Gromov - Witten Invariants

X = smooth projective variety $/\mathbb{C}$ or compact kähler manifold

$\overline{M}_{g,m}(X,d)$ = moduli stack of genus g, m-pointed stable maps to X of degree d

$m \in \mathbb{N}$, $g \in \mathbb{Z}_{\geq 0}$, $d \in H_2(X,\mathbb{Z})$

$$[\overline{M}_{g,m}(X,d)]^{vir} \in H_{2((1-g)\dim X + \int_d c_1(T_X) + 3g-3+m)}(\overline{M}_{g,m}(X,d), \mathbb{Q})$$

~ virtual fundamental class

$$\psi_i = c_1(L_i) \in H^2(\overline{M}_{g,m}(X,d), \mathbb{Q}), \quad L_i\Big|_{((C,p_1,\cdots,p_m)\xrightarrow{f}X)} := T^*_{p_i}C$$

~ descendant classes

$\boxed{\text{genus } g \text{ descendant potential:}}$

$$F_X^g(\{t_i\}) := \sum_{m,d} \frac{Q^d}{m!} \int_{[\overline{M}_{g,m}(X,d)]^{vir}} \prod_{i=1}^m \left(\sum_{k=0}^\infty (ev_i^* t_k)\psi_i^k\right)$$

where $ev_i : \overline{M}_{g,m}(X,d) \to X$; $ev_i((C,p_1,\cdots,p_m)\xrightarrow{f}X) = f(p_i)$

$t_0, t_1, \cdots \in H^*(X,\mathbb{Q})$

Q^d = element in (a completion of the) group ring of $H_2(X,\mathbb{Z})$

$\boxed{\text{Total descendant potential}}$

$$\boxed{D_X = \exp\left(\sum_{g\geq 0} \hbar^{g-1} F_X^g\right)}$$

Virasoro constraints

Basis: $\{\gamma_a\} \subset H^*(X, \mathbb{C})$ such that

- $\gamma_0 = 1 \in H^0(X, \mathbb{C})$
- γ_a is homogeneous w.r.t. the Hodge decomposition

  : $\gamma_a \in H^{p_a, q_a}(X) \subset H^*(X, \mathbb{C})$ for some p_a, q_a

Coordinates: $t_k = \sum_a t_k^a \gamma_a$

operators: $\rho = c_1(T_X) \cup - \ : H^*(X) \circlearrowleft$

ρ_a^b = matrix elements of ρ w.r.t. $\{\gamma_a\}$

: $\sum_b \rho_a^b \gamma_b = \rho(\gamma_a) = c_1(T_X) \cup \gamma_a$

μ = diagonal matrix in the basis $\{\gamma_a\}$ w/ entries $\mu_a = p_a - \frac{\dim X}{2}$

(Hodge grading operator)

Notation: $[x]_i^k = e_{k+1-i}(x, x+1, \cdots, x+k)$

e_k = k th elementary symmetric function of its arguments

$\left(\text{so } \sum_{i=0}^{k+1} s^i [x]_i^k = (s+x)(s+x+1)\cdots(s+x+k) \right)$

$\boxed{\text{Virasoro differential operators:}}$ $\quad k \geq -1$

$$L_k = \sum_{i=0}^{k+1} \Bigg(\frac{\hbar}{2} \sum_{m=i-k}^{-1} (-1)^m [\mu_a + m + \tfrac{1}{2}]^k_i (\rho^i)^{ab} \frac{\partial}{\partial t^a_{-m-1}} \frac{\partial}{\partial t^b_{m+k-i}}$$

$$- \left[\frac{3-\dim X}{2}\right]^k_i (\rho^i)^b_0 \frac{\partial}{\partial t^b_{k-i+1}}$$

$$+ \sum_{m=0}^{\infty} [\mu_a + m + \tfrac{1}{2}]^k_i (\rho^i)^b_a t^a_m \frac{\partial}{\partial t^b_{m+k-i}} \Bigg)$$

$$+ \frac{1}{2\hbar} (\rho^{k+1})_{ab} t^a_0 t^b_0 + \frac{\delta_{k,0}}{48} \int_X \Big((3-\dim X) c_{\dim X}(T_X) - 2 c_1(T_X) c_{\dim X-1}(T_X) \Big)$$

$\boxed{\text{Virasoro Conjecture (Eguchi-Hori-Xiong, S. Katz)}}$

$$L_k D_X = 0 \quad \text{for } k \geq -1$$

Remarks

① $L_{-1} D_X = 0$ is always true: it is equivalent to the string equation.

② $L_0 D_X = 0$, also known as Hori's equation, is always true

It follows from:
(a) dimension constraint
(b) divisor equation
(c) dilaton equation.

③ L_k satisfy the following commutation relation:

$$[L_k, L_\ell] = (k-\ell) L_{k+\ell}$$

This is the commutation relation for the algebra of polynomial vector fields on the line, whose central extension is the Virasoro algebra.

The proof of the commutation relation is a formal calculation. There is a geometric aspect: matching the constants in $[L_1, L_{-1}] = 2L_0$ yields

$$\mathrm{Str}(\mu^2) = \frac{1}{12}\int_X \left(\dim X\, c_{top}(X) + 2c_1(X)\, c_{top-1}(X)\right),$$

which can be proven by Hirzebruch-Riemann-Roch (Libgober-Wood)

Givental's formulation

Loop Space: $\mathcal{H} := H^*(X)((z^{-1}))$ ~ Laurent series in z^{-1} w/ coefficients in $H^*(X)$

Ω = symplectic form on $\mathcal{H}$

$$\Omega(f,g) = \mathrm{Res}_{z=0} (f(-z), g(z))\, dz$$

Poincaré pairing on X

Lagrangian polarizations: $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \cong T^*\mathcal{H}_+$

where $\mathcal{H}_+ = H^*(X)[z]$ $\quad \mathcal{H}_- = z^{-1} H^*(X)[[z^{-1}]]$

Darboux coordinates: $\{p_k^a, q_\ell^b\}$

$$\underbrace{\sum_{k\geq 0} p_k^a \gamma_a z^{-k-1}}_{p(z)=} + \underbrace{\sum_{\ell\geq 0} q_\ell^b \gamma_b z^\ell}_{=q(z)} \in \mathcal{H}$$

Convention: F_X^g and D_X are viewed as functions

on $\mathbf{t}(z) = t_0 + t_1 z + t_2 z^2 + \cdots$,

hence they are functions on $\mathcal{H}_+$

via the dilaton shift $q(z) = \mathbf{t}(z) - 1z$

The cone: $\mathcal{L}_X = \{ \mathbf{p} = d_q F_X^0 \} \subset \mathcal{H}$

is an overruled Lagrangian cone w/ vertex at origin in $\mathcal{H}$.

$\{$String + dilaton + TRR$\}$ $\Leftrightarrow$ "Overruled" means that each tangent space T of $\mathcal{L}_X$ is tangent to $\mathcal{L}_X$ exactly along zT.

$\boxed{\text{genus } 0:}$ $\quad l_{-1} = z^{-1} : \mathcal{H} \to \mathcal{H}$, $\quad l_0 = z\frac{d}{dz} + \frac{1}{2} + \mu + \frac{\rho}{z} : \mathcal{H} \to \mathcal{H}$.

$$l_1 = l_0 z l_0, \; l_2 = l_0 z l_0 z l_0, \; \cdots, \; l_k = l_0 (z l_0)^k : \mathcal{H} \to \mathcal{H}.$$

They satisfy $[l_m, l_n] = (n-m) l_{m+n}$ b/c $[l_0, l_{-1}] = -l_{-1}$

Thm (X. Liu – G. Tian) vector fields defined by l_m, $m \geq -1$ are all tangent to $\mathcal{L}_X$.

Hori's eq'n

pf Let $T = T_f \mathcal{L}_X$. Then $l_0 f \in T$. So $z l_0 f \in zT \subset \mathcal{L}_X$

So $\underbrace{l_0 z l_0}_{l_1} f \in T$. So $z l_0 z l_0 f \in zT \subset \mathcal{L}_X$, So $\underbrace{l_0 z l_0 z l_0}_{l_2} f \in T$, etc.

$\boxed{\text{Higher genus}}$ Differential operators can be obtained from l_k as follows. Write the quadratic Hamiltonian $f \mapsto \frac{1}{2}\Omega(l_k f, f)$ in Darboux coordinates $\{p_k^a, q_l^b\}$ and obtain a differential operator $\widehat{l_k}$ by the following quantization rule:

$$\widehat{q_m^a q_n^b} := \frac{q_m^a q_n^b}{\hbar}, \quad \widehat{q_m^a p_n^b} := q_m^a \frac{\partial}{\partial q_n^b}, \quad \widehat{p_m^a p_n^b} := \hbar \frac{\partial}{\partial q_m^a} \frac{\partial}{\partial q_n^b}$$

Note: a choice of ordering is made here

Then $L_k := \widehat{l_k} + c_k$, w/ $c_{k\neq 0} = 0$, $-c_0 = \frac{\chi(X)}{16} + \frac{\operatorname{str}(\mu \mu^*)}{4}$

commutes as $[L_m, L_n] = (n-m) L_{m+n}$

Note: The L_k here is the negative of the L_k written down before

Virasoro conjecture reads: $L_k D_X = 0, \quad k \geq -1$.

Proven cases

① X = point (Witten, Kontsevich)

② X = toric manifold (Givental, Iritani)

③ X = complete flag manifold of type A (Joe-Kim)

④ X = Grassmannian (Bertram-Ciocan-Fontanine-Kim)

⑤ X = compact Kähler manifold w/ (generic) semisimple QH^* (Teleman)

⑥ X = nonsingular curve (Okounkov-Pandharipande)

②, ③, ④, ⑤ are based on an approach due to Givental.

Givental's approach

An element $M : (\mathcal{H}, \Omega) \circlearrowleft$ of the twisted loop group (also known as the Givental group) yields a differential operator

$\hat{M} := \exp(\widehat{\ln M})$ which act on formal functions such as total descendant potentials.

<u>Proposition</u> (loop group covariance) Suppose D' and $D'' = \hat{M} D'$ both satisfy grading constraints $L_0' D' = 0$, $L_0'' D'' = 0$ for suitable grading operators l_0', l_0''. Suppose M respects the grading in the sense that $l_0'' = M l_0' M^{-1}$. Then D' satisfies Virasoro constraints if and only if D'' satisfies Virasoro constraints.

$\boxed{\text{Semi simple case}}$ The Givental - Teleman classification of semisimple CohFT implies that for target spaces X w/ (generic) semisimple quantum cohomology rings, we have the following formula

$$D_X = \hat{S}_X^{-1} \hat{R} D_{\text{point}}^{\otimes \dim H^*(X)}$$

Some scalar factors are omitted

S_X is defined explicitly in terms of genus 0 Gromov - Witten invariants. R is uniquely obtained from quantum differential eqn and grading condition for X.

Both S_X and R respect gradings, so loop group covariance and Witten - Kontsevich theorem imply Virasoro constraints for X.

Toric bundles

$X = \mathbb{C}^N /\!/ K$: smooth projective toric manifold.

$K = (S^1)^k \subset (S^1)^N$

B = compact Kähler manifold / smooth projective variety $/\mathbb{C}$

$L_1, \cdots, L_N \to B$: line bundles.

Replacing the fibers of $L_1 \oplus \cdots \oplus L_N \to B$ by X yields a toric bundle $E \to B$.

$T = (\mathbb{C}^\times)^N$ acts on E. The T-fixed locus E^T consists of n copies of B, which are sections of $E \to B$.
$(n = \operatorname{rank} H^*(X))$

The following results are joint work w/ T. Coates and A. Givental

Theorem (Coates - Givental - T.) $D_E = \widehat{M} D_B^{\otimes n}$ for a certain grading respecting loop group element M.

Loop group covariance then implies

Corollary Virasoro constraints hold for E if and only if they hold for B.

For example, Virasoro constraints hold for $\mathbb{P}^1$-bundles over curves (namely, ruled surfaces)

How to prove the Theorem

$\boxed{\text{Descendant / Ancestor correspondence}}$ For any smooth projective variety Y, we can consider the total ancestor potential:

$$A_Y(\tau; \mathbf{t}) = \exp\left(\sum_{g=0}^{\infty} \hbar^{g-1} \bar{F}_g(\tau; \mathbf{t})\right),$$

where $\bar{F}_g$ is the genus g ancestor potential:

$$\bar{F}_g(\tau; \mathbf{t}) = \sum_{d \in H_2(Y)} \sum_{m \geq 0} \sum_{n \geq 0} \frac{Q^d}{m!\, n!} \int_{[M_{g,m+n}(Y,d)]^{vir}} \prod_{i=1}^{m} \left(\sum_{k \geq 0} ev_i^*(t_k) \bar{\psi}_i^k\right) \prod_{i=m+1}^{m+n} ev_i^* \tau$$

where
- $\tau \in H^*(Y)$
- $\mathbf{t} = t_0 + t_1 z + t_2 z^2 + \cdots \in H^*(Y)[z]$
- $\overline{\psi}_i = ct^* \psi_i$ where $ct : \overline{M}_{g,m+n}(Y,d) \to \overline{M}_{g,m+n} \to \overline{M}_{g,m}$

forgets the map forgets the last n markings.

A relation between descendant and ancestor invariants, due to Kontsevich-Manin, can be formulated as follows:

$$D_Y = e^{F^1_Y(\tau)} \widehat{S_Y^{-1}} A_Y(\tau; \mathbf{t})$$

Here S_Y is the operator defined by

$$(a, S_Y b)_Y = (a,b) + \sum_{k \geq 0} \frac{1}{z^{1+k}} \sum_{m,d} \frac{Q^d}{m!} \int_{[\overline{M}_{0,m+2}(Y,d)]^{vir}} (ev_1^* a)\left(\prod_{i=2}^{m+1} ev_i^* \tau\right)(ev_{m+2}^* b)\psi_{m+2}^k$$

By dimension constraints of Gromov-Witten invariants, S_Y respects grading.

$\boxed{\text{Localization}}$ Because of descendant / ancestor correspondence, we can study $A_E(\tau; \mathbf{t})$ instead of D_E. We compute A_E by virtual localization w.r.t. the T-action on E.

The outcome is the following:

$$A_E(\tau;t) = \widehat{R(\tau)} \prod_{\alpha \in F} A_B^{\alpha, tw}$$

twisted Gromov-Witten theory associated to normal bundle of $E^\alpha \subset E$ and inverse Euler class

index for components of E^T

Some comments:

① Contributions from maps to the fixed component E^α assemble to $A_B^{\alpha, tw}$

② Node-smoothing terms in virtual normal bundles contain descendant classes. Replacing them by ancestor classes requires some care.

③ Contributions from maps to 1-dimensional T-orbits of E can be described by the action of $\widehat{R(\tau)}$.

$R(\tau)$ arises by studying S_E using localization:

$$S_E(\tau, z) = R(\tau, z)\left(\underbrace{\bigoplus_{\alpha \in F} S^{\alpha, tw}}_{=: S_{block}}\right)$$

$\boxed{\text{Non-equivariant limit}}$ $A_B^{\alpha, tw}$ can be related to D_B by applying (twisted) descendant / ancestor correspondence and quantum Riemann-Roch theorem.

Thus we have the following formula in T-equivariant Gromov-Witten theory:

$$A_E^{eq}(\tau) = \widehat{R(\tau)}\ \widehat{S_{block}(\tau)}\ \widehat{\Gamma_{block}^{-1}} \prod_{\alpha \in F} D_B$$

Some scalar factors are omitted

from descendant / ancestor correspondence

from quantum Riemann-Roch

- We need a formula relating A_E and $\prod_{\alpha \in F} D_B$ in non-equivariant Gromov-Witten theory of E. Therefore we want to show that the expression $R(\tau) S_{block}(\tau) \Gamma_{block}^{-1}$ has non-equivariant limit.
- The point is that $R(\tau) S_{block} \Gamma_{block}^{-1}$ is a solution to the T-equivariant quantum differential equation of E.
- J. Brown's work provides a fundamental solution to quantum differential equation of E using oscillating integrals and Gromov-Witten invariants of B.
- We show that $R(\tau) S_{block} \Gamma_{block}^{-1}$ can be obtained from stationary phase asymptotics of these oscillating integrals.
- Non-equivariant limit of this oscillating integral description can be seen to exist.
- Finally, the non-equivariant limit of the oscillating integral description can be shown to be grading-respecting.