**Math V1202. Calculus IV, Section 004, Spring 2007**

# Practice Final Exam

**Problem 1** Consider the integral

$$\int_1^2 \int_x^{x^2} 12x \, dy \, dx + \int_2^4 \int_x^4 12x \, dy \, dx$$

(a) Sketch the region of integration.

(b) Reverse the order of integration and evaluate the integral that you get.

**Problem 2** Consider the transformation of $\mathbb{R}^2$ defined by the equations given by $x = u/v$, $y = v$.

(a) Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ of the transformaion.

(b) Let $R$ be the region in the first quadrant bounded by the lines $y = x$, $y = 2x$ and the hyperbolas $xy = 1$, $xy = 2$. Sketch the region $S$ in the $uv$-plane corresponding to $R$.

(c) Evaluate $\iint_R y^4 dA$.

**Problem 3** Let $S$ be the boundary of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$, with outward orientation.

(a) Find the surface area of $S$. Note that the surface $S$ consists of a portion of the paraboloid $z = x^2 + y^2$ and a portion of the plane $z = 4$.

(b) Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = (x + y^2z^2)\mathbf{i} + (y + z^2x^2)\mathbf{j} + (z + x^2y^2)\mathbf{k}$.

**Problem 4** Let

$$\mathbf{F} = \frac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}.$$

Note that $\mathbf{F}$ is defined on $\{(x, y) \in \mathbb{R} \mid (x, y) \neq (0, 0)\}$.

(a) Evaluate $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$, where $C_1$ is the circle $x^2 + y^2 = 1$, oriented counterclockwise.

(b) Compute $\operatorname{curl} \mathbf{F}$.

(c) Use Green's Theorem to evaluate $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where $C_2$ is the circle $(x - 2)^2 + (y - 2)^2 = 1$, oriented counterclockwise.

(d) Is $\mathbf{F}$ conservative?

**Problem 5** Let $E$ be a solid in the first octant bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 1$. Evaluate $\iiint_E xyz^2 dV$.

**Problem 6** Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F} = e^{y^2}\mathbf{i} + (y + \sin(z^2))\mathbf{j} + (z - 1)\mathbf{k},$$

and $S$ is the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, oriented upward. Note that the surface $S$ does NOT include the bottom of the hemisphere.

**Problem 7** Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F} = x^2y\mathbf{i} - xy^2\mathbf{j} + z^3\mathbf{k},$$

and $C$ is the curve of intersection of the plane $3x + 2y + z = 6$ and the cylinder $x^2 + y^2 = 4$, oriented clockwise when viewed from above.

**Problem 8** Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F} = (\sin(y + z) - yx^2 - \frac{y^3}{3})\,\mathbf{i} + x\cos(y + z)\,\mathbf{j} + \cos(2y)\,\mathbf{k},$$

and $S$ consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outward.

**Problem 9** Write in the form of $a + bi$:

(a) Find all the fourth roots of $-4$.

(b) Evaluate $(1 - i)^{10}$.

(c) Find all the possible values of $(-2)^i$.

**Problem 10** Let $f(z) = e^{iz}$.

(a) Write $f(z)$ in the form $u + iv$.

(b) Is $f(z)$ analytic?

**Problem 11** Let $f(z)$ be an analytic function which only takes real values, i.e., $\mathrm{Im} f(z) = 0$. Show that $f(z)$ is a constant function. (Hint: Use Cauchy-Riemann equations.)