Quantum $K$-theory of flag varieties via non-abelian localization

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Outline

1. Introduction
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3. Grassmannian Case
4. Flag Variety Case
5. Applications
Some background

The study of a K-theoretic analogue of the quantum cohomology, namely the quantum K-theory, was initiated at the beginning of this century.

- Givental *On the WDVV equation in quantum K-theory*
- Lee *Quantum K-theory. I. Foundations*
- Givental-Lee *Quantum K-theory on flag manifolds, finite-difference Toda lattices and quantum groups*
About a decade later, relations of such invariants to integrable systems and representation theory were explored.

- Okounkov *Lectures on K-theoretic computations in enumerative geometry*
- Aganagic-Okounkov *Elliptic stable envelopes*
Main result (non-rigorous formulation)

The permutation-invariant **big $J$-function**, which is a generating function of the invariants, plays a crucial role in the theory.

- $X = \text{Flag}(v_1, \ldots, v_n; N)$: the flag variety ($v_1 < \cdots < v_n < N$),
- $V_i$: tautological bundles of $X$ ($1 \leq i \leq n$),
  $P_{ij}$: K-theoretic Chern roots of $V_i$ ($1 \leq i \leq n, 1 \leq j \leq v_i$),
- $Q_i(1 \leq i \leq n)$: Novikov variables of $X$ corresponding to the determinant bundles of $V_i$.

**Theorem (X.Y.)**

*The image of the big $J$-function of $X$ is covered by the orbit of $\widetilde{J}$ with respect to a family of pseudo-finite-difference operators, where*

\[
\widetilde{J} = (1 - q) \sum_{d_{ij} \geq 0} \prod_{i,j} Q_{ij}^{d_{ij}} \frac{\prod_{i=1}^{n} \prod_{1 \leq r, s \leq v_i} d_{is} - d_{ir}}{\prod_{i=1}^{n} \prod_{1 \leq r \leq v_{i+1}} d_{is} - d_{i+1,r}} \left(1 - \frac{P_{is}}{P_{i+1,r}} q^l\right). 
\]
Questions

- Permutation-invariant big $\mathcal{J}$-function?
- Pseudo-finite-difference operators??
- Why $\tilde{\mathcal{J}}$???
The main theorem can be regarded as a reconstruction theorem of the big $\mathcal{J}$-functions of flag varieties, generalizing the result of Givental [3] where the target variety is required to have its K-ring generated by line bundles (e.g. toric varieties and complete flag varieties).

Reconstruction of a different flavor is provided in Iritani-Milanov-Tonita [7], where the big quantum K-ring is recovered from the small J-function through analysis of $q$-shift operators.
Permutation-invariant quantum K-theory

Assume $X$ is a smooth projective variety and $d \in H_2(X; \mathbb{Z})$.

Definition

$\overline{M}_{g,m}(X, d)$ is the moduli of stable maps $f : (C; p_1, \cdots, p_m) \to X$ of homological degree $d$ and genus $g$ with $m$ marked points.

- **Stability:** $C$ connected, nodal and projective;
  $p_1, \cdots, p_m$ smooth points on $C$;
  $|\text{Aut}(f, (C; p_1, \cdots, p_m))| < \infty$.

- **Equivalence:** $(f, (C; p_1, \cdots, p_m)) \sim (f', (C'; p'_1, \cdots, p'_m)) \iff \exists \varphi : (C; p_1, \cdots, p_m) \simto (C'; p'_1, \cdots, p'_m)$ with $f' \circ \varphi = f$.

$S_m$ acts naturally on $\overline{M}_{g,m}(X, d)$ by permuting the marked points.
Correlators

With the virtual structure sheaf defined by Lee [8], one can define **K-theoretic permutation-invariant correlators** (of genus 0):

**Definition**

\[
\langle aL_1^k, \ldots, aL_m^k \rangle^{S_m}_{0,m,d} = \chi^{S_m}(\overline{M}_{0,m}(X, d), \mathcal{O}_{\text{virt}} \otimes \bigotimes_{l=1}^n \text{ev}_l^*(a)L_l^k),
\]

where \(a \in K(X), \text{ev}_l : \overline{M}_{0,m}(X, d) \to X\) is the evaluation map at the \(l\)-th marked point, and \(L_l\) is the universal cotangent bundle at the \(l\)-th marked point over the moduli space.

\(S_m\) in the above construction may be replaced by any subgroup.
Let $\{\phi_\alpha\}$ be an additive basis of $K(X)$ and $\{\phi^\alpha\}$ be its dual basis.

The **K-theoretic permutation-invariant big $\mathcal{J}$-function** is defined by

**Definition**

$$
\mathcal{J}^X(t; q) = 1 - q + t(q) + \sum_{m,d,\alpha} Q^d \phi^\alpha \langle \frac{\phi_\alpha}{1 - qL_0}, t(L_1), \cdots, t(L_n) \rangle_{0,m+1,d}^{S_m}
$$

where $Q^d = \prod_i Q_i^{d_i}$ with $\{Q_i\}$ are the Novikov variables, and Laurent polynomial $t = t(q) = \sum_k t_k q^k$ is the input (with coefficients $t_k \in K(X)[[Q_1, \cdots, Q_n]]$).
Loop space formalism

Denote

\[ \mathcal{K} = K(X)[[Q_1, \cdots, Q_n]](q^{\pm 1}) \]
\[ \mathcal{K}_+ = K(X)[[Q_1, \cdots, Q_n]][q, q^{-1}] \]
\[ \mathcal{K}_- = \{ f \in \mathcal{K} | f(0) \neq \infty, f(\infty) = 0 \} \]

Fact

\( \mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_- \) is a Lagrangian polarization under the symplectic pairing

\[ \Omega(f, g) = \text{Res}_{q \neq 0, \infty} \langle f(q^{-1}), g(q) \rangle \frac{dq}{q} \]

where \( \langle \cdot, \cdot \rangle \) is the K-theoretic Poincaré pairing.
Loop space formalism

Under this polarization,

\[
\mathcal{J}^X : \mathbf{t} \mapsto 1 - q + \mathbf{t}(q) + \sum_{m,d,\alpha} Q^d \phi^\alpha \left\langle \frac{\phi^\alpha}{1 - qL_0}, \mathbf{t}(L_1), \ldots, \mathbf{t}(L_n) \right\rangle_{0,m+1,d}^{S_m}
\]

is a map from $\mathcal{K}_+$ to $\mathcal{K}$.

**Fact**

*The image $\mathcal{L}^X$ of $\mathcal{J}^X$ is an overruled cone in $\mathcal{K}$.*

$\mathcal{J}^X(0) \in \mathcal{L}^X$ is called the **small $J$-function**.
Fact ([6][3][5])

Let $D$ be any Laurent polynomial. Then,

- ruling spaces of $\mathcal{L}^X$ are invariant under operators like $e^{D(Pq^Q, Q, q)}$;
- $\mathcal{L}^X$ is invariant under operators like $e^{\sum_{k>0} \frac{\psi_k(D(Pq^kQ^Q, Q, q))}{k(1-q^k)}}$.

Here $P$ represent line bundles and $Q$ represent the Novikov variables associated to $P$.

We denote by $\mathcal{P}$ the group generated by operators above. $\mathcal{L}^X$ is preserved by $\mathcal{P}$.
Recall our questions:

- Permutation-invariant big $\mathcal{J}$-function?
- Pseudo-finite-difference operators??
- Why $\tilde{\mathcal{J}}$???
Recall our questions:

- Permutation-invariant big $\mathcal{J}$-function?
- Pseudo-finite-difference operators??
- Why $\tilde{\mathcal{J}}$??

One-line answer:

- Abelian/Non-Abelian Correspondence (“Non-abelian localization”)

We obtain $\tilde{\mathcal{J}}$, the “starting” point to generate the overruled cone $\mathcal{L}^X$ of the flag variety (the non-abelian quotient), from a twisted quantum K-theory of $Y$, the (abelian quotient) associated to $X$. 
The abelian quotient $Y$

We regard the flag variety $X$ as a GIT quotient of vector space

$$X = R//G = \text{Hom}(\mathbb{C}^v_1, \mathbb{C}^v_2) \oplus \cdots \oplus \text{Hom}(\mathbb{C}^v_n, \mathbb{C}^N)//\text{GL}(v_1) \times \cdots \times \text{GL}(v_n).$$

Then the associated abelian quotient $Y$ is defined as

$$Y = R//S = \text{Hom}(\mathbb{C}^v_1, \mathbb{C}^v_2) \oplus \cdots \oplus \text{Hom}(\mathbb{C}^v_n, \mathbb{C}^N)//(\mathbb{C}^\times)^{v_1} \times \cdots \times (\mathbb{C}^\times)^{v_n}.$$

Here $S \subset G$ is the maximal torus.
The abelian quotient $Y$

We regard the flag variety $X$ as a GIT quotient of vector space

\[ X = R//G = \text{Hom}(\mathbb{C}^{v_1}, \mathbb{C}^{v_2}) \oplus \cdots \oplus \text{Hom}(\mathbb{C}^{v_n}, \mathbb{C}^N)// GL(v_1) \times \cdots \times GL(v_n). \]

Then the associated abelian quotient $Y$ is defined as

\[ Y = R//S = \text{Hom}(\mathbb{C}^{v_1}, \mathbb{C}^{v_2}) \oplus \cdots \oplus \text{Hom}(\mathbb{C}^{v_n}, \mathbb{C}^N)//(\mathbb{C}^\times)^{v_1} \times \cdots \times (\mathbb{C}^\times)^{v_n}. \]

Here $S \subset G$ is the maximal torus.

The torus $T = (\mathbb{C}^\times)^N$ acts naturally on both $X$ and $Y$ by acting on $\mathbb{C}^N$. We denote the characters by $\Lambda_1, \cdots, \Lambda_N$. 
Example

When $X = \text{Fl}(1, 2; 3)$,

$$Y = \text{Hom}(\mathbb{C}, \mathbb{C}^2) \oplus \text{Hom}(\mathbb{C}^2, \mathbb{C}^3)/\mathbb{C}^\times \times (\mathbb{C}^\times)^2.$$ 

\[\begin{array}{ccc}
\mathbb{C}P^1 & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\mathbb{C}P^2 \times \mathbb{C}P^2.
\end{array}\]

We denote by $P_{11}, P_{21}, P_{22}$ the tautological bundles of $Y$. These bundles generate the K-ring of $Y$. 

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In general, the picture of $Y$ is a tower of fiber bundles

$$(\mathbb{CP}^{v_2-1})^{v_1} \to Y$$

$$(\mathbb{CP}^{v_{n-1}-1})^{v_{n-2}} \to F_{n-2}$$

$$(\mathbb{CP}^{v_{n-1}-1})^{v_{n-1}} \to F_{n-1}$$

$$(\mathbb{CP}^{N-1})^{v_n}.$$

We denote by $P_{ij}$ the **tautological bundle** $\mathcal{O}(-1)$ on the $j$-th copy of $\mathbb{CP}^{v_{i+1}-1}$ in the $i$-th level ($1 \leq i \leq n, 1 \leq j \leq v_i$).

We denote by $\{Q_{ij}\}_{i=1}^n_{j=1}^{v_i}$ the corresponding **Novikov variables** of $Y$. 
The abelian quotient $Y$

$$R^s(G)/S \xleftarrow{\iota} Y = R^s(S)/S$$

$$X = R^s(G)/G$$

where $R^s(G)$ and $R^s(S)$ stands for the stable locus of the $G$- and $S$-action respectively.

**Fact**

*We have the following relations of the tautological bundles*

$$l^* \bigoplus_{k=1}^{\nu_i} P_{ik} = q^* V_i.$$
Grassmannian case: main result (rigorous formulation)

For the case of grassmannian $X = \text{Gr}(\nu, N)$, we simplify our notations as follows.

- $V$: the (only) tautological bundle of $X$;
- $P_1, \cdots, P_\nu$: the tautological bundles of $Y = (\mathbb{C}P^{N-1})^\nu$;
- $Q$ and $Q_i (1 \leq i \leq \nu)$: the Novikov variables of $X$ and $Y$ respectively.

**Theorem (Main theorem)**

The orbit of $\tilde{j}^{tw,Y}$ under the group $\mathcal{P}^W$ of Weyl-group-invariant pseudo-finite-difference operators covers $\mathcal{L}^X$ under the specialization $Q_i = Q$ and $y = 1$, where

$$
\tilde{j}^{tw,Y} = \sum_{0 \leq d_1, \ldots, d_\nu} \prod_{i=1}^\nu Q_i^{d_i} \prod_{1 \leq i, j \leq \nu} \prod_{m=1}^{d_i - d_j} (1 - yq^m P_i / P_j) \frac{\prod_{i=1}^\nu \prod_{m=1}^{d_i} (1 - q^m P_i)^N}{\prod_{i=1}^\nu \prod_{m=1}^{d_i} (1 - q^m P_i)^N}.
$$
Grassmannian case: main result (rigorous formulation)

In fact, we prove the $T$-equivariant version of the above theorem.

**Theorem (Main theorem’, Givental-X.Y.)**

The orbit of $\tilde{J}^{tw,Y}$ under the group $\mathcal{P}^W$ of Weyl-group-invariant pseudo-finite-difference operators cover the image $\mathcal{L}^X$ of the $T$-equivariant permutation-invariant big $J$-function of $X$ under the specialization $Q_i = Q$ and $y = 1$, where

$$
\tilde{J}^{tw,Y} = \sum_{0 \leq d_1, \ldots, d_v} \prod_{i=1}^v Q_i^{d_i} \frac{\prod_{1 \leq i, j \leq v} \prod_{m=1}^{d_i - d_j} (1 - yq^m P_i / P_j)}{\prod_{i=1}^v \prod_{j=1}^N \prod_{m=1}^{d_i} (1 - q^m P_i / \Lambda_j)}.
$$

Taking $\Lambda_i \to 1$ gives us the previous theorem back.
The theorem has two aspects:

- elements in the orbit of $\tilde{J}^{tw,Y}$ lie on $L^X$;
- all points on $L^X$ appear in the orbit $\tilde{J}^{tw,Y}$. 
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- elements in the orbit of $\tilde{j}^{tw,Y}$ lie on $L^X$;
- all points on $L^X$ appear in the orbit $\tilde{j}^{tw,Y}$. → save for later
Idea: abelian/non-abelian correspondence

elements in the orbit of $\tilde{J}_{tw, Y}$ lie on the image of big $J$-function of $X$

\[ \uparrow \]

$\tilde{J}_{tw, Y}$ lies on the image of big $J$-function of $X$

\[ \uparrow \]

$\tilde{J}_{tw, Y}$ lies on the image of big $J$-function of $Y$ twisted by $g/s$

$+$

big $J$-function of $Y$ twisted by $g/s$ “=” big $J$-function of $X$
Idea: abelian/non-abelian correspondence

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big $J$-function of $Y$ twisted by $g/s$ “=” big $J$-function of $X$

$\uparrow$

(Fixed point localization)
(Classical) Abelian/non-abelian correspondence

\[ \begin{array}{ccl}
R^s(G)/S & \overset{i}{\rightarrow} & Y = R^s(S)/S \\
\downarrow q & & \\
X = R^s(G)/G & & 
\end{array} \]

**Fact ([10])**

Let \( \sigma \in H^*_T(X) \) and \( \tilde{\sigma} \in H^*_T(Y) \) such that \( i^* \tilde{\sigma} = q^* \sigma \). Then,

\[
\frac{1}{|W|} \int_Y \omega \tilde{\sigma} = \int_X \sigma,
\]

where \( \omega = \text{Eu}(g/\mathfrak{s}) \).
Quantum abelian/non-abelian correspondence

Previous works using this idea:

- Bertram-Ciocan-Fontanine-Kim *Two proofs of a conjecture of Hori and Vafa, Gromov-Witten invariants for abelian and nonabelian quotients*
- Ciocan-Fontanine-Kim-Sabbah *The abelian/nonabelian correspondence and Frobenius manifolds*
- Webb *The abelian-nonabelian correspondence for I-functions*
- Wen *K-theoretic I-functions of \( V/\theta G \) and applications*
- González-Woodward *Quantum Witten localization and abelianization for qde solutions*, *Quantum Kirwan for quantum K-theory*
Idea: fixed point localization

\[ \tilde{J}_{tw,Y} \text{ lies on the image of big } J\text{-function of } Y \text{ twisted by } g/\mathfrak{s} \]

\[ + \]

big \( J \)-function of \( Y \) twisted by \( g/\mathfrak{s} \) \( \equiv \) big \( J \)-function of \( X \)
Idea: fixed point localization

\[ \tilde{J}^{tw,Y} \text{ lies on the image of big } J \text{-function of } Y \text{ twisted by } g/\mathfrak{s} \]

+ 

big \( J \)-function of \( Y \) twisted by \( g/\mathfrak{s} \) “=” big \( J \)-function of \( X \)

This may be proved by a recursive characterization of big \( J \)-functions based on fixed point localization.
Assume that $M$ has isolated fixed points under a torus action by $T$, and that the fixed points are connected by isolated one-dimensional $T$-orbits. Any $q$-rational function $f \in \mathcal{K}$ has the expansion

$$f = \sum_{a \in \mathcal{F}} f_a \phi^a$$

where $\{\phi^a\}_{a \in \mathcal{F}}$ are fixed point classes. Then, the following characterization of big $\mathcal{I}$-function holds [2]:

**Fact**

$f$ represents a value of $\mathcal{L}^M \ if \ and \ only \ if \ it \ satisfies \ Conditions \ (i) \ and \ (ii)$. 
Fixed point localization

(i) $f_a$, when expanded as meromorphic functions with **poles only at roots of unity**, lies in $\mathcal{L}^{pt}$, the cone of the permutation-invariant quantum K-theory for point target space with coefficient ring $K(M)[[Q]]$.

(ii) Outside 0, $\infty$ and roots of unity, $f_a$ has **poles only at values of the form** $\lambda^{1/m}$ with $\lambda$ a $T$-character of the tangent space $T_a M$ and $m$ a positive integer, and the residues satisfy the recursion relations

$$\text{Res}_{q=\lambda^{1/m}} f_a(q) \frac{dq}{q} = \frac{Q^{mD}}{m} \frac{\text{Eu}(T_a M)}{\text{Eu}(T_\phi \overline{M}_{0,2}(M, mD))} f_b(\lambda^{1/m}).$$
\( J_{tw,Y} \) is on twisted theory of \( Y \)

**Proposition**

\( J_{tw,Y} \) represents a value of the \((\text{Eu}, y^{-1}q/s)\)-twisted big \( J \)-function of the abelian quotient \( Y \).
Proposition

\[ \tilde{J}^{tw,Y} \text{ represents a value of the \((E_u, y^{-1}g/s)\)-twisted big } \mathcal{J} \text{-function of the abelian quotient } Y. \]

- One can directly check the recursion relations needed by the twisted theory.
- Alternatively, one could use the Quantum Adams-Riemann-Roch theorem [4] which describes the twisted big \(\mathcal{J}\) function in terms of the untwisted big \(\mathcal{J}\) function.
Twisted big $\mathcal{J}$-function of $Y$"=" big $\mathcal{J}$-function of $X$

Quantum K-theory of $Y = R//S$ twisted by $y^{-1}g/s$

$$\text{Res}_{q=\lambda^{1/m}} f_a \frac{dq}{q} = \frac{\prod_i Q_i^{mD_i}}{m} \frac{\text{Eu}(T_a Y)}{\text{Eu}(y^{-1}g/s)|_a} \frac{\text{Eu}((y^{-1}g/s)_{0,2,mD})|_\phi}{\text{Eu}(T_{\phi} Y_{0,2,mD})} f_b(\lambda^{1/m}).$$

$\downarrow$

Under the limit $Q_i = Q, y = 1$

$\downarrow$

Quantum K-theory of $X = R//G$

$$\text{Res}_{q=\lambda^{1/m}} f_a(q) \frac{dq}{q} = \frac{Q^{m\sum_i D_i}}{m} \frac{\text{Eu}(T_a X)}{\text{Eu}(T_{\phi} X_{0,2,m\sum_i D_i})} f_b(\lambda^{1/m}).$$

In other words, we check the recursion coefficients of the two theories coincide, under the specialization $Q_i = Q, y = 1$. 
Generating functions of quantum K-theory invariants of *symplectic* quiver varieties defined by quasi-map compactifications appear in the study of quantum integrable systems and representation theory. One often needs such functions to be **balanced** [11, 9] in order to apply rigidity arguments.

For the case of $T^*Gr(v, N)$, one may consider $I =$

$$
\sum_{0 \leq d_i} Q^{\sum_i d_i} \frac{ \prod_{i \neq j}^{1 \leq i, j \leq v} P_i - q^m P_j } { \prod_{i \neq j}^{1 \leq i, j \leq v} P_i - q^m P_j } \prod_{i=1}^{v} \prod_{j=1}^{N} \prod_{m=0}^{d_i-1} (1 - q^m P_i / \Lambda_j) \prod_{i=1}^{v} \prod_{j=1}^{N} \prod_{m=1}^{d_i} (1 - q^m P_i / \Lambda_j),
$$

where $\hbar$ denotes the equivariant parameter of an extra fiberwise $\mathbb{C}^\times$-action on $T^*Gr(v, N)$. 

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Question: Can $I$ be realized in terms of the language we introduced earlier?
Remarks

- Question: Can $I$ be realized in terms of the language we introduced earlier?
- Yes, but after certain *twistings*.

**Fact**

Let $X = \text{Gr}(\nu, N)$. Then $I/\text{Eu}(TX)$ lies on the image $\mathcal{L}^{\text{Eu},TX}$ of the big $\mathcal{J}$-function of $X$ twisted by its tangent bundle.

This may be proved using the same method.
Note however that $I =$

$$
\sum_{0 \leq d_i} Q^{\sum_i d_i} \frac{\prod_{i \neq j}^{1 \leq i,j \leq v} \prod_{m=1}^{d_i-d_j} (1 - q^m P_i/P_j)}{ \prod_{i \neq j}^{1 \leq i,j \leq v} \prod_{m=0}^{d_i-d_j-1} (1 - \hbar q^m P_i/P_j)} \frac{\prod_{i=1}^{v} \prod_{j=1}^{N} \prod_{m=0}^{d_i-1} (1 - \hbar q^m P_i/\Lambda_j)}{\prod_{i=1}^{v} \prod_{j=1}^{N} \prod_{m=1}^{d_i} (1 - q^m P_i/\Lambda_j)},
$$

is not the small J-function in the twisted theory. In other words, under the polarization $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$, $I = 1 - q + t + \mathcal{K}_-$ with $t \neq 0$.

This is due to the possible $\hbar$-terms in the denominator.
Recall that

\[ X = \text{Hom}(\mathbb{C}^{v_1}, \mathbb{C}^{v_2}) \oplus \cdots \oplus \text{Hom}(\mathbb{C}^{v_n}, \mathbb{C}^{N}) \text{//} \text{GL}(v_1) \times \cdots \times \text{GL}(v_n) \]

\[ Y = \text{Hom}(\mathbb{C}^{v_1}, \mathbb{C}^{v_2}) \oplus \cdots \oplus \text{Hom}(\mathbb{C}^{v_n}, \mathbb{C}^{N}) \text{//}(\mathbb{C}^{\times})^{v_1} \times \cdots \times (\mathbb{C}^{\times})^{v_n} \]
New flavor in the flag variety case

Recall that in grassmannian case,

- $T$-fixed points of $X$ are the coordinate subspaces and are isolated;
- $T$-fixed points of $Y = (\mathbb{C}P^{N-1})^v$ are also isolated.
New flavor in the flag variety case

Recall that in grassmannian case,

- $T$-fixed points of $X$ are the coordinate subspaces and are isolated;
- $T$-fixed points of $Y = (\mathbb{C}P^{N-1})^v$ are also isolated.

In flag variety case, however,

- $T$-fixed points of $X$ are the standard flags and are still isolated;
- but $T$-fixed points of $Y$ are no longer isolated!

For simplicity of notations, we will mainly consider the case $X = \text{Fl}(1, 2; 3)$. The method carries over to all partial flag varieties entirely.
New flavor in the flag variety case

Example

For $X = Fl(1, 2; 3)$, $Y$ is a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^2 \times \mathbb{CP}^2$.

- $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ are isolated $T$-fixed points of $Y$;

- $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are non-isolated $T$-fixed points of $Y$. In fact, the fixed point component containing $B$ and $C$ is isomorphic to $\mathbb{CP}^1$:

$$\left\{ \left( \begin{array}{cc} a \\ b \end{array} \right), \left( \begin{array}{cc} 0 \\ 1 \\ 0 \\ 0 \end{array} \right) \right\} | a, b \in \mathbb{C} \text{ not both zero} \right\}.$$
New flavor in the flag variety case

A (very sketchy) picture is shown below:
New idea

We will have to address the issue of “non-isolated recursion”.

\[ T^n \times \mathbb{P}^1 \times \mathbb{P}^1 \]
We will have to address the issue of “non-isolated recursion”.

The most direct idea is to enlarge the torus action on $Y$. We enlarge $T \to \tilde{T}$

- $\tilde{T} = T \times (\mathbb{C}^\times)^2$ with the extra action scaling the two entries of $\text{Hom}(\mathbb{C}, \mathbb{C}^2)$ (i.e. rotating the fibers $\mathbb{C}P^1$).
- We denote by $\Lambda_4, \Lambda_5$ the equivariant parameters of the extra action.
- Now, $A, B, C, D$ are all isolated fixed points of $\tilde{T}$-action.
Main result: flag variety case

We may follow the same idea of grassmannian case. Under the action of the enlarged torus $\tilde{T}$, we have

Theorem (Main theorem, X.Y.)

The orbit of $\tilde{\mathcal{J}}^{tw, Y}$ under the group of Weyl-group-invariant pseudo-finite-difference operators covers the entire image $L^X$ of the big $\mathcal{J}$-function of $X$ under the specialization $\Lambda_4 = \Lambda_5 = 1$, $Q_{ij} = Q_i$ and $y = 1$, where

$$\tilde{\mathcal{J}}^{tw, Y} = (1 - q) \sum_{d_{ij} \geq 0} \prod_{d_{ij} \geq 0} Q_{ij}^{d_{ij}} \cdot \prod_{l=1}^{d_{21} - d_{22}} (1 - y \frac{P_{21}}{P_{22}} q^l) \prod_{l=1}^{d_{22} - d_{21}} (1 - y \frac{P_{22}}{P_{21}} q^l) \prod_{s=1}^{2} \prod_{l=1}^{d_{11} - d_{2s}} (1 - \frac{P_{11}}{P_{2s} \Lambda_{s+3}} q^l) \cdot \prod_{r=1}^{3} \prod_{s=1}^{2} \prod_{l=1}^{d_{2r}} (1 - \frac{P_{2r}}{\Lambda_s} q^l).$$
Idea

Consider the 1-dim $\tilde{T}$-orbit $AD$ as an example.

**Step 1:** $\tilde{J}^{tw,Y}_A$ satisfies the recursion relations of the $\tilde{T}$-equivariant $(Eu, y^{-1}g/s)$-twisted big $J$-function of $Y$;

**Step 2:** Under the specialization $\Lambda_4 = \Lambda_5 = 1$, $Q_i = Q$ and $y = 1$, the twisted recursion along $AD$ of $Y$ descends correctly to the expected recursion along $AD$ of $X$. 
Consider the 1-dim $\tilde{T}$-orbit $AD$ as an example.

**Step 1:** $\tilde{j}^{tw,Y}|_A$ satisfies the recursion relations of the $\tilde{T}$-equivariant $(Eu, y^{-1}g/s)$-twisted big $J$-function of $Y$;

**Step 2:** Under the specialization $\Lambda_4 = \Lambda_5 = 1$, $Q_i = Q$ and $y = 1$, the twisted recursion along $AD$ of $Y$ descends correctly to the expected recursion along $AD$ of $X$.

- But are we done?
**Idea**

**NO!!** Both $AD$ and $AB$ contributes to the residue of of $\tilde{J}^{tw,Y}|_A$ at the pole $q = (\Lambda_2/\Lambda_1)^{1/m}$ as $\Lambda_4, \Lambda_5 \to 1$. 
Non-isolated recursion

Essentially, we are showing that the total non-isolated recursion from the component $BC$ vanishes as $\Lambda_4 = \Lambda_5 = 1$, $Q_i = Q$ and $y = 1$. 
Essentially, we are showing that the total **non-isolated recursion** from the component \( BC \) vanishes as \( \Lambda_4 = \Lambda_5 = 1, \ Q_i = Q \) and \( y = 1 \).

For partial flag varieties in general, at an isolated fixed point (like \( A \)), we prove the vanishing of recursion from the “degenerate” orbits (like \( AB \)) following this same idea:

- complete it into non-isolated recursion from a fixed-point component (like \( BC \)) by taking **balanced broken orbits** (like \( ADC \)) into consideration;
- prove that both the total non-isolated recursion and the added terms, which themselves are “lower non-isolated recursions”, vanish.
A special case of the main theorem:

**Corollary**

\[
J^X = (1 - q) \sum_{d_{ij} \geq 0} \prod_i Q_i^{\sum_j d_{ij}} \frac{\prod_{s=1}^{\nu_i} \prod_{r \neq s}^{1 \leq r \leq v_i} \prod_{l=1}^{d_i} \left(1 - \frac{P_{is}}{P_{ir}} q^l\right)}{\prod_{s=1}^{\nu_i} \prod_{r \leq v_i + 1}^{1 \leq r \leq \nu_i} \prod_{l=1}^{d_i - d_{i+1, r}} \left(1 - \frac{P_{is}}{P_{i+1, r}} q^l\right)}
\]

represents a value of the big $J$-function of $X$.

This is actually the small $J$-function.
Similar to the grassmannian case, we may consider balanced generating functions $I^X$ of K-theoretic quasi-map invariants of $T^*X$.

Denote by $J_d^X$ the coefficient of $Q^d$ in the small $J$-function $J^X$. Then, $I^X = \sum_d Q^d J_d^X$ takes the form

$$ I_d^X = J_d^X \cdot \frac{\prod_{i=1}^n \prod_{1 \leq r \leq v_i} \prod_{l=0}^{d_is-d_{i+1},r-1} (1 - \hbar \frac{P_{is}}{P_{i+1,r}} q^l)}{\prod_{i=1}^n \prod_{r \neq s} \prod_{l=0}^{d_is-d_{ir}-1} (1 - \hbar \frac{P_{is}}{P_{ir}} q^l)} $$

In fact, we have

**Fact**

$I/\text{Eu}(TX)$ represents a point on the image $L^{\text{Eu},TX}$ of the big $J$-function of $X$ twisted by its tangent bundle.
Recall the theorem has two aspects:

- elements in the orbit of $\tilde{J}^{tw,Y}$ lie on $L^X$;
- all points on $L^X$ appear in the orbit of $\tilde{J}^{tw,Y}$. → do this now
Surjectivity argument

Recall the theorem has two aspects:

- elements in the orbit of $\tilde{J}^{tw,Y}$ lie on $L^X$;
- all points on $L^X$ appear in the orbit of $\tilde{J}^{tw,Y}$. → do this now

Idea:

- We use the invariance of $L^X$ under pseudo-finite-difference operators to generate a family on it from $\tilde{J}^{tw,Y}$.
- We want to show that the projection of this family to $K_+$ covers the entire $K_+$: this is correct mod $Q$ by quantum K-theory of point target space, and is thus correct with $Q$ by Formal Implicit Function Theorem (Nakayama’s Lemma).
Applications: level structures

Recently, the level structures are introduced to quantum K-theory, inspiring new progress in the field.

- Ruan-Zhang *The level structure in quantum K-theory and mock theta functions*
- Ruan-Wen-Zhou *Quantum K-theory of toric varieties, level structures, and 3d mirror symmetry*
Level structure

Definition

Let $E$ be a vector bundle on $X$ and $l$ be an integer. The level structure $(E, l)$ is defined as the modification

$$\mathcal{O}^{\text{virt}} \rightarrow \mathcal{O}^{\text{virt}} \otimes \det^{-l}(ft_* \ev^* E)$$

to the virtual structure sheaf.

We consider the quantum K-theory of flag varieties with level structures.
Using similar techniques as before, we can prove the following proposition.

Write $J^X = \sum_{d \geq 0} Q^d J^X_d$ as before, then the $q$-rational function

$$J^X, V_i, l = \sum_{d \geq 0} Q^d \cdot \left[ \prod_{s=1}^{v_i} P_{is}^{d_{is}} q^{\frac{d_{is}(d_{is} - 1)}{2}} \right]^l \cdot J^X_d$$

represents a point on the overruled cone $\mathcal{L}^{X, V_i, l}$ of $X$ with level structure $(V_i, l)$.

Moreover, this is the small $J$-function as $|l|$ is small.
Level correspondence

A correspondence between level-twisted big $\mathcal{J}$-functions of dual grassmannians was observed in [1]. This may be generalized to the case of flag varieties as follows.

Consider the flag varieties

$$X = \text{Flag}(v_1, v_2, \cdots, v_n; N)$$

and

$$X' = \text{Flag}(N - v_n, N - v_{n-1}, \cdots, N - v_1; N).$$

There is a $T$-equivariant isomorphism which is explicitly given by

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_n \subset \mathbb{C}^N \longleftrightarrow 0 \subset (V_n)^\perp \subset (V_{n-1})^\perp \subset \cdots \subset (V_1)^\perp \subset (\mathbb{C}^N)^*.$$

Both $X$ and $X'$ have $n$ tautological bundles, and we name them $V_i$ and $V'_i$ respectively.
The following fact is not hard to prove.

**Fact**

\[ L^{X_i,V_i,l} = L^{X'_i,(V'_i)^\vee,-l}. \]

Therefore, combining the fact with what we have proved above, we have

**Corollary**

*When \(|l|\) is small,*

\[ j^{X_i,V_i,l} = j^{X'_i,(V'_i)^\vee,-l}. \]
Thank you!!!
H. Dong and Y. Wen.
Level correspondence of K-theoretic $I$-functions in grassmannian duality.

A. Givental.
Permutation-equivariant quantum K-theory II. Fixed point localization.

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