THE PROPER LG SUPERPOTENTIAL IS THE OPEN MIRROR MAP

↳ GRÄFNITZ, RUDDAT

This talk is about

$$M(Q) = 1 + 2Q + 5Q^2 + 32Q^3 + 286Q^4 + \dots$$

$$\left(= \left(\frac{Q}{z(Q)}\right)^{1/3}{}_{*} \text{, where } Q(z) = z\exp\left(\sum_{k\geq 1} \frac{(3k)!}{k(k!)^3} z^k\right) \right)$$

It has appeared in many settings

- Today: Superpotential of mirror of $(\mathbb{P}^2, E_{\mathrm{smth}})$

  || Thm [GRZ] 

- Open mirror map for AV branes in $K_{\mathbb{P}^2}$ [AKV, GZ]
- Open/Closed Picard-Fuchs equations for MS [M, L-M, F-L]

  closed mirror map   open mirror map M

  $K_{\mathbb{P}^2}$ mir:
  $$uv = 1 + x + y + \frac{z}{xy} \sim z^{-1/3}{}_{*} + x + y + \frac{1}{xy}$$
  $$x = z^{1/3}\tilde{x} \qquad 1 + z^{1/3}\left(\tilde{x} + \tilde{y} + \frac{1}{\tilde{x}\tilde{y}}\right)$$

- Slab function in G-S construction for mirror of $K_{\mathbb{P}^2}$ [G-S]
- Open invts of $K_{\mathbb{P}^2}$ and closed invts of $K_{\widehat{\mathbb{P}^2}}$ [C, CLL, CLT, CCLT, LLW]
  "CCLLTU"

  Fibers

  $$M(Q) = \sum_d n_{dL-C}(-Q)^d$$

Road Map

[CPS] : Tropical def. of W

Then : Tropical - Relative *

Relative / Log - Local

Local - Open [CCLLTW, F-L, L-M]

ALSO: W from wall crossing *

* also g>0

Some Context

- MS happens near toric degenerations

  $\xleftrightarrow[KS]{GS}$ affine mfld w/ sings

  ~~Moduli of Lag torus fibers~~

  dual $\cap$ cpx ✓

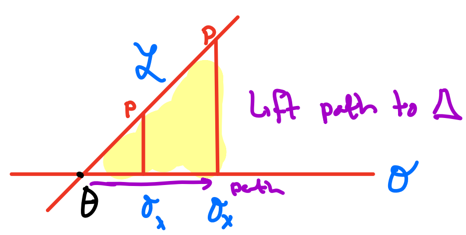
- Deformation from central sing

  incorporates disk instantons = hol disks bounding torus fibers

- Homog coord ring of family = Ring of Θ functions

  can be defined tropically GHKK, GS, CPS

Idea! $T^2 = \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}/\mathbb{Z} = B$

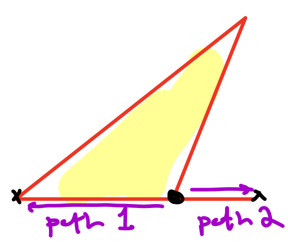


"$\theta \circ p = \theta(x)$" $\sum_n e^{2\pi i \tau (n+x)^2/2}$ nearly $\theta_{E_\tau}(x)$

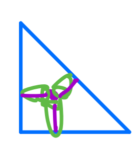
$\mathrm{Hom}(\mathcal{O}, \mathcal{L}) \otimes \mathrm{Hom}(\mathcal{L}, \mathcal{O}_x) \to \mathrm{Hom}(\mathcal{O}, \mathcal{O}_x)$ [PZ]

$\theta \qquad p \qquad \theta(x)\cdot x$

$\theta \circ \theta \longleftrightarrow$ pairs of paths



- In Fano case, θ prim = sup pot'l



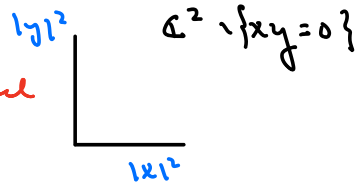
Hori-Vafa, Cho-Oh

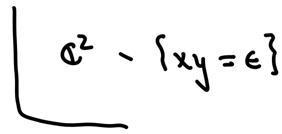
Fukaya, FOOO

$$W = x + y + \frac{t^3}{xy} \overset{x \to \frac{y}{x}t}{\underset{y \to ty}{\sim}} t\left(\frac{y}{x} + y + \frac{x}{y^2}\right)$$

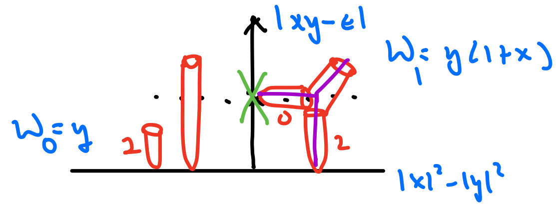
Auroux (Seidel) (Fukaya) explained role of singular fibers $\longleftrightarrow$ sings of affine structures

Local model





$\rightsquigarrow$



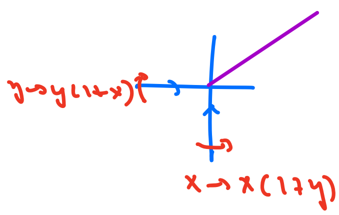
"  "

Lesson:

- Maslov-0 loci = walls
- $\Rightarrow$ chamber structure $\leftrightarrow$ charts for mirror
- coord $x f^{m}$* s.t. W well-defined
  "wall function"

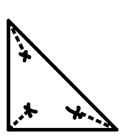
*cluster type

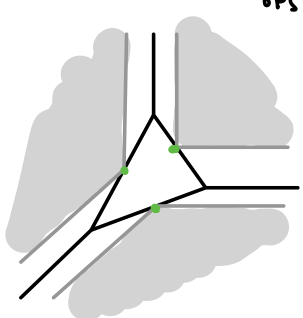
GPS: When walls collide



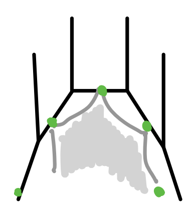
New ray forced by consistency : $1+xy$

Toric degeneration $\underset{GS}{\rightsquigarrow}$ Dual intersection cpx $\underset{GPS}{\rightsquigarrow}$ Consistent Scattering





unwrap [CPS,G] $\longrightarrow$



$\Rightarrow$ tropical def of θ-ring, W
[GS, CPS]

Explicitly for $\mathbb{P}^2, E_{sm}$

$$\mathcal{X} = \{ xyz - t(v + f_3(x,y,z)) \} \subset \mathbb{P}(1113) \times \mathbb{A}^1_t$$

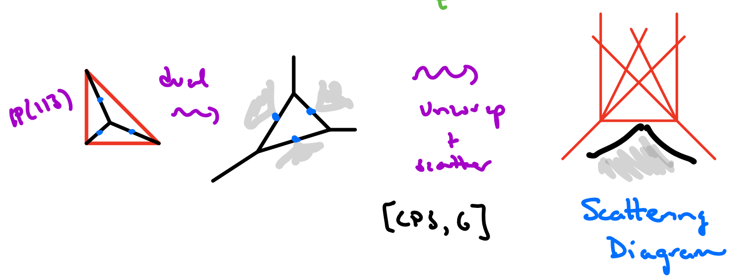
$$\cup$$

$$D = \{ v = 0 \}$$

$t \neq 0$ solve for $v \Rightarrow X_t \cong \mathbb{P}^2$

$D_t = E$

$t = 0$



Thm

$(\mathbb{P}^2, E)$

$Q^\beta = x^a y^b$

Unbounded chamber ($\Rightarrow a = 0$)

$$W = y M(Q), \quad Q = (t/y)^3$$

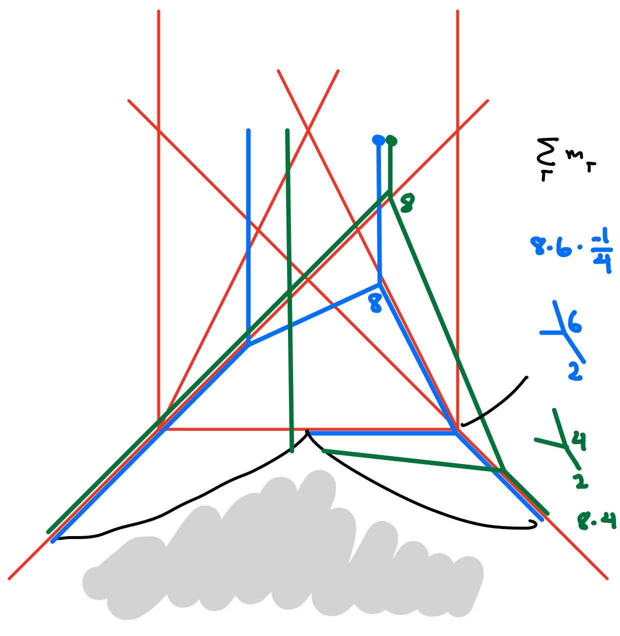
$$= y(1 + 2Q + 5Q^2 + 32Q^3 + \dots)$$

$$Q^d \leftrightarrow y^{-(3d-1)}$$

COMPUTE $32Q^3$

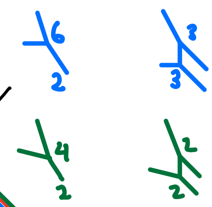
$(8 = 3\cdot 3 - 1)$

$m_\Gamma = \frac{1}{|\mathrm{Aut}\,\Gamma|}\cdot m_V\cdot m_E$



$\sum_\Gamma m_\Gamma =$

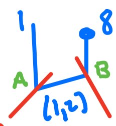
$8\cdot 6\cdot \frac{-1}{4} + \frac{1}{2}\cdot 8\cdot 3\cdot 3 = 24$



$8\cdot 4\cdot \frac{-1}{4} + \frac{1}{2}\cdot 8\cdot 2\cdot 2 = 8$

32

Or: Broken Lines



$1+t^3x^{-1}y^{-3} = f_A \qquad f_B = 1+3t^6xy^{-6}$

$$y \underset{A}{\rightsquigarrow} y^{-1} f_A^{(0,1)\wedge(1,2)} = \dots t^{3\cdot 1}xy^{-2} \underset{B}{\rightsquigarrow} t^3x^{-1}y^{-2} f_B^{(1,2)\wedge(0,8)} \Rightarrow 24\,t^9y^{-8}$$

By extending end upwards $\rightsquigarrow$, can relate broken lines/ tropical disks to 2-pointed relative invts.

DEFS:

- $R_{p,q} = \sum_{\beta.E = p+q} R_{p,q}(\beta)\, t^\beta$ (class $\beta \in H_2(X)$)

$$= \sum_\beta \sum_g R^g_{p,q}(\beta)\, t^\beta \hbar^{2g}$$

fixed pt mult p along E
(w likes q=1)

moving pt mult q along E

$$R^g_{p,q}(\beta) = \int_{[\mathcal{M}^{\log}_{g,2}(X,\beta)]} (-1)^g \lambda_g\, ev_1^*(pt)$$

- $R^{trop}_{p,q} = \sum_{T \text{ trop curves}} m_T(e^{i\hbar})\, t^{\beta(T)}$

Relations

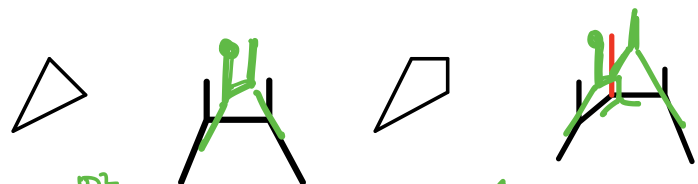
Tropical-relative and blow-up relations

$$\begin{array}{ccc} R^{trop}_{1,n}(X,\beta) & \overset{[GRZ]^*}{=} & R^{trop}_{n}(\hat{X},\hat{\beta}) \\ \text{[Grä]}\ \| & & \|\ \text{[Grä]} \\ R_{1,n}(X,\beta) & \overset{[GRZ]}{=} & R_n(\hat{X},\hat{\beta}) \end{array}$$

$\hat{\beta} = \beta - C$

tangency $\in \hat{E}$

$*$ CORRESPONDENCE BET. $\mathbb{P}^2$ TROP DISKS & F_1



$\mathbb{P}^2$ $\quad$ $F_1 = \hat{\mathbb{P}}^2$

Gathmann-Chen vGGR $\quad$ $R_{1,n} = n^2 R_{n,1}$

$R_n(\hat{X}) = (-1)^{n+1} n N(K_{\hat{X}})$

$\Downarrow$

Relative-Local : $N(K_{\hat{X}}, \hat{\beta}) = (-1)^{n+1} n R_{n,1}(X,\beta)$

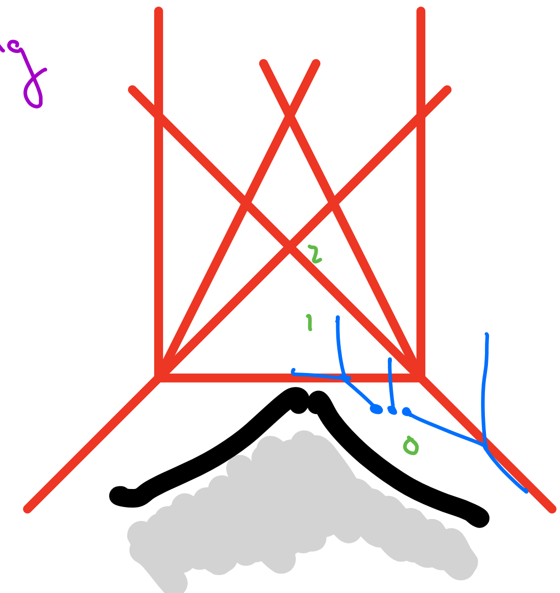
Open-Open : Inv'ts of AV branes $=$ Inv'ts of T^3 fibers
$\mu = 0$ $\qquad\qquad$ $\mu = 2$

Open-Local [CCLLTW]: $M(Q) = 1 + \sum_{d \geq 1} N^0(K_{F_1}, dL - C)(-Q)^d$

CONCLUSION:
$$= 1 + \sum n R_{n,1}(\mathbb{P}^2, dL)\, Q^d$$
$$= 1 + \sum R^{trop}_{n,1}(\mathbb{P}^2)\, Q^d$$
$$= y^{-1} W(Q), \quad Q = (t/y)^3$$

Rmk: $y^{-(3d-1)}$ ✓

Wall crossing



$$y + x + \frac{t^3}{xy}$$

$$\wr$$

$$W_0 = t\left(y + x^{-1}y + \frac{x}{y^2}\right)$$

W_1: $\underset{(0,-1)}{\downarrow}$ $f = t(1+x^{-1})$

$$y \to y f^{(0,-1)\cdot(0,1)}$$

$$x^{-1}y \to x^{-1}y f^{(0,-1)\cdot(-1,1)}$$

$$xy^{-2} \to xy^{-2} f^{(0,-1)\cdot(1,-2)}$$

$$t(1+x^{-1})y \to t(1+x^{-1})y f^{-1} = y$$

$$txy^{-2} \to txy^{-2}f^{2} = t^3xy^{-2}(1+2x^{-1}+x^{-2})$$

$$= t^3 2y^{-2} + \dots$$

$$W_1 = 1y + 2t^3y^{-2} + \dots$$

$$= y(1 + 2Q + \dots)$$