An extension of Taubes' Gromov invariant to Calabi–Yau 3-folds

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Symplectic manifolds and almost complex structures

- A (closed) symplectic manifold is pair (X, ω), consisting of a (closed) smooth manifold X and a differential 2-form ω on it which is closed and non-degenerate.
- An almost complex structure J on X (i.e. an endomorphism of TX such that $J^2 = -1$) is said to be
  1. tamed by ω if $\omega(v, Jv) > 0$ for all $0 \neq v \in T_xX$ and $x \in X$ and,
  2. compatible with ω if, in addition to being tamed, we also have $\omega(Jv, Jw) = \omega(v, w)$ for all $v, w \in T_xX$ and $x \in X$.
- The space of almost complex structures compatible with ω is denoted by $\mathcal{J}(X, \omega)$. It is a non-empty contractible space. (The same is true for tame almost complex structures as well.)
- Path connectedness of $\mathcal{J}(X, \omega)$ implies that the symplectic vector bundle (TX, ω) has well-defined Chern classes.

Gromov–Witten theory

- Fix a symplectic Calabi–Yau 3-fold (X, ω), i.e., X is compact without boundary, $\dim X = 6$ and $c_1(TX, \omega) = 0$.
- Given $J \in \mathcal{J}(X, \omega)$, $g \geq 0$ and $A \in H_2(X, \mathbb{Z})$, the moduli space $\overline{\mathcal{M}}_g(X, J, A)$ has virtual dimension 0.
  $\rightsquigarrow$ Gromov–Witten invariant $\mathrm{GW}_{A,g} \in \mathbb{Q}$, independent of J.
- Because of multiple covers and ghost components (which may have non-trivial automorphisms) these invariants are not $\mathbb{Z}$-valued!

Taubes' Gromov invariant

- For a closed symplectic 4-manifold Y, Taubes defined (in a 1996 paper) the so-called Gromov invariant

$$\mathrm{Gr} : H^2(Y, \mathbb{Z}) \to \mathbb{Z}$$

  for any $\mathrm{PD}(A) \in H^2(Y, \mathbb{Z})$ as a suitable $\mathbb{Z}$-weighted count of embedded J-holomorphic curves in class A.
- The definition involves a careful bifurcation analysis of the situation when a sequence of embedded tori converge to an unbranched double cover of another embedded torus. (More on this later.)
- Taubes later also proved the famous identity $\mathrm{SW} = \mathrm{Gr}$.

BPS invariants and GV formula

Conjecture (Gopakumar–Vafa '98)

Let X be a symplectic CY 3-fold. Then, there exist numbers $BPS_{A,h} \in \mathbb{Z}$ for all $h \geq 0$ and $A \in H_2(X,\mathbb{Z})$ satisfying the following identity

$$\sum_{A\neq 0, g\geq 0} GW_{A,g} t^{2g-2} q^A = \sum_{A\neq 0, h\geq 0} BPS_{A,h} \sum_{k=1}^{\infty} \frac{1}{k} \left(2\sin\left(\frac{kt}{2}\right)\right)^{2h-2} q^{kA}.$$

Moreover, for any $A \in H_2(X,\mathbb{Z})$, we have $BPS_{A,h} = 0$ for $h \gg 0$.

Theorem (Ionel–Parker, 2018)

Gopakumar–Vafa's integrality conjecture holds.

Theorem (Doan–Ionel–Walpuski arXiv:2103.08221)

Gopakumar–Vafa's finiteness conjecture holds.

Embedded curves

Fact

Away from a codimension 2 subset of $\mathcal{J}(X,\omega)$, all **simple** *holomorphic curves are* **embedded** *and have* **pairwise disjoint images**.

- Restrict attention to J as in the above fact for the remainder.
- Any non-constant J-holomorphic stable map $f' : \Sigma' \to X$ can then be factored uniquely as

$$\Sigma' \xrightarrow{\varphi} \Sigma \xrightarrow{f} X$$

  where Σ is a smooth closed Riemann surface, f is a J-holomorphic embedding and φ is holomorphic.

Super-rigidity

Definition

$J \in \mathcal{J}(X, \omega)$ is called **super-rigid** if, for all non-constant J-holomorphic stable maps

$$\Sigma' \xrightarrow{\varphi} \Sigma \subset X,$$

we have $\ker(\varphi^* D^N_{\Sigma,J}) = 0$, where

$$D^N_{\Sigma,J} : \Omega^0(\Sigma, N_\Sigma) \to \Omega^{0,1}(\Sigma, N_\Sigma)$$

is the **normal deformation operator** of the embedded J-curve $\Sigma \subset X$.

- If J is super-rigid, then given any sequence of embedded J_n-curves $\Sigma_n \subset X$ (of bounded genus and ω-area), with $J_n \to J$, we can find a subsequence converging to an embedded J-curve $\Sigma \subset X$.
- This allows us to separate embedded curves from multiple covers!

Generic super-rigidity

Conjecture (Bryan–Pandharipande '01)

Super-rigid almost complex structures in $\mathcal{J}(X,\omega)$ form a Baire subset.

- This conjecture has recently been resolved.

Theorem (Wendl 2019, arXiv:1609.09867)

The subset of $\mathcal{J}(X,\omega)$ where super-rigidity fails has codimension ≥ 1.

- Wendl actually proves a more precise statement which determines the codimensions of the various strata of this subset (corresponding to the Galois group of the covers involved and their representations).
- Doan–Walpuski arXiv:2006.01352 have provided a more general perspective on this result and clarified the conditions that make Wendl's proof work for Cauchy–Riemann operators.

Motivation

- The proofs of the GV conjecture (integrality as well as finiteness) do not show how to directly interpret the integers $\mathrm{BPS}_{A,h}$ enumeratively.
- Motivated by Taubes' construction of Gr in dimension 4 (and the GV formula), one may ask the following question.

Question

Is there a $\mathbb{Z}$-valued invariant analogous to Taubes' Gromov invariant for CY 3-folds? If so, how is it related to the BPS invariants?

- Zinger '11 and Doan–Walpuski arXiv:1910.12338 answer this question for primitive homology classes.
- Our recent paper arXiv:2106.01206 provides an answer to this question in the first non-trivial case when multiple covers are possible.

Main result

Theorem 1 (Bai–S., 2021)

Let (X, ω) be a CY 3-fold. Fix an ω-primitive class $A \in H_2(X, \mathbb{Z})$ and an integer $h \geq 0$. For a super-rigid $J \in \mathcal{J}(X, \omega)$, define the *virtual count of embedded genus* h *curves of class* $2A$ to be the integer

$$\mathrm{Gr}_{2A,h}(X, \omega, J) = \sum_{C' : 2A, h} \mathrm{sgn}(C') + \sum_{g \leq h} \sum_{C : A, g} \mathrm{sgn}(C) \cdot \mathrm{w}_{2,h}(D^N_{C,J})$$

where the first sum is over embedded J-curves C' of genus h and class $2A$, the second sum is over all genera $0 \leq g \leq h$ and J-curves C of genus g and class A and $\mathrm{w}_{2,h}$ are suitably defined integer weights.
Then, $\mathrm{Gr}_{2A,h}(X, \omega, J)$ is independent of the choice of J and the resulting integer $\mathrm{Gr}_{2A,h}(X, \omega)$ is a symplectic deformation invariant. Moreover, for $h = 0$, it coincides with the corresponding BPS invariant.

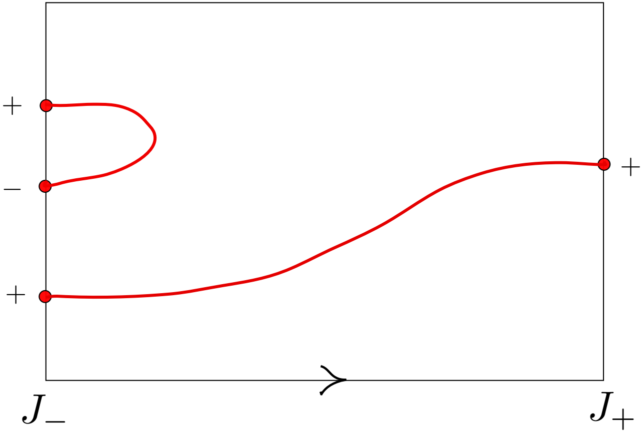
Outline of proof

- Super-rigid $J \Rightarrow$ finitely many embedded J-curves in any fixed homology class and genus. Therefore, the sums defining $\mathrm{Gr}_{2A,h}(X,\omega,J)$ are finite.
- A stronger finiteness result (assuming only an area bound) has been proved by Doan–Walpuski '21, but we do not need this for our purposes.
- To show symplectic invariance, we take a *generic* path (J_t) joining two given super-rigid almost complex structures J_- and J_+. In view of Wendl's theorem, we just need to explicitly understand the bifurcations that occur at the discrete set of times t for which J_t fails to be super-rigid.

Outline of proof

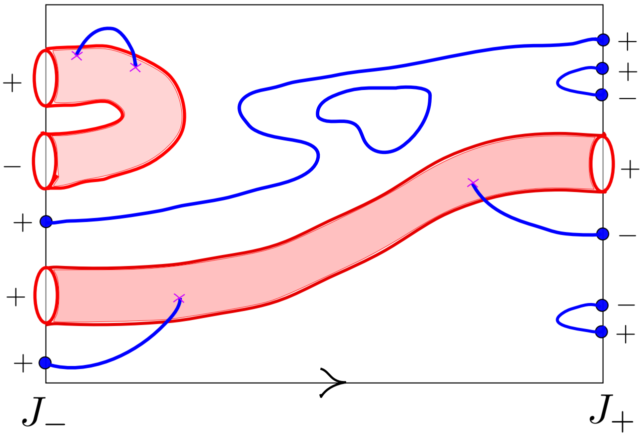
- The invariance argument has three key ingredients.
  1. A necessary condition for bifurcations to occur.
  2. A sufficient condition for bifurcations to occur.
  3. The construction of the "linear wall-crossing correction" $\mathrm{w}_{2,h}$.
- Proving the identity $\mathrm{Gr}_{2A,0} = \mathrm{BPS}_{2A,0}$ requires us to study how the (virtual) Euler numbers of certain obstruction bundles change for a generic 1-parameter family of $\mathbb{R}$-linear Cauchy–Riemann operators.

Stable maps of class A and genus $\leq h$ (preview)



- Embedded curves of class A are shown in **red**.

Stable maps of class $2A$ and genus h (preview)



- Double covers of embedded curves of class A are shown in **red**.
- Embedded curves of class $2A$ are shown in **blue**.
- Bifurcations are shown in **magenta**.

Necessary condition for bifurcations: statement

Theorem 2 (Bai–S., 2021)

Let $\gamma : [-1, 1] \to \mathcal{J}(X, \omega)$ be a smooth path and $t_n \to t$ be a convergent sequence in $[-1, 1]$. Set $J_n := \gamma(t_n)$ and $J := \gamma(t)$. Assume that there exists a sequence $\Sigma_n \subset X$ of smooth embedded J_n-curves of a fixed genus $h \geq 0$ which Gromov converges to the stable J-holomorphic map $\Sigma \xrightarrow{\varphi} C \subset X$ where C is a smooth embedded J-curve of genus $g \leq h$ and $(\Sigma, \varphi) \in \overline{\mathcal{M}}_h(C, d)$ for some $d \geq 1$. Then, exactly one of the following is true.

1. Either φ is an isomorphism ("*no bifurcation occurs*"),
2. or, there exists an element in $\ker \varphi^* D^N_{C,J}$ which is not pulled back from C via φ ("*a multi-valued normal deformation exists*").

Necessary condition for bifurcations: comments

- A weaker version of this is implicit in Wendl's paper arXiv:1609.09867. (There is an *extra assumption* that $\ker D^N_{C,J} = 0$.)
- Our result places a necessary condition on any multiply covered curve which is realized as a limit of embedded ones. In the situation of Theorem 1, it allows us to rule out degenerations into nodal stable maps (possibly with ghost components). The weaker version alluded to above does not suffice for Theorem 1.
- Our result also implies that a sequence of embedded curves of a fixed genus lying in a primitive homology class can't limit to an embedded curve with some ghost component(s) attached to it. This recovers the primitive class case by a simpler argument avoiding any appeal to GW theory (as in Zinger '11) or delicate gluing analysis (as in Doan–Walpuski arXiv:1910.12338).
- Theorem 2 is quite general. It is independent of any transversality assumptions on γ and works for *any* almost complex manifold.

Necessary condition for bifurcations: proof idea

- The proof is by a careful examination of the infinitesimal deformations and obstructions of the two moduli spaces

$$\overline{\mathcal{M}}_h(\mathcal{M}_g^{\mathrm{emb}}(X,\gamma,[C]),d) \subset \overline{\mathcal{M}}_h(X,\gamma,d[C])$$

  at the point $(t,\Sigma \xrightarrow{\varphi} C \subset X)$. Here, the latter is the usual stable map moduli space along γ while the former consists of those stable maps which factor through an embedded genus g curve in class $[C]$.
- The key assertion to prove is the following. The fibre of the normal bundle of the above inclusion, after suitably "thickening" both moduli spaces, at the point $(t,\Sigma \xrightarrow{\varphi} C \subset X)$ is canonically isomorphic to the vector space

$$\frac{\ker \varphi^* D^N_{C,J}}{\ker D^N_{C,J}}.$$

Sufficient condition for bifurcations: statement

The next result provides a converse to Theorem 2 by giving a precise description of what happens to the number of embedded curves when a bifurcation occurs (under an additional technical assumption).

Theorem 3 (Bai–S., 2021)

Let (J_t) be a *generic* path in $\mathcal{J}(X, \omega)$. Assume that for $t = 0$ there exists an embedded rigid J_0-curve $C \subset X$ along with a d-fold genus h branched multiple cover $\varphi : \Sigma \to C$ along which a non-trivial multi-valued normal deformation exists. If φ determines an **elementary wall type**, then

1. $\mathrm{Aut}(\varphi) \subset \mathbb{Z}/2\mathbb{Z}$ and,
2. the change in the signed count of embedded curves of genus h and class $d[C]$ near φ, when going from $t < 0$ to $t > 0$, is given by $\pm 2/|\mathrm{Aut}(\varphi)|$.

Sufficient condition for bifurcations: comments

- The technical notion of *elementary wall type* covers a large class of branched covers. For example, this includes all d-fold covers $\Sigma \xrightarrow{\varphi} C$ for which Σ is smooth, φ has the expected number of distinct branch points and the Galois group of φ is S_d.
- Theorem 3 comes out of an explicit analysis of local Kuranishi models along paths which are sufficiently *generic*.

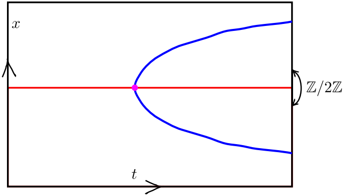
Sufficient condition for bifurcations: comments

- The notion of *generic* path in Theorem 3 deserves some elaboration.
  1. Firstly, γ should be transverse to the codimension 1 walls in $\mathcal{J}(X, \omega)$. This is not hard to arrange using Wendl's theorem.
  2. Secondly, γ should miss a certain codimension 1 subset of the wall (the *degeneracy locus*). This needs some work and is the place where the *elementary* nature of φ is used.
- The second condition is easiest to understand in Taubes' original setting of 2:1 cover bifurcations of embedded tori in a symplectic 4-manifold.

Sufficient condition for bifurcations: proof idea

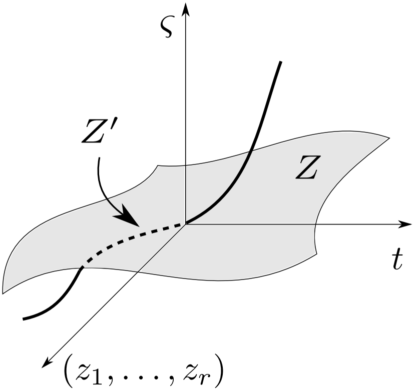
Focus on Taubes' setting of a 2:1 cover $T' \xrightarrow{\varphi} T$ of a torus by a torus.

- One has to argue that the moduli space locally looks like this:



  Here, $x \in \mathbb{R}$ is a coordinate on $\ker \varphi^* D^N_{T,J_0}$, double covers close to φ are shown in **red** and embedded curves are shown in **blue** and picture represents the locus $x(t - x^2) = 0$ modulo the non-trivial action of $\mathbb{Z}/2\mathbb{Z}$ on x.
- More precisely, one argues that the Kuranishi map is given by $f(t,x) = x(\alpha t + \beta x^2 + \cdots)$ with $\alpha \neq 0$ corresponding to genericity condition 1 and $\beta \neq 0$ corresponding to genericity condition 2.

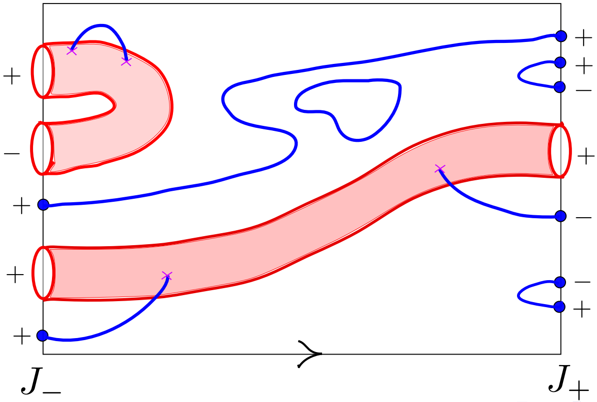
Schematic picture of the local Kuranishi model



In the picture, $(z_1, \ldots, z_r) \in \mathbb{C}^r$ are coordinates on $T_\varphi \mathcal{M}_h(C, d)$, $\varsigma \in \mathbb{R}$ is a coordinate on $\ker(\varphi^* D^N_{C,J})$ and Z, Z' are the local irreducible components of the moduli space near $(0, \Sigma \xrightarrow{\varphi} C \subset X)$.

Summary of bifurcation analysis

Theorems 2 and 3 together show that the following picture, introduced earlier, is indeed a faithful depiction of the moduli space of stable maps of class $2A$ and genus h along a generic path γ. (Showing finiteness of the number of bifurcation points also requires an appeal to Theorem 2 – it is not automatic from Wendl's theorem.)



Linear wall-crossing invariant

- As the previous picture shows, embedded curves (of class $2A$ and genus h) may appear or disappear when bifurcations occur. We need to account for this phenomenon by using a suitably defined *correction term* to get a symplectic invariant.
- For this purpose, we define the algebraic invariant $\mathrm{w}_{2,h}(\cdot) \in \mathbb{Z}$ for the normal deformation operator of any embedded pseudo-holomorphic curve of genus $g \leq h$ in class A. It can be viewed as a kind of *equivariant spectral flow*. Showing that it is well-defined takes some work and the proof uses Theorem 2 (as well as a result on generic embeddedness of normal multi-valued deformations).

Linear wall-crossing invariant

- Such constructions probably first originated in gauge theory (e.g. in Walker '92, Boden–Herald '98, Mrowka–Ruberman–Saveliev '11).
- Putting the definition of the *linear wall-crossing invariant* $\mathrm{w}_{2,h}$ together with the bifurcation analysis (Theorems 2 and 3) yields the proof of invariance in Theorem 1.

Comparison to BPS

- We can't expect $\mathrm{Gr}_{2A,h}(X,\omega)$ to exactly equal the BPS invariants for $h > 0$. Instead, we suspect that the following is true.

Conjecture

$\mathrm{Gr}_{2A,h}(X,\omega)$ coincides with the *virtual count of embedded clusters*, denoted $e_{2A,h}(X,\omega)$, defined by Ionel–Parker '18.

- Showing $\mathrm{Gr}_{2A,0}(X,\omega) = e_{2A,0}(X,\omega) = \mathrm{BPS}_{2A,0}(X,\omega)$ translates to a relation between $\mathrm{w}_{2,0}(D^N_{C,J})$ and the Euler number of a suitable obstruction bundle associated to $D^N_{C,J}$ (for each embedded J-curve C of genus 0 and class A).
- This is proved using a more general result about the change of the (virtual) Euler number of an obstruction bundle along a generic path of $\mathbb{R}$-linear Cauchy–Riemann operators. We use the virtual fundamental class (VFC) package developed by Pardon '16.

Obstruction bundles

Theorem 4 (Bai–S., 2021)

Let $\mathcal{D} = \{D_t\}_{t\in[-1,1]}$ be a *generic* 1-parameter family of $\mathbb{R}$-linear Cauchy–Riemann operators on $N \to \Sigma$ (which is a rank 2 complex vector bundle on a genus g Riemann surface Σ, with $c_1(N)\cdot[\Sigma] = 2g-2$). Suppose $([\varphi : \Sigma' \to \Sigma], t) \mapsto \operatorname{coker}(\varphi^* D_t)$ gives a vector bundle of the expected rank on the space

$$\overline{\mathcal{M}}_h(\Sigma, d) \times [-1,1] \setminus \Delta \times \{0\}$$

where $\Delta \subset \mathcal{M}_h(\Sigma, d)$ is a finite set where the rank jumps. Then, denoting the Euler number of this vector bundle over $t = \pm 1$ by $e_{d,h}(D_\pm)$, we have

$$e_{d,h}(D_+) - e_{d,h}(D_-) = \sum_{p\in\Delta} \frac{2\cdot \operatorname{sgn}(\mathcal{D}, p)}{|\operatorname{Aut}(p)|}$$

with $\operatorname{sgn}(\mathcal{D}, p) \in \{-1, +1\}$ determined by the behavior of $\mathcal{D}$ near p.

Further directions

- We do not give a complete answer to the motivating question due to the following obstacle: our sufficient condition for bifurcations (Theorem 3) only deals with *elementary wall types*. We hope to address this limitation in future work.
- Motivated by the present work, we may pose the following question.

Question

Is it possible to give an algebro-geometric construction of this analogue of Taubes' Gromov invariant for CY 3-folds?

Since one doesn't expect generic super-rigidity to hold in the algebro-geometric context, answering this question probably requires a novel compactification of the moduli space of embedded curves.

- Our bifurcation analysis arguments might be useful in other contexts e.g., in counting associative submanifolds in G_2 manifolds or Doan–Walpuski's proposal for a differential-geometric construction of the Pandharipande–Thomas invariants.

Thank you!