Gromov–Witten Invariants

\( X = \text{smooth projective variety} \) or compact Kähler manifold

\( \overline{M}_{g,m}(X,d) = \text{moduli stack of genus } g, m\text{-pointed stable maps} \)

to \( X \) \& degree \( d \) \( \in \mathbb{Z}_{\geq 0} \) \( \in H_2(X, \mathbb{Z}) \)

\( \left[ \overline{M}_{g,m}(X,d) \right]^{vir} \in H_{2((1-g)\dim X + \sum d_c(T_x) + 3g-3+m)} \left( \overline{M}_{g,m}(X,d), \mathbb{Q} \right) \)

\( \sim \text{virtual fundamental class} \)

\( \psi_i = c_i(L_i) \in H^2(\overline{M}_{g,m}(X,d), \mathbb{Q}), \quad L_i \bigg|_{(C, p_1, \ldots, p_m) \mapsto X} := T^*_{p_i} C \)

\( \sim \text{descendant classes} \)

\begin{align*}
\text{Genus } g \text{ descendant potential:} \\
F_{g,X}(t_0, t_1, \ldots) := \sum_{m,d} \frac{Q^d}{m!} \int_{[\overline{M}_{g,m}(X,d)]^{vir}} \prod_{i=1}^{\infty} (\sum_{k=0}^{\infty} t_i^{g-1} \psi_i^k) \\
\text{where } ev_i : \overline{M}_{g,m}(X,d) \to X \quad \text{ev}_i(C, p_1, \ldots, p_m) \mapsto X = f(p_i) \quad t_0, t_1, \ldots \in H^*(X, \mathbb{Q}) \quad Q^d \text{ = element in (a completion of the) group ring of } H_2(X, \mathbb{Z}) \\
\end{align*}

Total descendant potential

\( D_X = \exp \left( \sum_{g \geq 0} \frac{\bar{g}^{-1} g^2}{g} F_g \right) \)
Virasoro constraints

Basis: \( \{ \gamma_a \} \subset H^*(X, \mathbb{C}) \) such that

- \( \gamma_0 = 1 \in H^0(X, \mathbb{C}) \)
- \( \gamma_a \) is homogeneous w.r.t. the Hodge decomposition
  \( \gamma_a \in H^{P_a, \bar{q}_a}(X) \subset H^*(X, \mathbb{C}) \) for some \( P_a, \bar{q}_a \)

Coordinates: \( t_K = \sum_a t_K^a \gamma_a \)

Operators: \( \rho = c_i(T_X) \cup -: H^*(X) \)

\( \rho_a^b = \text{matrix elements of } \rho \text{ w.r.t. } \{ \gamma_a \} \)

\( \sum_b \rho_a^b \gamma_b = \rho(\gamma_a) = c_i(T_X) \cup \gamma_a \)

\( M = \text{diagonal matrix in the basis } \{ \gamma_a \} \)

with entries \( \mu_a = P_a - \frac{\text{dim}X}{2} \)

(Hodge grading operator)

Notation: \( [x]_i^k = e_{k+1-i}(x, x+1, \ldots, x+k) \)

\( e_k = k^{th} \text{ elementary symmetric function of its arguments} \)

\( (\text{so } \sum_{i=0}^{k+1} s^i [x]_i^k = (s+x)(s+x+1) \cdots (s+x+k) ) \)
Virasoro differential operators: \( K \geq -1 \)

\[
L_K = \sum_{i=0}^{k+1} \left( \frac{\hbar}{2} \sum_{m=i-K}^{-1} (-1)^m [M_a + m + \frac{1}{2}]^k_i (\rho^i)_{ab} \frac{\partial}{\partial t_a^m} \frac{\partial}{\partial t_b^{m+k-i}} \right) \\
- \left[ \frac{3-dimX}{2} \right]^k_i (\rho^i)_{0} b \frac{\partial}{\partial t_{k-i+1}} \\
+ \sum_{m=0}^{\infty} \left[ M_a + m + \frac{1}{2} \right]^k_i (\rho^i)_{a} c_m \frac{\partial}{\partial t_{m+k-i}} \\
+ \frac{1}{2\hbar} (\rho^{k+1})_{ab} t_a^b + \delta_{k,0} \left( (3-dimX) c_{dimX}(T_X) - 2 c_{c_{dimX}}(T_X) \right)
\]

Virasoro Conjecture (Eguchi-Hori-Xiong, S. Katz)

\( L_K D_X = 0 \) for \( K \geq -1 \)

Remarks

1. \( L_{-1} D_X = 0 \) is always true; it is equivalent to the string equation.

2. \( L_0 D_X = 0 \), also known as Hori's equation, is always true. It follows from:
   - (a) dimension constraint
   - (b) divisor equation
   - (c) dilaton equation.
3. Let satisfy the following commutation relation:

\[ [L_k, L_l] = (k-l) L_{k+l} \]

This is the commutation relation for the algebra of polynomial vector fields on the line, whose central extension is the Virasoro algebra.

The proof of the commutation relation is a formal calculation. There is a geometric aspect: matching the constants in \([L_1, L_{-1}] = 2L_0\) yields

\[
\text{Str} \left( \mu^2 \right) = \frac{1}{12} \int_{X} \left( \dim X \cdot C_{\text{top}}(X) + 2c_1(X) C_{\text{top}, -1}(X) \right),
\]

which can be proven by Hirzebruch–Riemann–Roch (Libgober–Wood).

**Givental's formulation**

Loop Space: \( \mathcal{H} := H^*(X)(\mathbb{C}[z^{-1}]) \sim \text{Laurent series in } z^{-1} \) with coefficients in \( H^*(X) \)

\[ \Omega = \text{symplectic form on } \mathcal{H} \]

\[ \Omega(f, g) = \text{Res}_{z=0} (f(-z), g(z)) \, dz \]

\( \Omega \) is the Poincaré pairing on \( X \)

Lagrangian polarizations: \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \cong T^* \mathcal{H}_+ \)
where $\mathcal{H}_+ = H^*(X)[z^2]$  \hspace{2cm} \mathcal{H}_- = z^{-1}H^*(X)[z^{-1}]$.

Darboux coordinates: $\{ p_k^a, q_k^b \} \subseteq \mathcal{H}$

$$p(z) = \sum_{k \geq 0} p_k^a z^{-k-1} + \sum_{l \geq 0} q_l^b z^l \in \mathcal{H}$$

Convention: $g^d_X$ and $F_X$ are viewed as functions on $\mathcal{H}$.

$$s(z) = t_0 + t_1 z + t_2 z^2 + \ldots$$

hence they are functions on $\mathcal{H}_+$

via the dilaton shift $\tilde{s}(z) = s(z) - 1z$.

The cone: $L_x = \{ p = d_q F_X^0 \} \subseteq \mathcal{H}$

is an overruled Lagrangian cone w/ vertex at origin in $\mathcal{H}$.

The "overruled" means that each tangent space $T$ of $L_x$ is tangent to $L_x$ exactly along $zT$.

**Genus 0:**

$L_1 = z^{-1}: \mathcal{H} \rightarrow \mathcal{H}$, $L_0 = \frac{2}{z^2} + \frac{1}{2} + M + \frac{P}{z}$.

$L_1 = L_0 \circ L_0$, $L_2 = L_0 \circ L_0 \circ L_0$, \ldots, $L_k = L_0(2L_0)^k : \mathcal{H} \rightarrow \mathcal{H}$.

They satisfy $[L_m, L_n] = (n-m) L_{m+n}$ b/c $[L_0, L_1] = -L_1$.
Thm (X. Liu - G. Tian) Vector fields defined by $l_m$, $m \geq 1$ are all tangent to $L_x$.

Proof: Let $T = T_x L_x$. Then $l_0 f \in T$. So $z l_0 f \in z T C L_x$.
So $l_0 z l_0 f \in T$. So $z l_0 z l_0 f \in z T C L_x$, so $l_0 z l_0 z l_0 f \in T$, etc.

Higher genus: Differential operators can be obtained from $l_k$ as follows. Write the quadratic Hamiltonian $f \mapsto \frac{1}{2} \Omega(l_k f, f)$ in Darboux coordinates $\{ p^a, q^b \}$ and obtain a differential operator $\hat{l}_k$ by the following quantization rule:

$\hat{g}^a_{mn} = \frac{g^a_{mn}}{\hbar}, \quad \hat{g}^a_{m} \hat{p}^b_{n} = g^a_{mn} \frac{\partial}{\partial q^b_{m}}, \quad \hat{p}^a_{m} \hat{p}^b_{n} = \hbar \frac{\partial}{\partial q^a_{m}} \frac{\partial}{\partial q^b_{n}}$.

Note: a choice of ordering is made here.

Then $L_k := \hat{l}_k + c_k$, w/ $c_{k=0} = 0, -c_0 = \frac{\lambda (x)}{16} + \text{str} (\mu \mu^*)$ commutes as $[L_m, L_n] = (n-m) L_{mn}$.

Note: The $L_k$ here is the negative of the $L_k$ written down before.

Virasoro conjecture reads: $L_k D_x = 0$, $k \geq -1$. 
Proven cases

1. $X = \text{point}$ (Witten, Kontsevich)
2. $X = \text{toric manifold}$ (Givental, Iritani)
3. $X = \text{complete flag manifold of type } A$ (Joe-Kim)
4. $X = \text{Grassmannian}$ (Bertram-Ciocan-Fontanine-Kim)
5. $X = \text{compact Kähler manifold w/ (generic) semisimple } \mathbb{Q}H^*$ (Teleman)
6. $X = \text{non-singular curve}$ (Okounkov-Pandharipande)

2), 3), 4), 5) are based on an approach due to Givental.

Givental’s approach

An element $M: \mathfrak{g}_{(k,2)}$ of the twisted loop group (also known as the Givental group) yields a differential operator $\hat{M} := \exp(\ln M)$ which act on formal functions such as total descendant potentials.

Proposition (loop group covariance) Suppose $D'$ and $D'' = \hat{M}D'$ both satisfy grading constraints $L_0'D' = 0$, $L_0''D'' = 0$ for suitable grading operators $L_0'$, $L_0''$. Suppose $M$ respects the grading in the sense that $L_0'' = ML_0'M^{-1}$. Then $D'$ satisfies Virasoro constraints if and only if $D''$ satisfies Virasoro constraints.
Semi simple case: The Givental - Teleman classification of semi simple CohFT implies that for target spaces $X$ w/ (generic) semisimple quantum cohomology rings, we have the following formula:

$$D_x = \hat{S}^{-1} \hat{R} \otimes \dim H^*(x)$$

Some scalar factors are omitted.

$S_x$ is defined explicitly in terms of genus 0 Gromov-Witten invariants. $R$ is uniquely obtained from quantum differential eqn and grading condition for $X$.

Both $S_x$ and $R$ respect gradings, so loop group covariance and Witten-Kontsevich theorem imply Virasoro constraints for $X$.

Toric bundles:

$X = \mathbb{C}^N/K$ : smooth projective toric manifold.

$K = (S^1)^K < (S^1)^N$

$B$ = compact Kähler manifold / smooth projective variety $C$

$L_1, \ldots, L_N \to B$ : line bundles.

Replacing the fibers of $L_1 \oplus \ldots \oplus L_N \to B$ by $X$ yields a toric bundle $E \to B$.
\( T = (\mathbb{C}^\times)^N \) acts on \( E \). The \( T \)-fixed locus \( E^T \) consists of \( n \) copies of \( B \), which are sections of \( E \to B \).

\((n = \text{rank } H^*(X))\)

The following results are joint work w/ T. Coates and A. Givental

**Theorem (Coates - Givental - T.)**  \( \hat{D}_E = \hat{M} \hat{D}^\otimes \) \( \otimes n \) for a certain grading respecting loop group element \( M \).

Loop group covariance then implies

**Corollary** Virasoro constraints hold for \( E \) if and only if they hold for \( B \).

For example, Virasoro constraints hold for \( \mathbb{P}^1 \)-bundles over curves (namely, ruled surfaces).

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**How to prove the Theorem**

**Descendant/Ancestor correspondence** For any smooth projective variety \( Y \), we can consider the total ancestor potential:

\[
A_Y(\tau; t) = \exp \left( \sum_{g=0}^{\infty} \tau^{g-1} \overline{F}_g(\tau; t) \right),
\]

where \( \overline{F}_g \) is the genus \( g \) ancestor potential:

\[
\overline{F}_g(\tau; t) = \sum_{d \in H_2(Y)} \sum_{m \geq 0} \sum_{n \geq 0} \frac{\tau^d}{m! n!} \int \left[ \overline{\text{vir}} \right] \prod_{i=m+1}^{m+n} (\sum_{k=0}^{\infty} e_{i+k \circ}(t) \overline{v}_i^k) \prod_{i=m+1}^{m+n} e_{i+k \circ} \cdot c_i
\]
where
\[ \tau \in H^*(\Sigma) \]
and
\[ t^t = t_0 + t_1 z + t_2 z^2 + \ldots \in H^*(\Sigma)[z] \]

\[ \Phi_i = c_t^* \psi_i \], where \( c_t : \overline{M}_{g, m+n}(Y, d) \rightarrow \overline{M}_{g, m+n} \rightarrow \overline{M}_{g, m} \)

A relation between descendant and ancestor invariants, due to Kontsevich-Manin, can be formulated as follows:

\[ D_Y = e^{F_Y(t)} \sum_{i} \Phi_i A_{\psi_i}(\tau; t) \]

Here \( S_Y \) is the operator defined by

\[ (a, S_Y b) = (a, b) + \sum_{k \geq 2} \sum_{m,d} \frac{1}{k!} \sum_{m_1, \ldots, m_k} \int \left( \prod_{i=1}^{m+k} (e_i^* a) \right) \left( \prod_{i=1}^{m+k} (e_i^* b) \right) \gamma_{m+k}^{m+k} \]

By dimension constraints of Gromov-Witten invariants, \( S_Y \) respects grading.

Localization: Because of descendant/ancestor correspondence, we can study \( A_{\psi_i}(\tau; t) \) instead of \( D_E \). We compute \( A_{\psi_i} \) by virtual localization w.r.t. the \( T \)-action on \( E \).

The outcome is the following:
$A_E(\tau; t) = \bigcap_{\alpha, t} A_{\beta}^{\alpha, tw}$

Some comments:

1. Contributions from maps to the fixed component $E^x$ assemble to $A_{\beta}^{\alpha, tw}$

2. node-smoothing terms in virtual normal bundles contain descendant classes. Replacing them by ancestor classes requires some care.

3. Contributions from maps to 1-dimensional $T$-orbits of $E$ can be described by the action of $R(\tau)$. $R(\tau)$ arises by studying $S_E$ using localization:

   $S_E(\tau, z) = R(\tau, z) \left( \bigoplus_{\alpha, t} S_{\alpha, tw}^{\text{def}} \right)$

Non-equivariant limit $A_{\beta}^{\alpha, tw}$ can be related to $D_B$ by applying (twisted) descendant/ancestor correspondence and quantum Riemann-Roch theorem. Thus we have the following formula in $T$-equivariant Gromov-Witten theory:

$A^E_E(\tau) = \widehat{R}(\tau) \widehat{S}_{\text{block}}(\tau) \prod_{\text{block}}^{-1} \bigoplus_{\text{def}} D_B$

Some scalar factors are omitted from quantum Riemann-Roch.
We need a formula relating $A_E$ and $\Pi_{def}^D_B$ in non-equivariant Gromov-Witten theory of $E$. Therefore we want to show that the expression $R(\tau)S_{block}(\tau)\Gamma_{block}^{-1}$ has non-equivariant limit.

The point is that $R(\tau)S_{block}\Gamma_{block}^{-1}$ is a solution to the $T$-equivariant quantum differential equation of $E$.

J. Brown's work provides a fundamental solution to quantum differential equation of $E$ using oscillating integrals and Gromov-Witten invariants of $B$.

We show that $R(\tau)S_{block}\Gamma_{block}^{-1}$ can be obtained from stationary phase asymptotics of these oscillating integrals.

Non-equivariant limit of this oscillating integral description can be seen to exist.

Finally, the non-equivariant limit of the oscillating integral description can be shown to be grading-respecting.