$$\int_0^4 \int_{\sqrt{y}}^{2} e^{x^3} dx dy.$$

Solution:

$$\int_0^4 \int_{\sqrt{y}}^{2} e^{x^3} dx dy = \int_0^2 \int_0^{x^2} e^{x^3} dy dx = \int_0^2 x^2 e^{x^3} dx = \frac{e^{x^3}}{3} \Bigg|_{x=0}^{x=2} = \frac{e^8 - 1}{3}$$

Problem 3. Evaluate the integral

$$\iint_R \cos\left(\frac{y - x}{y + x} \right) dA,$$

where R is the triangular region with vertices $(2, 0)$, $(0, 2)$, and $(0, 0)$.

Solution: We use the transformation $u = y - x$, $v = y + x$. Then

$$x = \frac{v - u}{2}, \quad y = \frac{u + v}{2}, \quad \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2},$$

so

$$dA = dx dy = \left| -\frac{1}{2} \right| du dv = \frac{1}{2} du dv.$$

$$\begin{aligned} \iint_R \cos\left(\frac{y - x}{y + x} \right) dA &= \int_0^2 \int_{-v}^{v} \cos \frac{u}{v} \frac{1}{2} du dv = \int_0^2 \frac{v}{2} \sin \frac{u}{v} \Bigg|_{u=-v}^{u=v} \\ &= \int_0^2 v \sin 1 dv = \frac{v^2}{2} \sin 1 \Bigg|_{v=0}^{v=2} = 2 \sin 1 \end{aligned}$$

Problem 4. Evaluatee
$$\iiint_E \frac{1}{x^2+y^2+z^2}\, dV,$$
where E is the region bounded by the spheres with center the origin and radii 1 and 3.

Solution: $x = \rho \sin\varphi \cos\theta, \quad y = \rho \sin\varphi \sin\theta, \quad z = \rho \cos\varphi$

$$dV = dxdydz = \rho^2 \sin\phi d\rho d\varphi d\theta$$

$$\begin{aligned} \iiint_E \frac{1}{x^2+y^2+z^2}\, dV &= \int_0^{2\pi}\int_0^{\pi}\int_1^3 \sin\varphi d\rho d\varphi d\theta = \int_0^{2\pi}\int_0^{\pi} 2\sin\varphi d\varphi d\theta \\ &= \int_0^{2\pi} (-2\cos\varphi)|_{\varphi=0}^{\varphi=\pi} d\theta = 8\pi \end{aligned}$$

Problem 5. Evaluate $\iiint_E xzdV$, where E is bounded by the cylinder $x^2+y^2=4$ and the planes $x=0$, $y=0$, $z=0$, and $y=z$ in the first octant.

Solution: $x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$

$$dV = dxdydz = rdzdrd\theta$$

$$\begin{aligned} \iiint_E xzdV &= \int_0^{\pi/2}\int_0^2\int_0^{r\sin\theta} r^2\cos\theta z dz dr d\theta \\ &= \int_0^{\pi/2}\int_0^2 \left(r^2\cos\theta \frac{z^2}{2} \right)\Bigg|_{z=0}^{z=r\sin\theta} drd\theta \\ &= \int_0^{\pi/2}\int_0^2 \frac{1}{2} r^4 \sin^2\theta\cos\theta drd\theta \\ &= \int_0^{\pi/2} \left(\frac{1}{10} r^5 \sin^2\theta \cos\theta \right)\Bigg|_{r=0}^{r=2} d\theta \\ &= \int_0^{\pi/2} \frac{16}{5}\sin^2\theta\cos\theta d\theta = \left(\frac{16}{15}\sin^3\theta \right)\Bigg|_0^{\pi/2} = \frac{16}{15} \end{aligned}$$

Problem 6. Find the volume of the solid E that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=4$.

Solution: $x = \rho \sin\varphi \cos\theta, \quad y = \rho \sin\varphi \sin\theta, \quad z = \rho \cos\varphi,$

$$dV = dxdydz = \rho^2 \sin\varphi d\rho d\varphi d\theta$$

$$\begin{aligned}
\text{volume}(E) &= \iiint_W 1dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^2 \rho^2 \sin\varphi d\rho d\varphi d\theta \\
&= \int_0^{2\pi} 1d\theta \int_0^{\pi/4} \sin\varphi d\varphi \int_0^2 \rho^2 d\rho \\
&= 2\pi \cdot (-\cos\varphi)\Big|_{\varphi=0}^{\varphi=\pi/4} \cdot \frac{\rho^3}{3}\Big|_{\rho=0}^{\rho=2} = \frac{16\pi}{3}(1 - \frac{\sqrt{2}}{2}).
\end{aligned}$$

Problem 7. Find the area of the region inside the circle $r = 2\cos\theta$ and outside the circle $r = \sqrt{2}$.

Solution: Note that the polar curves $r = 2\cos\theta$ and $r = \sqrt{2}$ intersect when $2\cos\theta = \sqrt{2}$, or equivalently, $\theta = \pm\pi/4$. The area is given by

$$\begin{aligned}
\int_{-\pi/4}^{\pi/4}\int_{\sqrt{2}}^{2\cos\theta} rdrd\theta &= \int_{-\pi/4}^{\pi/4} \frac{r^2}{2}\Big|_{r=\sqrt{2}}^{r=2\cos\theta} d\theta = \int_{-\pi/4}^{\pi/4}(2\cos^2\theta - 1)d\theta \\
&= \int_{-\pi/4}^{\pi/4} \cos(2\theta)d\theta = \frac{1}{2}\sin(2\theta)\Big|_{-\pi/4}^{\pi/4} = 1.
\end{aligned}$$