§1 GLSMs
a) examples
b) state space
§2 Main theorem
a) proof —— 3 ingredients
- def, w/ light pts
- comparison of invts
- generating fctns
b) further questions

Def: A GLSM is the data (V, G, θ, w)
- A graded $\mathbb{C}$x vector space V
- A linearly reductive group $G \subseteq GL(V)$
- A choice of character $\theta \in \mathrm{Hom}(G, \mathbb{C}^*)$
- A G-invariant polynomial $w: V \to \mathbb{C}$,

homogeneous of deg $d > 0$ w.r.t. the grading

Given a GLSM, define
$$Y := [V /\!/_\theta G]$$
the GIT (stack) quotient.
w descends to a function $w: Y \to \mathbb{C}$
the pair (Y, w) is a "Landau-Ginzburg model"

we require, among other things, that the critical locus
$$\mathrm{Crit}(w) := \{dw = 0\} \subset Y$$
is compact.

One can define curve counting invariants for a GLSM, analogous to Gromov-Witten theory.

a) Examples

(i) If G is finite,

the LG model (Y, w) represents a singularity

$$w : [\mathbb{C}^N/G] \to \mathbb{C}$$

Curve counting invariants $\longleftrightarrow$ FJRW theory.

(ii) let X be a smooth projective variety

let E be a vector bundle on X,

and $s \in \Gamma(X, E)$ a regular section,

$\rightsquigarrow$ subvariety $Z = \{s = 0\} \subseteq X$

can construct a LG model of Z,

let $Y = \mathrm{tot}(E^\vee)$, the section $\check{s} : E^\vee \to \mathcal{O}_X$

induces a function $w : Y \to \mathbb{C}$

Philosophy: The LG model (Y, w) gives equiv. information to Z: eg. $Z \simeq \{dw = 0\} \subset Y$, $H^*(Z) = H^*(Y; \mathrm{Re}(w)^{-1}((M, \infty)))$

etc...

Often $Y = [V /\!/_\theta G]$, then

GW theory (Z) = GLSM theory (V, G, θ, w)

(Chang-Li, Kim-Oh, Ciocan-Fontanine-Kim-Guéré-S–)

So GLSM invariants generalize FJRW theory, GW theory of hypersurfaces and complete intersections.

A new example:

(iii) non-commutative resolutions

let $f, g \in \mathbb{C}[x_1, \dots, x_5]$ be general deg 4 homog. polys, let $s = x_1 \cdot f + x_2 \cdot g$.

$X = \mathbb{P}^4$, $Z = \{s = 0\} \subseteq \mathbb{P}^4$ has singularities at $x_1 = x_2 = f = g = 0 \rightsquigarrow$ 16 points

to resolve singularities: let $\tilde{X} = \mathrm{Bl}_{\{x_1 = x_2 = 0\}} X$

$\tilde{Z}$ = proper transform of Z under $\tilde{X} \to X$

$\tilde{Z} \to Z$ is a crepant resolution

Or use LG models: $s \in \Gamma(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5))$,

let $Y = \mathrm{tot}(\mathcal{O}_{\mathbb{P}^4}(-5))$, $s \rightsquigarrow w : Y \to \mathbb{C}$.

(Y, w) is a singular LG model

this is because $\mathrm{Crit}(w) \subset Y$ is non-compact

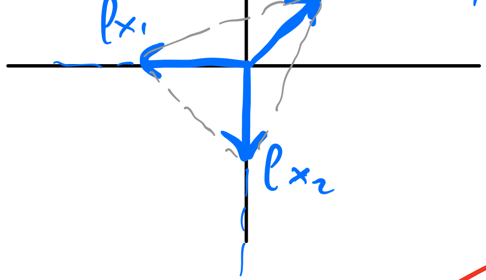
$$\mathrm{Crit}(w) = Z \cup \underbrace{\{x_1 = x_2 = f = g = 0\}}_{\text{16 fibers of } Y \to X}$$

We can "resolve singularities" by partially compactifying Y.

(natural way to do this)

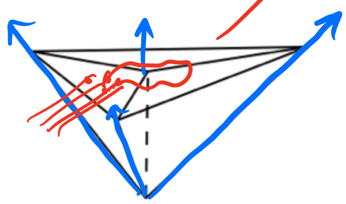
Toric picture

fan for $X = \mathbb{P}^4$



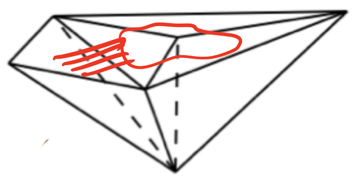
$\mathrm{Crit}(W)$

fan for Y



fan for $\bar{Y}$

($\mathbb{C}$-fibers of $Y \to X$ over $\{x_1 = x_2 = 0\}$ become $\mathbb{P}^1$'s)



W extends to $\bar{w} : \bar{Y} \to \mathbb{C}$

$(\bar{Y}, \bar{w})$ is a LG crepant resolution of Z.

Fact: $MF(\bar{Y}, \bar{w}) \simeq D(\tilde{Z})$

- categorical crepant resolution
- related to non-commutative crepant resolutions (Kuznetsov) (Van den Bergh)
- "exoflop" (Aspinwall)

Conj: GLSM invariants of $(\bar{Y}, \bar{w})$ are related to Gromov-Witten theory of $\tilde{Z}$ via analytic continuation / symplectic transformation.

"non-commutative crepant resolution conjecture"

b) State spaces

Let (Y, w) be an LG model.

for Gromov-Witten theory of Y, state space is $H^*(Y)$.

consider $\phi: H^*_{cs}(Y) \to H^*(Y)$

define the compact-type subspace to be

$H^*_{ct}(Y) := \mathrm{im}(\phi) \subset H^*(Y)$

has a perfect pairing!

the GLSM state space is $\mathcal{H}(Y, w) := H^*(Y; w^{+\infty})$

$w^{+\infty} := \mathrm{Re}(w)^{-1}((M, \infty))$ for $M \gg 0$

Given V a smooth proper subvariety of $w^{-1}(0)$,

$\exists$ pushforward $j'_*: H^*(V) \to \mathcal{H}(Y, w)$.

define $\mathcal{H}_{ct}(Y, w)$ to be the image of j'_* for all such V.

Rmk: If (Y, w) is an LG model for $Z = \{s = 0\} \subseteq X$ in the example, then $\mathcal{H}(Y, w) \cong H^*(Z)$

$\mathcal{H}_{ct}(Y, w) \cong$ cohomology pulled back from X.

Pullback via the map of pairs $(Y, \phi) \hookrightarrow (Y, w^{+\infty})$
induces $\phi^w : \mathcal{H}(Y,w) \to H^*(Y)$

$\mathcal{H}_{ct}(Y,w) \mapsto H^*_{ct}(Y)$

surjective on the compact-type subspace under mild assumptions.

let $\sigma_w : H^*_{ct}(Y) \to \mathcal{H}_{ct}(Y,w)$
be a choice of splitting $(\phi^w \circ \sigma_w = \mathrm{id})$.

§2 Main theorem.

Our main result is that in genus zero, the compact-type GLSM theory of (Y,w) can be obtained from GW theory of Y.
GLSM I-functions arise as derivatives of Gromov-Witten I functions of Y.

a) Proof: has 3 main steps:

(i) define the necessary GLSM invts (with light point insertions)

(ii) compare invariants

(iii) obtain generating functions

(i) defining GLSM invariants.

As usual, they are integrals against a virtual class on a moduli space $QLG(Y,d)$. The moduli space is usually not compact.

The various approaches to addressing this involve constructing a virtual class with compact support.

Approaches

1) Fan–Jarvis–Ruan, ct
2) Polishchuk–Vaintrob,
3) Ciocan-Fontanine–Favero–Guéré–Kim–S—,
4) Chang–Li–Li,
5) Favero–Kim very general, hard to compute

all invts, but not all GLSMs

Key Lemma: If evaluation maps

$$QLG_{g,n}(Y,d) \xrightarrow[ev_i]{} Y$$

are proper, and if $\alpha_1, \alpha_2 \in H^*_{ct}(Y)$, then

$ev_1^*(\alpha_1) \cup ev_2^*(\alpha_2)$ gives a well-defined class in $H^*_{cs}(QLG_{g,n}(Y,d))$

compact support.

In this case, can define GLSM invts for $\gamma_1, \dots, \gamma_n \in \mathcal{H}_{ct}(Y,w)$, $n \geq 2$ by

$$(*) \quad \langle \gamma_1, \dots, \gamma_n \rangle^{(Y,w)}_{g,n,d} := \int_{[QLG_{g,n}(Y,d)]^{vir}} \prod ev_i^*(\phi^w(\gamma_i)).$$

Rmk 1: $(*)$ agrees w/ 1), 2), 3) when both are defined.

Rmk 2: broad vanishing: $(*) = 0$ if $\gamma_i \in \ker(\phi^w)$.

Rmk 3: the evaluation maps are rarely proper. But they <u>are</u> if $g=0$, $n=2$, $\varepsilon = 0^+$, for <u>any</u> GLSM, (also if we add light points).

the ones needed for a mirror theorem.

(ii) comparing invts:

<u>Thm</u>: For $\gamma_1, \gamma_2 \in H_{ct}(Y,w)$,

$$\langle \gamma_1 \psi_1^{k_1}, \gamma_2 \psi_2^{k_2} \rangle_{0,2,d}^{(Y,w)} = \langle \phi^w(\gamma_1)\psi_1^{k_1}, \phi^w(\gamma_2)\psi_2^{k_2} \rangle_{0,2,d}^{Y}$$

GLSM invt (0^+-stable)

quasi-map invt

for $\alpha_1, \dots, \alpha_k \in H^*([V/G])$,

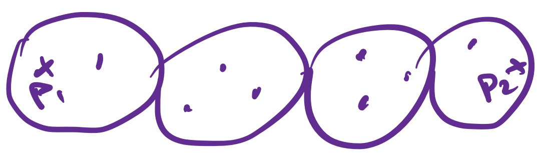
$$\langle \gamma_1 \psi_1^{k_1}, \gamma_2 \psi_2^{k_2} \mid \tau(\alpha_1), \dots, \tau(\alpha_k) \rangle_{0,2|k,d}^{(Y,w)} = \langle \phi^w(\gamma_1)\psi_1^{k_1}, \phi^w(\gamma_2)\psi_2^{k_2} \mid \alpha_1, \dots, \alpha_k \rangle_{0,2|k,d}^{Y}$$

proof by example: say $V = \mathbb{C}^n$, $G = \mu_d$, $w = \sum_{i=1}^{N} x_i^d$

$$QLG_{0,2|k}([\mathbb{C}^N/\mu_d], d) = \left\{ \left(C, \begin{smallmatrix} L \\ \downarrow \\ C \end{smallmatrix}, s \in \Gamma(C, L^{\oplus N})\right) \;\middle|\; \begin{matrix} g(C) = 0 \\ 2 \text{ marked pts, } p_1, p_2 \\ k \text{ light pts} \\ L^{\otimes d} = \omega_C(p_1+p_2) \end{matrix} \right\}$$

$$Q_{0,2|k}([\mathbb{C}^N/\mu_d], d) = \left\{ \left(C, \begin{smallmatrix} L \\ \downarrow \\ C \end{smallmatrix}, s \in \Gamma(C, L^{\oplus N})\right) \;\middle|\; L^{\otimes d} = \mathcal{O}_C \right\}$$

$g=0$, $n=2$, stability forces the source curve to be



on such curves, $\omega_C(p_1+p_2) = \mathcal{O}_C$ canonically via residue map.

(iii) generating functions

GW (quasimap) invariants of Y

Define

$$\langle\langle \phi_1, \phi_2 \rangle\rangle^Y(t) := \sum_d \frac{q^d}{K!} \langle \phi_1, \phi_2 \,|\, t, \ldots, t \rangle^Y_{0,2|K,d}$$

$$S^Y(q,t,z)(\gamma) := \sum_i \phi_i \langle\langle \frac{\phi^i}{z-\psi}, \gamma \rangle\rangle^Y(t)$$

Define an I-function of Y to be any

$$I^Y(q,t,z) = S^Y(q,f(t),z)\, P(q,t,z)$$

with $P \in H^*(Y)[[q,t^i]][z]$ (only positive powers of z!)

If $P = 1 + O(t^i)$ then I^Y lies on Givental's Lagrangian cone $\mathcal{L}_Y$. In this generality, I^Y lies on $T_\tau \mathcal{L}_Y$ for some τ.

For $\gamma_1, \gamma_2 \in \mathcal{H}_{ct}(Y,w)$, define

GLSM invariants

$$\langle\langle \gamma_1, \gamma_2 \rangle\rangle^{Y,w}(t) := \sum_d \frac{q^d}{K!} \langle \gamma_1, \gamma_2 \,|\, t, \ldots, t \rangle^{Y,w}_{0,2|K,d},$$

$$S^{Y,w}(q,t,z)\gamma = \sum_i \gamma_i \langle\langle \frac{\gamma^i}{z-\psi}, \gamma \rangle\rangle^{Y,w}$$

$\{\gamma_i\}$ basis for $\sigma_w(H^*_{ct}(Y)) \subset \mathcal{H}_{ct}(Y)$.

(this determines all $g=0$, $n=2$ invts w/ compact type insertions by broad vanishing)

Define an I-function for (Y, w) to be anything of the form

$$I^{Y,w}(q,t,z) = S^{Y,w}(q, f(t), z)\, P(q,t,z)$$

for $P \in \mathcal{H}_{ct}(Y,w)[[q, t^i]][z]$.

Lemma: If $\gamma \in H^*_{ct}(Y)$, then

$$S^Y(q,t,z)\,\gamma \in H^*_{ct}(Y)$$

(only involves only compact type invariants)

Corollary: w/ $\sigma_w : H^*_{ct}(Y) \to \mathcal{H}_{ct}(Y, w)$ as before, for $\gamma \in H^*_{ct}(Y)$,

$$\sigma_w\left(S^Y(q,t,z)\,\gamma \right) = S^{Y,w}(q, \tau(t), z)\, \sigma_w(\gamma).$$

Corollary: If $I^Y = S^Y \cdot P$ is such that $P \in H^*_{ct}(Y)$ $\left(\Leftrightarrow I^Y \in H^*_{ct}(Y)\right)$ then $I^{Y,w} := \sigma_w(I^Y)$ an I function for (Y, w).

final challenge, the standard I functions for Y are never supported in $H^*_{ct}(Y)$. However derivatives of them are, and are still I functions.

$$\left(z\,\partial/\partial t^i \,(S^Y \cdot P) = S^Y \cdot \underbrace{z\nabla_i P}_{\text{nonnegative powers of } z!} \right)$$

(b) Further directions

(i) How do these invariants (w/ light points) compare to more standard definitions

(joint w/ Yang Zhou)

(ii) There is a mirror construction for toric GLSMs (Hori-Vafa, Clarke, Gross-Katzarkov-Ruddat) and non-abelian (Gu-Parsian-Sharpe)

Would like to show these I-functions correspond to periods of the mirror GLSM.