Gromov-Witten Potentials of Local Banana Manifold
and Siegel Modular Forms

- Joint with Jim Bryan from ~2019.
- Connects with recent mirror symmetry work of Azam-Cannizzo-Lee-Lin

Outline:

§1. Curve-counting theories for a local banana manifold
§2. Quick survey of Siegel modular forms and Jacobi forms
§3. Main computations
§4. Comments on mirror symmetry

§1. Curve-counting theories for local banana manifold

Let $S \to D$ be an elliptically fibered surface over a disk such that the central fiber is a nodal cubic, and all other fibers are smooth elliptic curves.

For this talk, the local banana manifold

$$Y = \text{Bl}_\Delta (S \times_D S)$$

is a small resolution of singularities of $S \times_D S$, blowing up the diagonal $\Delta$. 
So $Y \to \mathbb{P}$ is a non-compact Calabi-Yau threefold fibered by Abelian surfaces

- Smooth fibers: $E \times E$, for smooth elliptic curve $E$.
- The central fiber is a non-normal toric surface:

Normalization of the singular fiber is a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at two torus-invariant points.

Much of enumerative geometry is concerned with defining curve-counting invariants for Calabi-Yau threefolds. So we want to understand curves in $Y$:

$$\langle C_1, C_2, C_3 \rangle: \text{Lattice of proper curves in } Y$$

For any effective curve class $\beta \in H_2(Y, \mathbb{Z})$, we can write $\beta = d_1 C_1 + d_2 C_2 + d_3 C_3$.

But to see modularity later on, it is convenient to use a different basis:

$$\langle C_1 + C_3, C_2 + C_3, C_3 \rangle$$
Introduce formal variables $Q, q, y$ such that the above class corresponds to monomial:

$$Q^m q^n y^l$$

**Donaldson–Thomas Partition Function:**

$$Z_{\text{DT}}(Y) = \sum_{m,n,k} \sum_{\epsilon \in \mathbb{Z}} \text{DT}_{\beta, m,n,k}(Y) (-p)^k Q^m q^n y^l$$

where

$$\text{DT}_{\beta, m,n,k}(Y) = \#_{\text{virtual}} \left\{ \text{1-dimensional subschemes} \right\}$$

$$\left[ \text{Supp} (\mathcal{O}_2) \right] = \beta, [\mathcal{O}_2] \in \text{Hilb} (Y), \chi (\mathcal{O}_2) = k$$

**Theorem:** (Bryan) The Donaldson–Thomas partition function of $Y$ is given by the infinite product:

$$Z_{\text{DT}}(Y) = \prod \left( 1 - p^k Q^m q^n y^l \right)^{c(4nm-l^2, K)}$$

where $c(4nm-l^2, K)$ are Fourier coefficients of the equivariant elliptic genus of $\mathbb{C}^2$:

$$\text{Ell}_{Q,y}(\mathbb{C}^2; p) = \frac{\Theta_i(q,yp): \Theta_i(q,yp')}{\Theta_i(q,p) \cdot \Theta_i(q,p')} = \sum_{n,k} c(4nm-l^2, K) q^n y^l p^k$$
Remark: The above theorem looks analogous to the "Igusa cusp form conjecture" for \( K3 \times E \). (Oberdieck, Pandharipande, Pixton, Shen)

\[
\Theta_1(q, y) = - \sum_{k \in \mathbb{Z} + \frac{1}{2}} q^k (-y)^k
\]

\[
Z_{\text{DT}}(K3 \times E) = \frac{1}{X_0(6, q, y)} = \frac{1}{q^{3g} y} \prod_{m,n,k} (1 - q^m y^n)^{c(y_{mn} - k^2)}
\]

Igusa cusp form

\( c(y_{mn} - k^2) \): Fourier coefficients of \( \text{Ell}_{y/k}(K3) \).

Gromov-Witten Potentials:

\[
F_g(Y) = \sum_{m,n,k} \text{GW}_{g, \beta_{m,n,k}}(Y) Q^m y^n
\]

Genus \( g \) GW potential

GW invariants: virtual counts of stable maps into \( Y \).

**Theorem:** (Bryan, Pandharipande) For \( g \geq 2 \), the genus \( g \) Gromov-Witten potential \( F_g(Y) \) are meromorphic genus two Siegel modular forms of weight \( 2g - 2 \). Moreover

\[
E_{2g}(q) = 1 - \frac{4g}{B_{2g}} \sum_{n=1}^{\infty} q^{2g-1} q^{n^2} \sim E_{2g}(q) \cdot \phi \sim F_g(Y)
\]

"Maass Lift"

Eisenstein series
(modular weight \( 2g, g \geq 2 \))

So each \( F_g(Y) \) is built essentially from \( E_{2g} \)

§2. Quick Survey of Siegel Modular Forms and Jacobi Forms:

Define the genus two Siegel upper-half space to be:
\[ H_2^+ := \left\{ \Omega = \left( \begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{array} \right) \in M_{2 \times 2}(\mathbb{H}) \mid \text{Im}(\Omega) > 0 \right\} \]

and the integral symplectic group:

\[ \text{Sp}_4(\mathbb{Z}) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in M_{4 \times 4}(\mathbb{Z}) \mid \begin{array}{l} a b^T = b a^T \\ c d^T = d c^T \\ a d^T - b c^T = 2 \end{array} \right\} \]

Notice that \( H_2^+ \) generalizes the ordinary upper half plane, and \( \text{Sp}_4(\mathbb{Z}) \) generalizes \( \text{SL}_2(\mathbb{Z}) \). We have an action of \( \text{Sp}_4(\mathbb{Z}) \) on \( H_2^+ \) generalizing that of \( \text{SL}_2(\mathbb{Z}) \) on \( H_1^+ \):

\[ \Omega \mapsto \gamma(\Omega) = (a \Omega + b)(c \Omega + d)^{-1}, \quad \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{Sp}_4(\mathbb{Z}) \]

(Genus two) Siegel modular forms are functions on \( H_2^+ \) transforming under \( \text{Sp}_4(\mathbb{Z}) \):

More specifically, a \underline{Siegel modular form} of weight \( K \) is a holomorphic function \( f: H_2^+ \to \mathbb{C} \) such that:

\[ f(\gamma(\Omega)) = \det(c \Omega + d)^K f(\Omega) \]

for all \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{Sp}_4(\mathbb{Z}) \).

**Geometric Interpretation:**

\[ \mathbb{A}_2 := \text{moduli space of principally polarized Abelian surfaces} \]

\[ \mathbb{A}_2 \cong H_2^+/\text{Sp}_4(\mathbb{Z}) \]

(Principally polarized Abelian surface / isom. \( \sim \) Period matrix \( \Omega \in H_2^+ \), up to \( \text{Sp}_4(\mathbb{Z}) \).)

There exists a bundle \( \mathcal{E} \to \mathbb{A}_2 \) called the \underline{Hodge bundle} with fiber \( H^0(X, \Omega_X) \cong \mathbb{C}^2 \) over \( X \in \mathbb{A}_x \). Explicitly, we construct \( \mathcal{E} \) as follows:
Universal Family:

\[
\begin{align*}
\mathcal{X}_2 & \rightarrow A_2 \\
\downarrow s & \downarrow \\
\mathcal{A}_2 & \rightarrow \mathcal{H}_2 \times \mathbb{C}^2 / \text{Sp}_4(\mathbb{Z}) \times \mathbb{Z}^4
\end{align*}
\]

\[
E = s^* \Omega_{\mathcal{X}/A}
\]

Remark: I'm ignoring that \(\mathcal{X}_2 \rightarrow A_2\) is really an orbifold family, and \(E\) an orbifold bundle. Need to take an honest family \(\mathcal{X}_2(n) \rightarrow A_2(n)\) for principal congruence subgroup \(\Gamma(n) \subset \text{Sp}_4(\mathbb{Z})\) and quotient by \(\text{Sp}_4(\mathbb{Z}/n\mathbb{Z})\).

Proposition: A weight \(K\) Siegel modular form \(f\) can be viewed as a section of line bundle:

\[
f \in H^0(A_2, \det(E)^K)
\]

(Roughly: think of transformation law \(f(\gamma(z)) = \det(c\Omega + d)^K f(z)\) as a transition function between charts.)

This is part of the much bigger story of automorphic forms on Shimura varieties \(X/\Gamma\).

Maass Lifting Jacobi Forms to Siegel Modular Forms:

Standard Change of Variables:

\[
\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} \mapsto Q = e^{2\pi i \tau}, \quad q = e^{2\pi i z}, \quad y = e^{2\pi i \sigma}
\]

Theorem: (Eichler-Zagier) A Siegel modular form \(f\) of weight \(K\) has a "Fourier-Jacobi expansion":

\[
f(Q, q; y) = \sum_{m=0}^{\infty} \phi_{K, m}(q; y) Q^m
\]

where \(\phi_{K, m}(q; y)\) are Jacobi forms of weight \(K\) and index \(m\).

For all \((a, b, c, d) \in SL_2(\mathbb{Z})\) and \(\lambda, \mu \in \mathbb{Z}:

\[
\phi_{K, m} \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = (c\tau + d)^K \exp \left( \frac{2\pi imc z^2}{c\tau + d} \right) \phi_{K, m}(\tau, z)
\]

modular transformation
Example: The unique weight -2 index 1 (weak) Jacobi form:

\[ \phi_{-2,1}(q,y) = (y^{1/2} - y^{-1/2})^2 \prod_{n=1}^{\infty} \frac{(1- y^{1/2} q^n)^2 (1- y^{-1/2} q^n)^2}{(1-q^n)^2} \]

Given Siegel modular form \( f(Q,q,y) \), the Jacobi forms \( \{ \phi_{K,m} \}_{m=0}^{\infty} \) are in general, unrelated. But using “Hecke operators” \( V_m \) we can build a Siegel modular form from an index 1 Jacobi form:

Theorem: (Eichler–Zagier, Aoki) If \( \phi_{K,1} \) is an index 1 Jacobi form, then

\[ \text{ML}(\phi_{K,1}) := \sum_{m=0}^{\infty} V_m(\phi_{K,1}) Q^m \]

is a Siegel modular form of weight \( K \) called the Maass lift of \( \phi_{K,1} \).

Here \( V_m : \text{Jac}_{K,1} \to \text{Jac}_{K,m} \) is the Hecke operator, raising the index of a Jacobi form. For \( m > 0 \),

\[ V_m(\phi_{K,1}) = m^{K-1} \sum_{a \mid m} \sum_{b=0}^{d-1} (-1)^k \phi_{K,1}( \frac{a \tau + b}{d} , a \tau ) \]

Fact: Fourier coefficients only depend on the quantity \( 4m-l^2 \).

Proposition: If \( \phi_{K,1}(q,y) = \sum_{n,k} \sum_{l} c(4n-l^2) q^n y^l \) is the Fourier expansion of \( \phi_{K,1} \), then:

\[ \text{ML}(\phi_{K,1}) = -c(0) \frac{B_K}{2K} + \sum_{m,n} c(4mn-l^2) \sum_{r=1}^{\infty} r^{-K} \frac{R_m}{Q^m} \frac{R_n}{y^n} y^l \]

where (R. Borcherds)

\[ V_0(\phi_{K,1}) := -c(0) \frac{B_K}{2K} + \sum_{m,n} c(-l^2) \sum_{r=1}^{\infty} r^{-K} \frac{R_m}{Q^m} \frac{R_n}{y^n} y^l \]
§3. Main Computation:

I want to sketch a proof of the following main theorem:

**Theorem:** (Bryan, P–) For $g > 2$, the genus $g$ Gromov–Witten potential $F_g(Y)$ are meromorphic genus two Siegel modular forms of weight $2g - 2$. Moreover

$$F_g(Y) = ML \left( a_{2g} E_{2g}(q) \phi_{-2,1}(q') \right)$$

where $a_{2g} = \frac{(-1)^g}{(2g-2)!} \frac{B_{2g}}{2g}$ is a constant.

We use the GW/DT Correspondence: Under the change of variables $p = e^{2\lambda}$

$$\log Z'_\text{DT}(Y) = \sum_{g=0}^{\infty} F_g(Y) \lambda^{2g-2}$$

reduced DT partition function (divide by power of MacMahon function)

For $g > 2$, we have nice formula for degree zero contributions:

$$F^{(0)}_g(Y) = \frac{(-1)^g}{(2g-2)!} \frac{B_{2g}}{2g} \frac{B_{2g-2}}{2g-2}$$

$$Z'_\text{DT}(Y) = \prod_{(m,n,k) > 0} \prod_{k \in \mathbb{Z}} (1 - p^{km} q^m y^k)^{c\left(4m-2^2, k\right)}$$

$$\Rightarrow \log Z'_\text{DT}(Y) = \sum_{(m,n,k) > 0} \sum_{k \in \mathbb{Z}} c\left(4m-2^2, k\right) \log (1 - p^{km} q^m y^k)$$

$$= \sum_{(m,n,k) > 0} \sum_{k \in \mathbb{Z}} c\left(4m-2^2, k\right) \sum_{r=1}^{\infty} \frac{1}{r} p^{rk} q^m q^n y^k r^l$$
Equivariant Elliptic Genus of $\mathbb{C}^2$:

Under the change of variables \( p = e^{i\lambda} \), we have (Zhou):

\[
\text{Ell}_{\phi}(\mathbb{C}^2; p) = \sum_{g=0}^{\infty} \lambda^{2g-2} \left( a_{2g} \cdot E_{2g}(\phi) \cdot \phi_{_{2,1}}(\tau, \sigma) \right)
\]

Let $C_{2g-2}(4n-2)$ be the Fourier coefficients of $\phi$. Then we have:

\[
\sum_{K \in \mathbb{Z}} C(4m-2, K) p^r K = \sum_{g=0}^{\infty} C_{2g-2}(4n-2) (r \lambda)^{2g-2}
\]

\[
\log Z'_{DT}(Y) = \sum_{g=0}^{\infty} \lambda^{2g-2} \left( \sum_{(m, n, l) \geq 0} C_{2g-2}(4m-2) \sum_{n=1}^{\infty} r^{g-3} Q^{m} q^{n} y^{l} \right)
\]

\[
F_g'(Y), \text{ by GW/DT Correspondence.}
\]

\[
M_{\phi}(a_{2g} E_{2g}(\phi) \cdot \phi_{_{2,1}}(\tau, \sigma)) = -C_{2g-2}(0) \cdot \frac{B_{2g-2}}{2(2g-2)} + F_g'(Y)
\]

\[
C_{2g-2}(0) = -2a_{2g} \quad \text{(b/c the $g^0$ coefficient of $\phi_{_{2,1}}(\tau, \sigma)$)}
\]

The key is that $a_{2g} \frac{B_{2g-2}}{2g-2}$ is precisely the degree zero contributions to $F_g(Y)$. So:

\[
M_{\phi}(a_{2g} E_{2g}(\phi) \cdot \phi_{_{2,1}}(\tau, \sigma)) = F_g(Y).
\]

§ 4. Mirror Symmetry:

General Comments:

• It is expected that mirror symmetry places serious constraints on the
Gromov-Witten potentials $F_g(X)$ of a Calabi-Yau threefold $X$.

- A mathematician might think of $F_g(X)$ as just a formal series in curve variables $\mathbf{Q} = (Q_1, \ldots, Q_r)$.

- In physics, you treat $\mathbf{Q}$ as coordinates on the complexified Kähler cone, and $F_g(X)$ as an expansion "at large volume."

- But via the mirror map, $F_g(X)$ should live (at least locally) on the moduli space $\mathcal{M}$ of complex structures on the mirror $\hat{X}$:
  - Conjecturally, $F_g(X)$ should be a (local) section of a degree $2g-2$ line bundle (vacuum bundle) on $\mathcal{M}$.
    (See e.g. Dijkgraaf's "Mirror Symmetry and Elliptic Curves")
  - Or, $F_g(X)$ inherits monodromy action present on B-model side.

(Abozaid-Auroux-Katzarkov) There is a sense in which the "mirror" of a banana configuration is a smooth genus two curve. (Really, two mirror Landau-Ginsburg models)

Roughly: Symplectic volumes of three bananas $\leadsto$ Three complex periods, $\Omega \in H^2$.
Global Kähler moduli – known in physics as the stringy Kähler moduli space $M_{Kah}$ – is not well understood in general.

It is conjecturally related to the Bridgeland stability manifold.

But some recent progress has been made for the local banana manifold:

**Theorem: (Azam-Cannizzo-Lee-Liu)**

$$M_{Kah} = \frac{H_2}{G}$$

where $G = \{ (a, b) \in Sp_4(\mathbb{Z}) \mid c = 0 \} \subset Sp_4(\mathbb{Z})$.

The action of $G$ on $H_2$ is generated by:

- $\Omega \mapsto \Omega + b$, $b$: symmetric $2 \times 2$ integral matrix
- $\Omega \mapsto a \Omega a^T$, $a \in GL_2(\mathbb{Z})$

**Fact:** $G \subset Sp_4(\mathbb{Z})$ is the subgroup under which $F_g$ transforms trivially.
• All Siegel modular forms are invariant under $\Omega \mapsto \Omega + b$.

Necessary to have Fourier expansion: $Q = \exp(2\pi i\Omega)$, $\Omega_i = \exp(2\pi i\Pi)$,

• $F_g(a\Omega a^T) = \det(a)^{2g-2} \quad F_g(\Omega) = \frac{F_g(\Omega)}{(\pm 1)^{2g-2}}$

So pulling back $F_g(\Omega)$ by the covering map $H_2/G \longrightarrow \frac{H_2}{Sp(2, \mathbb{Z})}$ defines a function on $H_2/G$, i.e. a section of the trivial bundle.